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Note on the number of semistar operations, XIV

Ryûki Matsuda*

Abstract

We determine conditions for a grading monoid to have only a finite number of semistar operations.

This is a note on the number of semistar operations, and is a continuation of [M3]. The notions of star operations, semistar operations, and their Kronecker function rings of integral domains have been well known. We refer to Fontana-Loper([FL]) and its references for them. Let $G$ be a torsion-free abelian additive group, and let $S$ be an additively closed subset containing 0 of $G$. Then $S$ is called a grading monoid (or, a $g$-monoid). We refer to [M1] for notions of star operations, semistar operations, and their Kronecker function rings for $g$-monoids.

Let $\Sigma(S)$ be the set of star operations on $S$, and let $\Sigma_0(S)$ be the set of semistar operations on $S$. In §1 of this paper, we are interested in the cardinalities $|\Sigma(S)|$, and $|\Sigma_0(S)|$, especially, in when $|\Sigma_0(S)| < \infty$? We will determine conditions for $|\Sigma_0(S)| < \infty$. §2 is an another note on $|\Sigma'_0(D)|$ for i-local domains $D$.

§1 The conditions for $|\Sigma'_0(S)| < \infty$

Let $G$ be the quotient group of $S$, and let $\bar{S}$ be the integral closure of $S$. If $S$ is a group, we have $|\Sigma_0(S)| = 1$ trivially. Thus, throughout the section, let $S$ be a $g$-monoid which is not a group, let $M$ (resp. $N$) be the maximal ideal of $S$ (resp. $\bar{S}$), let $H$ (resp. $L$) be the group of units of $S$ (resp. $\bar{S}$). In [M2, Theorem 14] we proved the following fact: Assume that $M = N$. Then we have that $|\Sigma'_0(S)| < \infty$ if and only if $\dim(S) < \infty$, $\bar{S}$ is a valuation semigroup, and $\bar{S}/S$ is a finite set modulo $H$.

In this section, we will prove the following,

**Theorem 1** Assume that $M \neq N$. Then the following conditions are equivalent.

1. $|\Sigma'_0(S)| < \infty$.
2. $\dim(S) < \infty$, $\bar{S}$ is a valuation semigroup, and $\bar{S}/S$ is a finite set modulo $H$.

1.1 (cf. [M2, Proposition 1]) Let $V$ be a valuation semigroup with maximal ideal $N$. If $N$ is a principal ideal of $V$, then $|\Sigma(V)| = 1$, and if $N$ is not a principal ideal $N$. If $N$ is a principal ideal of $V$, then $|\Sigma(V)| = 1$, and if $N$ is not a principal
ideal of $V$, then $|\Sigma(V)| = 2$.

**1.2** Assume that $|\Sigma'(S)| < \infty$. Then there is only a finite number of oversemigroups of $S$.

Proof. Let $T$ be an oversemigroup of $S$. Then there arises a semistar operation $I \mapsto I + T$ on $S$.

**1.3** Assume that $|\Sigma'(S)| < \infty$. Then $L/H$ is a finite group.

Proof. If $L/H$ is an infinite group, then there is an infinite number of subgroups $K$ of $L$ containing $H$. Set $T = K \cup N$. Then $T$ is an oversemigroup of $S$.

Let $A$ be a subset of $G$. Then $S[A]$ denotes the oversemigroup of $S$ generated by $A$.

**1.4** Assume that $|\Sigma'(S)| < \infty$. Then $\dim(S) < \infty$, $S$ is a valuation semigroup, and $\bar{S} = S$ is a finite set modulo $H$.

Proof. Suppose that $\dim(S) = \infty$. Then there is an infinite number of oversemigroups of $S$. Then $|\Sigma'(S)| = \infty$ by (1.2).

Suppose that $\bar{S}$ is not a valuation semigroup. Then there is an element $x \in G - \bar{S}$ such that $-x \notin \bar{S}$. We have $S[2^n x] \supset S[2^{n+1} x]$ for each positive integer $n$. Then $|\Sigma'(S)| = \infty$ by (1.2).

