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Some properties of slowly increasing functions

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Abstract

A family of slowly increasing functions has been introduced in [1]. The main purpose of this article is to study further properties of slowly increasing functions. More specifically, we concentrate on constructing a family of differential equations involving slowly increasing functions in their coefficients.

1. Introduction

Let m and j be nonnegative integers and $m + j \geq 1$. In this article, we study the following ordinary differential equations:

$$-U''(r) = \frac{1}{4} r^{-2} \left(1 + S_j^{\langle m \rangle}(r)\right) U(r) \qquad (r \geq 1), \tag{1.1}$$

where the quantity $S_j^{\langle m \rangle}$ is defined by using slowly increasing functions which were introduced in [1]. It is known that slowly increasing functions in [1] are infinitely differentiable. For convenience, we recall their elementary properties in Section 3.

The definition of the functions $S_j^{\langle m \rangle}$ is a little bit complicated. For example,

$$S_0^{\langle 1 \rangle}(r) = \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} F^k(ar) \right)^{-2} \qquad (r \geq 1, a > 1),$$

where $F : [a, \infty) \to [a, \infty)$ is the function defined by

$$F(u) = a - \log a + \log u = a + \int_a^u \frac{1}{t}\, dt \qquad (u \geq a) \tag{1.2}$$

and

$$F^0(u) = u, \qquad F^k(u) = F(F^{k-1}(u)) \qquad (k \in \mathbb{N} = \{1, 2, \ldots\}).$$

We will give the definition of the functions $S_j^{\langle m \rangle}$ for general m and j in Section 2.

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For $u \geq a > 1$, let

$$\tilde{F}(u) = a \prod_{k=0}^{\infty} \frac{F^k(u)}{a} \tag{1.3}$$

and

$$\phi(u) = a + \int_a^u \frac{1}{\tilde{F}(t)}\, dt. \tag{1.4}$$

Then ϕ is a slowly increasing function defined in [1] and it has the following properties:

$$\lim_{u\to\infty} \phi(u) = \infty \quad \text{and} \quad \lim_{u\to\infty} \frac{\phi(u)}{F^k(u)} = 0 \quad \text{for all } k \in \{0\} \cup \mathbb{N}.$$

In this paper, we construct solutions $U = U_j^{\langle m\rangle}$ to the equation (1.1) by using these functions $\tilde{F}$ and ϕ. Then we have the following:

Theorem 1.1. *Let $a > 1$. Let $m, j \in \{0\} \cup \mathbb{N}$ and $m + j \geq 1$. Then the ordinary differential equation* (1.1) *has the general solutions*

$$U(r) = C_1 U_j^{\langle m\rangle}(r) + C_2 U_{j+1}^{\langle m\rangle}(r) \left(F^j(\phi^m(ar))\right)^{1/2} \qquad (r \geq 1)$$

with arbitrary constants $C_1, C_2 \in \mathbb{R}$, where $\phi^0(u) = u$ and $\phi^m(u) = \phi(\phi^{m-1}(u))$ $(m \in \mathbb{N})$ for $u \geq a$.

For example,

$$U_0^{\langle 1\rangle}(r) = \left(\frac{1}{a}\tilde{F}(ar)\right)^{1/2}, \quad U_1^{\langle 1\rangle}(r) = \frac{1}{a}\left(\tilde{F}(ar)\phi(ar)\right)^{1/2} \quad (r \geq 1).$$

We also remark that the above result can be applied to improve the classical weighted Hardy type inequalities. We give this application in the paper [2]. Note that the definitions of the functions $U_j^{\langle m\rangle}$ and $S_j^{\langle m\rangle}$ are the same as in the paper [2]. Using the functions $U_j^{\langle m\rangle}$ and $S_j^{\langle m\rangle}$, we have the following proposition as a special case of Theorem 1.1, which is used to improve the classical weighted Hardy type inequalities by adding infinitely many missing terms.

Proposition 1.1 ([2, Proposition 3.2])**.** *Let $m, j \in \{0\} \cup \mathbb{N}$ and $m + j \geq 1$. Then the equality*

$$-\left(U_j^{\langle m\rangle}(r)\right)'' = \frac{1}{4} r^{-2} \left(1 + S_j^{\langle m\rangle}(r)\right) U_j^{\langle m\rangle}(r) \qquad (r \geq 1)$$

holds.

Moreover by the direct calculation we can see the following.