Conferring (1.3), let $\alpha_1, \cdots, \alpha_k$ be a complete representative system of $L$ modulo $H$. Let $v$ be a valuation belonging to $\bar{S}$. By (1.2), we have $\{S[x] \mid x \in S - S\} = \{S[x_1], \cdots, S[x_m]\}$ for some positive integer $m$. Let $x \in \bar{S} - S$. Then $S[x] = S[x_i]$, for some $i$. Hence $v(x) = v(x_i)$. Then $x - x_i = \alpha_j + h$ for some $j$ and $h \in H$.

We have seen that (1) implies (2) in Theorem 1.

Thus, in the rest of the section, we assume that $M \neq N$, $\dim(S) < \infty$, $\bar{S}$ is a valuation semigroup, and $\bar{S} = S$ is a finite set modulo $H$. Set $\bar{S} = V$, let $v$ be a valuation belonging to $V$, and let $\Gamma$ be the value group of $v$.

**1.5** (1) Let $I \in F'(S)$ so that $v(I)$ is not bounded below. Then $I = G$.

(2) $F'(S) = F(S) \cup \{G\}$.

(3) Each star operation $\ast$ on $S$ can be extended uniquely to a semistar operation on $S$.

(4) $|\Sigma(S)| \leq |\Sigma'(S)|$.

(5) $L/H$ is a finite group.

(6) Let $I \in F(S)$ so that there does not exist $\inf v(I)$. Then we have $\{x \in G \mid v(x) \geq v(a)\}$ for some $a \in I \subseteq I$.

Proof. (1) Let $x \in G$. There are $x_1, x_2, x_3, \cdots$ in $I$ so that $v(x) > v(x_1) > \cdots$. Then $x - x_i \in N$ for each $i$, and $x - x_i \notin x - x_j$ modulo $H$ for each $i < j$. It follows that $x - x_n \in S$ for some $n$, and hence $x \in I$.
(2) follows from (1).
(3) follows from (2).
(4) follows from (3).
(5) follows from the fact that $V - S$ is a finite set modulo $H$.
(6) Suppose the contrary. There are $x \notin I$ and $a_0 \in I$ such that $v(a_0) \leq v(x)$.
There are elements $a_1, a_2, a_3, \cdots$ in $I$ so that $v(a_0) > v(a_1) > v(a_2) > \cdots$. Then $x - a_i \in V - S$ for each $i$, and $x - a_i \neq x - a_j$ modulo $H$ for each $i < j$; a contradiction.

Let $\alpha_1, \cdots, \alpha_n$ be a complete representative system of $L$ modulo $H$. And let $z_1, \cdots, z_n$ be a complete representative system of $N - M$ modulo $H$. We may assume that $v(z_1) \leq \cdots \leq v(z_n)$.

(1.6) Let $T$ be an oversemigroup of $S$ with $T \in F(S)$, and let $*$ be either a star operation or a semistar operation on $S$. Then $T^*$ is an oversemigroup of $S$.

Proof. Let $x, y \in T^*$. Then $x + y \in T^* + T^* \subset (T^* + T^*)^* = (T + T)^* = T^*$.

(1.7) There is min $v(N)$.

Proof. Suppose that $0 < v(s) < v(z_1)$ for some $s \in S$. We have $z_1 - s \equiv z_i$ modulo $H$ for some $i$. Then $v(z_1) = v(z_i)$, and hence $v(s) = 0$; a contradiction.

Let $x \in N - M$. Then $x \equiv z_i$ modulo $H$ for some $i$. Hence $v(x) = v(z_i) \geq v(z_1)$.

We may assume that is the rank 1 convex subgroup of $\Gamma$. Take an element $\pi \in N$ such that $v(\pi) = 1$.