Remark 1.1. Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Let $m, j \in \{0\} \cup \mathbb{N}$ and $m + j \geq 1$. Let $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$, and let $\mathcal{U}$ be the function defined by $\mathcal{U}(x) = |x|^{-\alpha-(n-1)/2} U_j^{\langle m \rangle}(|x|)$ for $x \in \Omega \cup \partial\Omega$. Then the function $\mathcal{U}$ is the solution to the boundary value problem

$$-\mathrm{div}\left(|x|^{2\alpha}\nabla\mathcal{U}\right) = |x|^{2\alpha-2}\left\{\left(\alpha + \frac{n-2}{2}\right)^2 + \frac{1}{4}\, S_j^{\langle m \rangle}(|x|)\right\}\mathcal{U} \quad \text{in} \ \ \Omega,$$

$$\mathcal{U} = 1 \qquad \text{on} \ \ \partial\Omega.$$

In Section 2, we construct solutions $U = U_j^{\langle m \rangle}$ to the equation (1.1) for $m, j \in \{0\} \cup \mathbb{N}$ with $m + j \geq 1$ and prove Theorem 1.1. For the self-containedness of this paper, in Section 3, we provide the elementary review of slowly increasing functions studied in [1].

2. Main results

In this section we state our main results precisely and prove them. We first prepare several functions. Let $a > 1$, and let $F, \tilde{F}$ and ϕ be the functions defined by (1.2), (1.3) and (1.4) respectively.

Definition 2.1. Let $m, j \in \{0\} \cup \mathbb{N}$. For $u \geq a$, let

$$V_{j+1}^{\langle 0 \rangle}(u) = \sum_{k=0}^{j}(F^{k+1}(u) - a) = \log\left(\prod_{k=0}^{j}\frac{F^k(u)}{a}\right), \tag{2.1}$$

$$V_0^{\langle m+1 \rangle}(u) = \sum_{l=0}^{m}\sum_{k=0}^{\infty}(F^{k+1}(\phi^l(u)) - a) = \sum_{l=0}^{m}\log\frac{\tilde{F}(\phi^l(u))}{a}, \tag{2.2}$$

$$\begin{aligned} V_{j+1}^{\langle m+1 \rangle}(u) &= V_0^{\langle m+1 \rangle}(u) + \sum_{k=0}^{j}(F^{k+1}(\phi^{m+1}(u)) - a) \\ &= \sum_{l=0}^{m}\log\frac{\tilde{F}(\phi^l(u))}{a} + \log\left(\prod_{k=0}^{j}\frac{F^k(\phi^{m+1}(u))}{a}\right) \end{aligned} \tag{2.3}$$

and

$$Y_j^{\langle m \rangle}(u) = \exp\left(\frac{1}{2}\, V_j^{\langle m \rangle}(u)\right) \qquad (m + j \geq 1). \tag{2.4}$$

Namely, for $u \geq a$ we have

$$\exp\left(V_{j+1}^{\langle 0 \rangle}(u)\right) = \prod_{k=0}^{j}\frac{F^k(u)}{a}, \tag{2.5}$$

$$\exp\left(V_0^{\langle m+1 \rangle}(u)\right) = \prod_{l=0}^{m}\frac{\tilde{F}(\phi^l(u))}{a}, \tag{2.6}$$

$$\exp\left(V_{j+1}^{\langle m+1 \rangle}(u)\right) = \prod_{l=0}^{m}\frac{\tilde{F}(\phi^l(u))}{a}\prod_{k=0}^{j}\frac{F^k(\phi^{m+1}(u))}{a} \tag{2.7}$$

and

$$Y_{j+1}^{\langle 0\rangle}(u) = \left(\prod_{k=0}^{j} \frac{F^k(u)}{a} \right)^{1/2}, \tag{2.8}$$

$$Y_{0}^{\langle m+1\rangle}(u) = \left(\prod_{l=0}^{m} \frac{\tilde{F}(\phi^l(u))}{a} \right)^{1/2}, \tag{2.9}$$

$$Y_{j+1}^{\langle m+1\rangle}(u) = \left(\prod_{l=0}^{m} \frac{\tilde{F}(\phi^l(u))}{a} \prod_{k=0}^{j} \frac{F^k(\phi^{m+1}(u))}{a} \right)^{1/2}. \tag{2.10}$$

Definition 2.2. Let $j \in \{0\} \cup \mathbb{N}$. For $u \geq a$, let

$$D_{j+1}^{\langle 0\rangle}(u) = \sum_{i=0}^{j} \left(\prod_{k=0}^{i} F^k(u) \right)^{-2}, \tag{2.11}$$

$$D_{0}^{\langle 1\rangle}(u) = \sum_{i=0}^{\infty} \left(\prod_{k=0}^{i} F^k(u) \right)^{-2} \tag{2.12}$$

and, for $m \in \mathbb{N}$,

$$D_{j+1}^{\langle m\rangle}(u) = D_0^{\langle m\rangle}(u) + \sum_{i=0}^{j} \left(\prod_{l=0}^{m-1} \tilde{F}(\phi^l(u)) \prod_{k=0}^{i} F^k(\phi^m(u)) \right)^{-2}, \tag{2.13}$$

$$\begin{aligned} D_{0}^{\langle m+1\rangle}(u) &= D_0^{\langle m\rangle}(u) + \sum_{i=0}^{\infty} \left(\prod_{l=0}^{m-1} \tilde{F}(\phi^l(u)) \prod_{k=0}^{i} F^k(\phi^m(u)) \right)^{-2} \\ &= \sum_{i=0}^{\infty} \left(\prod_{k=0}^{i} F^k(u) \right)^{-2} + \sum_{\nu=1}^{m} \sum_{i=0}^{\infty} \left(\prod_{l=0}^{\nu-1} \tilde{F}(\phi^l(u)) \prod_{k=0}^{i} F^k(\phi^\nu(u)) \right)^{-2}. \end{aligned} \tag{2.14}$$

Further we define the functions $U_j^{\langle m\rangle}$, $Q_j^{\langle m\rangle}$ and $S_j^{\langle m\rangle}$ in the following way.

Definition 2.3. Let $m, j \in \{0\} \cup \mathbb{N}$ and $m + j \geq 1$. For $r \geq 1$, let

$$U_j^{\langle m\rangle}(r) = Y_j^{\langle m\rangle}(ar), \tag{2.15}$$

$$Q_j^{\langle m\rangle}(r) = a^2 D_j^{\langle m\rangle}(ar) \tag{2.16}$$

and

$$S_j^{\langle m\rangle}(r) = r^2 \left(Q_j^{\langle m\rangle}(r) - Q_1^{\langle 0\rangle}(r) \right) = r^2 \left(Q_j^{\langle m\rangle}(r) - r^{-2} \right), \tag{2.17}$$

that is,

$$Q_j^{\langle m\rangle}(r) = r^{-2} \left(1 + S_j^{\langle m\rangle}(r) \right). \tag{2.18}$$

Then for each $m, j \in \{0\} \cup \mathbb{N}$ with $m+j \geq 1$ the explicit form of $S_j^{\langle m \rangle}$ is as follows.

$$S_1^{\langle 0 \rangle}(r) = 0, \tag{2.19}$$

$$S_{j+1}^{\langle 0 \rangle}(r) = (ar)^2 \sum_{i=1}^{j} \left(\prod_{k=0}^{i} F^k(ar) \right)^{-2} = \sum_{i=1}^{j} \left(\prod_{k=1}^{i} F^k(ar) \right)^{-2} \qquad (j \in \mathbb{N}), \tag{2.20}$$

$$S_0^{\langle 1 \rangle}(r) = \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} F^k(ar) \right)^{-2} \tag{2.21}$$

and, for $m \in \mathbb{N}$,

$$S_{j+1}^{\langle m \rangle}(r) = S_0^{\langle m \rangle}(r) + \sum_{i=0}^{j} \left((ar)^{-1} \prod_{l=0}^{m-1} \tilde{F}(\phi^l(ar)) \prod_{k=0}^{i} F^k(\phi^m(ar)) \right)^{-2}, \tag{2.22}$$

$$\begin{aligned} S_0^{\langle m+1 \rangle}(r) &= S_0^{\langle m \rangle}(r) + \sum_{i=0}^{\infty} \left((ar)^{-1} \prod_{l=0}^{m-1} \tilde{F}(\phi^l(ar)) \prod_{k=0}^{i} F^k(\phi^m(ar)) \right)^{-2} \\ &= \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} F^k(ar) \right)^{-2} + \sum_{\nu=1}^{m} \sum_{i=0}^{\infty} \left((ar)^{-1} \prod_{l=0}^{\nu-1} \tilde{F}(\phi^l(ar)) \prod_{k=0}^{i} F^k(\phi^\nu(ar)) \right)^{-2}. \end{aligned} \tag{2.23}$$

Now we recall Theorem 1.1 and Proposition 1.1.

Theorem 1.1 *Let $m, j \in \{0\} \cup \mathbb{N}$ and $m + j \geq 1$. Then the ordinary differential equation* (1.1)*:*

$$-U''(r) = \frac{1}{4} r^{-2} \left(1 + S_j^{\langle m \rangle}(r) \right) U(r) \qquad (r \geq 1)$$

has the general solutions

$$U(r) = C_1 U_j^{\langle m \rangle}(r) + C_2 U_{j+1}^{\langle m \rangle}(r) \left(F^j(\phi^m(ar)) \right)^{1/2} \qquad (r \geq 1)$$

with arbitrary constants $C_1, C_2 \in \mathbb{R}$.