(1.8) Let $T$ be an oversemigroup of $S$. Then $T \supset V$ or $T \subset V$.

Proof. Assume that $T \not\subset V$, and take an element $x_0 \in T - V$. Then $-x_0 \in N$.
Let $x \in V$. We have $x - kx_0 \in V$ for each $k \geq 0$. If $0 < i < j$, then $x - ix_0 \neq x - jx_0$ modulo $H$. Therefore $x - mx_0 \in S$ for some $m$. Then $x \in S[x_0]$, and hence $V \subset T$.

(1.9) There is only a finite number of oversemigroups of $S$.

Proof. It follows from (1.8), $\dim(V) < \infty$, and the hypothesis that $V - S$ is a finite set modulo $H$.

(1.10) Let $\dim(S) = 1$. Then $V^* = V$, that is, $V$ is a divisorial fractional ideal of $S$.

Proof. $V^*$ is an oversemigroup of $V$ by (1.6), and we have $\dim(V) = 1$. Suppose that $V^* \neq V$. Then $V^* = G$. Take an element $x_0 \in M$. Let $1 \leq i \leq b$, and let $0 < j < k$. Then $z_i + jx_0 \neq z_i + kx_0$ modulo $H$. Hence there is $m_i$ so that $z_i + mx_0 \in S$ for each $m \geq m_i$, that is, $z_i \in (-mx_0)$. Similarly, there is $n_j$ so that $\alpha_j \in (-mx_0)$ for each $m \geq n_j$. Let $\max\{m_i, n_j \mid i, j\} = m_0$. Then $V \subset (-mx_0)$ for each $m \geq m_0$. Since $V^* = \cap\{x \mid (x) \supset V\}$, we have $(-mx_0) = G$ for each $m \geq m_0$.
this is clearly impossible.

(1.11) We have $V^c = V$.

Proof. By (1.10), we may assume that $\dim(S) \geq 2$. Let $Q$ be a prime ideal of $V$ with $\text{ht}(Q) = \text{ht}(N) - 1$, and let $P = S \cap Q$, where $\text{ht}(N)$ (resp. $\text{ht}(Q)$) means the height of $N$ (resp. the height of $Q$). Suppose that $V^c \neq V$. Then $V^c \supseteq V_Q$. Take an element $x_0 \in M - P$. The similar argument to the proof of (1.10) shows that $(-mx_0) \supseteq V_Q$ for all sufficiently large $m$. Since $(m+1)x_0 \notin Q$, we have $-(m+1)x_0 \in V_Q \subset (-mx_0)$, and hence $-x_0 \in S$; a contradiction.

(1.12) Let $I \in F(S)$ so that there does not exist $\inf v(I)$. Then $I^c = I$.

Proof. Suppose that $I^c \supseteq I$. Take an element $x \in I^c - I$. Then $v(x)$ is a lower bound of $v(I)$ by (1.5)(6). There is a lower bound $v(y)$ of $v(I)$ with $v(x) < v(y)$. Set $I - y = J$. Since $J \subset V$, we have $J^c \subset V$ by (1.11). We have $x - y \in I^c = J^c$, and $v(x - y) = 0$. Hence $J^c \not\subset V$; a contradiction.

(1.13) $|\Sigma(S)| < \infty$.

Proof. Let $I \in F(S)$ with $S \subset I \subset V$. Then $I$ is generated on $S$ by a subset of $\{a_i, z_j \mid i, j\}$. Therefore the set $\{I \in F(S) \mid S \subset I \subset V\} = X$ is a finite set.

Let $* \in \Sigma(S)$ and let $I \in X$. Set $I^* = g_*(I)$. Then $g_*$ is a mapping from $X$ to $X$ by (1.11), that is, $g_* \in X^X$. Then $g$ is a mapping from $\Sigma(S)$ to $X^X$.