Proposition 1.1 *Let $m, j \in \{0\} \cup \mathbb{N}$ and $m + j \geq 1$. Then the equality*

$$-\left(U_j^{\langle m \rangle}(r) \right)'' = \frac{1}{4} r^{-2} \left(1 + S_j^{\langle m \rangle}(r) \right) U_j^{\langle m \rangle}(r) \qquad (r \geq 1)$$

holds.

By Definition 2.3 and a simple change of variable, Theorem 1.1 and Proposition 1.1 clearly follow from the next Theorems 2.1 and 2.2 respectively.

Theorem 2.1. *Let $m, j \in \{0\} \cup \mathbb{N}$ and $m + j \geq 1$. Then the ordinary differential equation*

$$-Y''(u) = \frac{1}{4} D_j^{\langle m \rangle}(u) Y(u) \qquad (u \geq a) \tag{2.24}$$

has the general solutions

$$Y(u) = C_1 Y_j^{\langle m\rangle}(u) + C_2 Y_{j+1}^{\langle m\rangle}(u)\left(F^j(\phi^m(u))\right)^{1/2} \qquad (u \geq a) \tag{2.25}$$

with arbitrary constants $C_1, C_2 \in \mathbb{R}$.

Theorem 2.2. *Let* $m, j \in \{0\} \cup \mathbb{N}$ *and* $m + j \geq 1$. *Then the equality*

$$-\left(Y_j^{\langle m\rangle}(u)\right)'' = \frac{1}{4} D_j^{\langle m\rangle}(u) Y_j^{\langle m\rangle}(u) \qquad (u \geq a) \tag{2.26}$$

holds.

Proof of Theorem 2.2. For $u \geq a$, from (2.4) it follows that

$$\left(Y_j^{\langle m\rangle}(u)\right)' = \frac{1}{2}\left(V_j^{\langle m\rangle}(u)\right)' Y_j^{\langle m\rangle}(u),$$
$$\left(Y_j^{\langle m\rangle}(u)\right)'' = \frac{1}{4}\left\{2\left(V_j^{\langle m\rangle}(u)\right)'' + \left(\left(V_j^{\langle m\rangle}(u)\right)'\right)^2\right\} Y_j^{\langle m\rangle}(u).$$

Therefore it is enough for the verification of (2.26) to show the equality

$$2\left(V_j^{\langle m\rangle}(u)\right)'' + \left(\left(V_j^{\langle m\rangle}(u)\right)'\right)^2 = -D_j^{\langle m\rangle}(u). \tag{2.27}$$

To do so we make useful observations. We use the formulas made in the following section. Note that

$$\left(F^j(\phi^m(u))\right)' = \frac{1}{a^{m+j}} \exp\left(-V_j^{\langle m\rangle}(u)\right). \tag{2.28}$$

In fact, for $m \geq 1$, since

$$(\phi^m(u))' = \left(\phi(\phi^{m-1}(u))\right)' = \phi'(\phi^{m-1}(u))\left(\phi^{m-1}(u)\right)' = \frac{\left(\phi^{m-1}(u)\right)'}{\tilde{F}(\phi^{m-1}(u))}$$

by (1.4), from (2.6) it follows that

$$(\phi^m(u))' = \frac{1}{\prod_{l=0}^{m-1} \tilde{F}(\phi^l(u))} = \frac{1}{a^m \exp\left(V_0^{\langle m\rangle}(u)\right)}, \tag{2.29}$$

which shows (2.28) with $j = 0$. (2.28) for $m = 0$ follows from (3.16) and (2.5). For $m, j \geq 1$, from (3.16) and (2.29) it follows that

$$\left(F^j(\phi^m(u))\right)' = (F^j)'(\phi^m(u))\,(\phi^m(u))' = \frac{1}{\prod_{k=0}^{j-1} F^k(\phi^m(u))} \frac{1}{\prod_{l=0}^{m-1} \tilde{F}(\phi^l(u))}, \tag{2.30}$$

which implies (2.28) by (2.7). Thus we saw (2.28) for sure. Furthermore, letting $V_0^{\langle 0\rangle}(u) \equiv 0$, from (2.3) and (2.28) it follows that

$$
\begin{aligned}
\left(V_{j+1}^{\langle m\rangle}(u)\right)' &= \left(V_j^{\langle m\rangle}(u)\right)' + \left(F^{j+1}(\phi^m(u))\right)', \\
\left(V_{j+1}^{\langle m\rangle}(u)\right)'' &= \left(V_j^{\langle m\rangle}(u)\right)'' + \left(F^{j+1}(\phi^m(u))\right)'' \\
&= \left(V_j^{\langle m\rangle}(u)\right)'' - \left(V_{j+1}^{\langle m\rangle}(u)\right)' \left(F^{j+1}(\phi^m(u))\right)' \\
&= \left(V_j^{\langle m\rangle}(u)\right)'' - \left(\left(V_j^{\langle m\rangle}(u)\right)' + \left(F^{j+1}(\phi^m(u))\right)'\right) \left(F^{j+1}(\phi^m(u))\right)'.
\end{aligned}
$$

Hence, for each m and j, we have the recurrence relation

$$
\begin{aligned}
&2\left(V_{j+1}^{\langle m\rangle}(u)\right)'' + \left(\left(V_{j+1}^{\langle m\rangle}(u)\right)'\right)^2 \\
&\quad = 2\left(V_j^{\langle m\rangle}(u)\right)'' + \left(\left(V_j^{\langle m\rangle}(u)\right)'\right)^2 - \left(\left(F^{j+1}(\phi^m(u))\right)'\right)^2. \qquad (2.31)
\end{aligned}
$$

Now we show (2.27) by induction on m. When $m = 0$, from (2.31), (3.16) and (2.11) it follows that

$$
\begin{aligned}
2\left(V_{j+1}^{\langle 0\rangle}(u)\right)'' + \left(\left(V_{j+1}^{\langle 0\rangle}(u)\right)'\right)^2 &= -\sum_{k=0}^{j}\left(\left(F^{k+1}(u)\right)'\right)^2 \\
&= -\sum_{k=0}^{j}\left(\prod_{i=0}^{k} F^i(u)\right)^{-2} = -D_{j+1}^{\langle 0\rangle}(u),
\end{aligned}
$$

which shows (2.27) for all j. Suppose that (2.27) holds for all j with some fixed m. Then from (2.2), (2.3) and the assumption of induction it follows that

$$
\begin{aligned}
2\left(V_0^{\langle m+1\rangle}(u)\right)'' + \left(\left(V_0^{\langle m+1\rangle}(u)\right)'\right)^2 &= \lim_{j\to\infty}\left\{2\left(V_j^{\langle m\rangle}(u)\right)'' + \left(\left(V_j^{\langle m\rangle}(u)\right)'\right)^2\right\} \\
&= -\lim_{j\to\infty} D_j^{\langle m\rangle}(u) \\
&= -D_0^{\langle m+1\rangle}(u). \qquad (2.32)
\end{aligned}
$$

In addition, from (2.31) and (2.32) it follows that

$$
\begin{aligned}
&2\left(V_{j+1}^{\langle m+1\rangle}(u)\right)'' + \left(\left(V_{j+1}^{\langle m+1\rangle}(u)\right)'\right)^2 \\
&\quad = -D_0^{\langle m+1\rangle}(u) - \sum_{k=0}^{j}\left(\left(F^{k+1}(\phi^{m+1}(u))\right)'\right)^2. \qquad (2.33)
\end{aligned}
$$

According to (2.33), (2.30) and (2.13), we obtain

$$2\left(V_{j+1}^{\langle m+1\rangle}(u)\right)''+\left(\left(V_{j+1}^{\langle m+1\rangle}(u)\right)'\right)^2=-D_{j+1}^{\langle m+1\rangle}(u). \tag{2.34}$$

(2.32) and (2.34) indeed show that (2.27) holds for $m+1$ and all j. Consequently (2.27) was shown by induction. That completes the proof of Theorem 2.2. □

Proof of Theorem 2.1. From (2.26) and (2.24) it follows that

$$\begin{aligned}0&=Y_j^{\langle m\rangle}(u)Y''(u)-\left(Y_j^{\langle m\rangle}(u)\right)''Y(u)\\&=\left\{Y_j^{\langle m\rangle}(u)Y'(u)-\left(Y_j^{\langle m\rangle}(u)\right)'Y(u)\right\}'\\&=\left\{\left(Y_j^{\langle m\rangle}(u)\right)^2\left(\frac{Y(u)}{Y_j^{\langle m\rangle}(u)}\right)'\right\}',\end{aligned}$$

and so

$$\left(\frac{Y(u)}{Y_j^{\langle m\rangle}(u)}\right)'=C\left(Y_j^{\langle m\rangle}(u)\right)^{-2}$$

with an arbitrary constant $C\in\mathbb{R}$. In addition, by (2.4) and (2.28) we have

$$\left(Y_j^{\langle m\rangle}(u)\right)^{-2}=\exp\left(-V_j^{\langle m\rangle}(u)\right)=a^{m+j}\left(F^j(\phi^m(u))\right)'.$$

Hence it follows that

$$\frac{Y(u)}{Y_j^{\langle m\rangle}(u)}=C_1+C_2F^j(\phi^m(u))$$

with arbitrary constants $C_1,C_2\in\mathbb{R}$, which implies (2.25) by the representations (2.8) $\sim$ (2.10) of functions $Y_j^{\langle m\rangle}$. That concludes the proof of Theorem 2.1. □

3. Appendix

In this section we review some elementary results of slowly increasing functions studied in [1]. Let $a>1$ be fixed.

Definition 3.1. Let $\mathcal{F}_a$ be the set of all continuous, increasing and bijective functions from $[a,\infty)$ to itself.

If $f\in\mathcal{F}_a$, then $f(a)=a$ and $\lim_{u\to\infty}f(u)=\infty$. For $f,g\in\mathcal{F}_a$, let $g\circ f$ be a composite function of f and g defined by $(g\circ f)(u)=g(f(u))$. Then $g\circ f\in\mathcal{F}_a$.

Let $k\in\{0\}\cup\mathbb{N}$. For a function $f\in\mathcal{F}_a$, let $f^0(u)=u$ and $f^{k+1}(u)=f(f^k(u))$. Then $f^k\in\mathcal{F}_a$.

We recall the definition (1.2) of function F. Namely, we define a function $F \in \mathcal{F}_a$ as

$$F(u) = F_a(u) = a - \log a + \log u = a + \log \frac{u}{a} = a + \int_a^u \frac{1}{t}\, dt \qquad (u \geq a). \tag{3.1}$$

We easily see the following.

Lemma 3.1. *It holds that*

$$\lim_{u\to\infty} \frac{F(u)}{u^{\varepsilon}} = 0 \qquad \text{for any} \quad \varepsilon > 0. \tag{3.2}$$

Moreover it holds that

$$\lim_{u\to\infty} \frac{F^k(u)}{\log^k u} = 1 \qquad \text{for each} \quad k \in \{0\} \cup \mathbb{N}, \tag{3.3}$$

where $\log^0 u = u$ *and* $\log^k u = \log(\log^{k-1} u)$ $(k \in \mathbb{N})$.

For $u > a$, since

$$F(u) = a + \int_a^u \frac{1}{t}\, dt < a + \int_a^u \frac{1}{a}\, dt = a + \frac{u-a}{a} = u - \frac{a-1}{a}(u-a),$$

we have the relations

$$F(u) < u \tag{3.4}$$

and

$$a(F(u) - a) < u - a, \tag{3.5}$$

which imply the properties

$$F^{k+1}(u) < F^k(u) \tag{3.6}$$

and

$$a\left(F^{k+1}(u) - a\right) < F^k(u) - a \tag{3.7}$$

respectively. From (3.7) it follows that

$$F^{k+1}(u) - a \leq \frac{1}{a^k}(F(u) - a) = \frac{1}{a^k} \log \frac{u}{a} \qquad (u \geq a).$$

Hence the infinite product

$$\prod_{k=0}^{\infty} \frac{F^k(u)}{a} = \exp\left(\sum_{k=0}^{\infty} \log \frac{F^k(u)}{a} \right) = \exp\left(\sum_{k=0}^{\infty} \left(F^{k+1}(u) - a\right) \right)$$

converges locally uniformly in $[a, \infty)$ and the estimate

$$\prod_{k=0}^{\infty} \frac{F^k(u)}{a} \leq \exp\left(\sum_{k=0}^{\infty} \frac{1}{a^k} \log \frac{u}{a} \right) = \left(\frac{u}{a}\right)^{a/(a-1)} \qquad (u \geq a) \tag{3.8}$$

holds by $\sum_{k=0}^{\infty} 1/a^k = a/(a-1)$.

Then we recall the definition (1.3) of function $\tilde{F}$.

Definition 3.2. Let $j \in \{0\} \cup \mathbb{N}$. For $u \geq a$, let

$$\tilde{F}_j(u) = a \prod_{k=0}^{j} \frac{F^k(u)}{a} \tag{3.9}$$

and

$$\tilde{F}(u) = a \prod_{k=0}^{\infty} \frac{F^k(u)}{a}. \tag{3.10}$$

It is clear that $\tilde{F}_j \in \mathcal{F}_a$ for each $j \in \{0\} \cup \mathbb{N}$, $\tilde{F} \in \mathcal{F}_a$ and, from (3.8),

$$\tilde{F}(u) \leq a \left(\frac{u}{a}\right)^{a/(a-1)} \qquad (u \geq a). \tag{3.11}$$

Moreover the function $\tilde{F}$ is infinitely differentiable and $\left(\dfrac{d}{du}\right)^n \log \tilde{F}(u)$ is bounded in $[a, \infty)$ for each $n \in \mathbb{N}$. See Theorem 2.1 in [1].

For $u > a$, in view of

$$\lim_{j\to\infty} F^j(u) = \lim_{j\to\infty} \left(a \, \frac{\tilde{F}_j(u)}{\tilde{F}_{j-1}(u)} \right) = a \, \frac{\tilde{F}(u)}{\tilde{F}(u)} = a \tag{3.12}$$

it holds that

$$\lim_{j\to\infty} (F^j)^{-1}(u) = \infty. \tag{3.13}$$

In fact, from (3.12), for any $G > a$ there exists a $j_G \in \mathbb{N}$ such that $F^j(G) < u$ $(j \geq j_G)$, which is equivalent to $(F^j)^{-1}(u) > G$ $(j \geq j_G)$ by the increasingness of F^j. This shows (3.13).

Now we also recall the definition (1.4) of slowly increasing function ϕ.

Definition 3.3. For $u \geq a$, let

$$\phi(u) = a + \int_a^u \frac{1}{\tilde{F}(t)} \, dt. \tag{3.14}$$

Then the function ϕ is in $\mathcal{F}_a$ and infinitely differentiable. Actually, from the next lemma it follows that $\lim_{u\to\infty} \phi(u) = \infty$.

Lemma 3.2. *It holds that*

$$\int_a^{\infty} \frac{1}{\tilde{F}(t)} \, dt = \infty. \tag{3.15}$$

Proof. For $u \geq a$, since

$$\left(F^{k+1}(u)\right)' = \left(F(F^k(u))\right)' = F'(F^k(u)) \left(F^k(u)\right)' = \frac{\left(F^k(u)\right)'}{F^k(u)},$$

the relation

$$\left(F^{k+1}(u)\right)' = \frac{1}{\prod_{i=0}^{k} F^i(u)} = \frac{1}{a^k \tilde{F}_k(u)} \tag{3.16}$$

holds. Combining (3.16) and the relation

$$\begin{aligned}\tilde{F}(u) &= a \prod_{k=0}^{j} \frac{F^k(u)}{a} \prod_{k=j+1}^{\infty} \frac{F^k(u)}{a} \\ &= \tilde{F}_j(u) \prod_{k=j+1}^{\infty} \frac{F^{k-j-1}(F^{j+1}(u))}{a} = \tilde{F}_j(u)\, \frac{\tilde{F}(F^{j+1}(u))}{a},\end{aligned}$$

we have

$$\begin{aligned}\int_a^u \frac{1}{\tilde{F}(t)}\, dt &= \int_a^u \frac{a}{\tilde{F}_j(t)} \frac{1}{\tilde{F}(F^{j+1}(t))}\, dt \\ &= a^{j+1} \int_a^u \frac{\left(F^{j+1}(t)\right)'}{\tilde{F}(F^{j+1}(t))}\, dt = a^{j+1} \int_a^{F^{j+1}(u)} \frac{1}{\tilde{F}(s)}\, ds.\end{aligned}$$

Hence it holds that

$$\int_a^u \frac{1}{\tilde{F}(t)}\, dt = a^j \int_a^{F^j(u)} \frac{1}{\tilde{F}(t)}\, dt \qquad (j \in \mathbb{N} \cup \{0\}). \tag{3.17}$$

Substituting $(F^j)^{-1}(2a)$ for u in (3.17), we obtain the equality

$$\int_a^{(F^j)^{-1}(2a)} \frac{1}{\tilde{F}(t)}\, dt = a^j \int_a^{2a} \frac{1}{\tilde{F}(t)}\, dt \qquad (j \in \mathbb{N} \cup \{0\}), \tag{3.18}$$

which implies the assertion (3.15) because of $\lim_{j\to\infty} (F^j)^{-1}(2a) = \infty$ by (3.13). □

Finally we show the basic properties of slowly increasing functions $\phi, F^k \circ \phi^m$ and $\tilde{F} \circ \phi^m$ for $m, k \in \{0\} \cup \mathbb{N}$.

Lemma 3.3. *For each $k \in \{0\} \cup \mathbb{N}$ and for any $\varepsilon > 0$ it holds that*

$$\lim_{u\to\infty} \frac{\phi(u)}{\left(F^k(u)\right)^{\varepsilon}} = 0. \tag{3.19}$$

Proof. For $u > a$, by using

$$\tilde{F}(u) > \tilde{F}_k(u) \geq F^k(u) \tag{3.20}$$

and (3.16), we have the estimate

$$\begin{aligned}\int_a^u \frac{1}{\tilde{F}(t)}\, dt < \int_a^u \frac{1}{\tilde{F}_k(t)}\, dt &= \int_a^u a^k \left(F^{k+1}(t)\right)' dt \\ &= a^k \left(F^{k+1}(u) - a\right) \leq a^k F^{k+1}(u) - a,\end{aligned}$$

and so

$$\phi(u) < a^k F^{k+1}(u). \tag{3.21}$$

Hence from (3.21), (3.2) and $\lim_{u\to\infty} F^k(u) = \infty$ it follows that

$$0 < \frac{\phi(u)}{(F^k(u))^\varepsilon} < a^k \frac{F^{k+1}(u)}{(F^k(u))^\varepsilon} = a^k \frac{F(F^k(u))}{(F^k(u))^\varepsilon} \longrightarrow 0 \qquad \text{as} \quad u \to \infty,$$

which shows (3.19). □

Proposition 3.1. *For each $m, k \in \{0\} \cup \mathbb{N}$ it holds that $F^k \circ \phi^m \in \mathcal{F}_a$ and, for $u > a$,*

$$F^{k+1}(\phi^m(u)) < F^k(\phi^m(u)), \tag{3.22}$$

$$F^k(\phi^m(u)) < \frac{\tilde{F}(\phi^m(u))}{\phi^m(u)} \qquad (k \geq 1), \tag{3.23}$$

$$F^k(\phi^m(u)) < \tilde{F}(\phi^m(u)), \tag{3.24}$$

$$\tilde{F}(\phi^{m+1}(u)) < \tilde{F}(\phi^m(u)). \tag{3.25}$$

Moreover for each $m, k \in \{0\} \cup \mathbb{N}$ it holds that

$$\lim_{u\to\infty} \frac{F^k(\phi^m(u))}{\log^k \phi^m(u)} = 1 \tag{3.26}$$

and, for any $\varepsilon > 0$,

$$\lim_{u\to\infty} \frac{\phi^{m+1}(u)}{(F^k(\phi^m(u)))^\varepsilon} = 0, \tag{3.27}$$

$$\lim_{u\to\infty} \frac{F^k(\phi^{m+1}(u))}{\left(\tilde{F}(\phi^m(u))\right)^\varepsilon} = \lim_{u\to\infty} \frac{F^k(\phi^{m+1}(u))}{\left(\tilde{F}(\phi^m(u))/\phi^m(u)\right)^\varepsilon} = 0, \tag{3.28}$$

$$\lim_{u\to\infty} \frac{\tilde{F}(\phi^{m+1}(u))}{\left(\tilde{F}(\phi^m(u))\right)^\varepsilon} = \lim_{u\to\infty} \frac{\tilde{F}(\phi^{m+1}(u))}{\left(\tilde{F}(\phi^m(u))/\phi^m(u)\right)^\varepsilon} = 0. \tag{3.29}$$

Proof. Since $\lim_{u\to\infty} \phi^m(u) = \infty$ by $\phi^m \in \mathcal{F}_a$, it suffices to show all the assertions for the case of $m = 0$. Then (3.22), (3.24), (3.26) and (3.27) coincide (3.6), (3.20), (3.3) and (3.19) respectively. (3.23) follows from (3.20) and

$$\tilde{F}_k(u) = F^k(u)\tilde{F}_{k-1}(u) \geq F^k(u)F^0(u) = u\, F^k(u) \qquad (k \geq 1).$$

Combining (3.21) with $k = 0$ and (3.4), we have

$$\phi(u) < u \qquad (u > a), \tag{3.30}$$

which implies (3.25) due to the increasingness of $\tilde{F}$. Finally we show (3.28) and (3.29). By (3.20) we have

$$\frac{F^k(\phi(u))}{\left(\tilde{F}(u)\right)^\varepsilon} < \frac{F^k(\phi(u))}{\left(\tilde{F}(u)/u\right)^\varepsilon} < \frac{\tilde{F}(\phi(u))}{\left(\tilde{F}(u)/u\right)^\varepsilon}, \qquad \frac{\tilde{F}(\phi(u))}{\left(\tilde{F}(u)\right)^\varepsilon} < \frac{\tilde{F}(\phi(u))}{\left(\tilde{F}(u)/u\right)^\varepsilon} \qquad (u > a).$$

In addition, from (3.11) and (3.23) it follows that

$$\frac{\tilde{F}(\phi(u))}{\left(\tilde{F}(u)/u\right)^{\varepsilon}} < \frac{a(\phi(u)/a)^{a/(a-1)}}{(F^k(u))^{\varepsilon}}$$

$$= a^{-1/(a-1)} \left(\frac{\phi(u)}{(F^k(u))^{(a-1)\varepsilon/a}} \right)^{a/(a-1)} \longrightarrow 0 \quad \text{as} \quad u \to \infty$$

because of Lemma 3.3. Hence we obtain (3.28) and (3.29). That concludes the proof. □

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