Let $*, *' \in \Sigma(S), I \in F(S)$, and assume that $g_* = g_{*'}$. If there does not exist $\inf v(I)$, then $I^* = I'^* = I$ by (1.12). If there is $\inf v(I) = v(x)$, then $\min v(I - x) = 0$ by (1.7). Hence $S \subset I - y \subset V$ for some $y \in I$. Since $g_*, g_{*'}$, we have $(I - y)^* = (I - y)^{*'}$, and hence $I^* = I'^*$. We have proved that $* = *'$, and hence $g$ is an injection. It follows that $|\Sigma(S)| < \infty$.

(1.14) Let $T$ be an oversemigroup of $S$ with $T \subset V$. Then $|\Sigma(T)| < \infty$.

Proof. Let $M'$ be the maximal ideal of $T$, and let $H'$ be the group of units of $T$. We have that $T = V$, $\dim(T) = \dim(S) < \infty$, and $L/H'$ is a finite group. If $M' = N$, we have $|\Sigma(T)| < \infty$ by [M2, Theorem 14], and hence $|\Sigma(T)| < \infty$ by (1.5)(4). If $M' \neq N$, we have $|\Sigma(T)| < \infty$ by (1.13).

(1.15) Let $T$ be an oversemigroup of $S$. Then $|\Sigma(T)| < \infty$.

Proof. We may assume that $T \subset V$ by (1.14). Then $T \supset V$ by (1.8). Then $|\Sigma(T)| \leq 2$ by (1.1).

Conferring (1.9), let $\{T_1, \cdots, T_c\}$ be the set of oversemigroups of $S$. For each $1 \leq i \leq c, * \in \Sigma(T_i)$ and $I \in F(S)$, set $(I + T_i)^* = I^\sigma(*)$ and $G = G^\sigma(*)$.

(1.16) (1) If $i \neq j$, then $\Sigma(T_i) \cap \Sigma(T_j) = \emptyset$. 
(2) There is a canonical mapping \( \sigma \) from \( \bigcup_{\mathcal{T}} \Sigma(T) \) to \( \Sigma'(S) \).

**Proof.** (1) We have \( F(T_i) \neq F(T_j) \), and \( \Sigma(T_i) \) (resp. \( \Sigma(T_j) \)) is a set of mappings from \( F(T_i) \) to \( F(T_i) \) (resp. from \( F(T_j) \) to \( F(T_j) \)).

(2) We see easily that \( \sigma(*) \) satisfies the conditions of a semistar operation on \( S \).

| (1.17) | The mapping \( \sigma \) is a bijection onto \( \Sigma'(S) \). |

**Proof.** Let \( * \in \Sigma'(S) \). Then \( S\sigma(*) = T_i \) for some \( T_i \). There is a star operation \( * : J \rightarrow J' \) on \( T_i \). Then we have \( \sigma(*) = T_i \).

| (1.18) | \( |\Sigma'(S)| < \infty \). |

**Proof.** It follows from (1.15), (1.16), and (1.17).

The proof of Theorem 1 is complete.

### §2 Another note

In [M4], we determined conditions for \( |\Sigma'(D)| < \infty \) for any APVD (or, an almost pseudo-valuation domain) \( D \), and in §1, we determined conditions for \( |\Sigma'(S)| < \infty \) for any g-monoid \( S \). Every g-monoid that is not a group has a unique maximal ideal, and every APVD \( D \) has the property that \( D \) and its integral closure \( \bar{D} \) has a unique maximal ideal. We refer to [BH] for APVD’s. Thus it is natural to consider the class of domains \( D \) such that \( \bar{D} \) has a unique maximal ideal. We call such a domain an i-local domain. In §2, we will study \( |\Sigma'(D)| \) for i-local domains \( D \).

(2.1) Let \( D \) be an i-local domain. Assume that \( \bar{D} \) is a valuation domain with maximal ideal \( M \), \( v \) be a valuation belonging to \( \bar{D} \), and \( M^n \subset D \) for some positive integer \( n \). Then either \( D \) is a PVD (or, a pseudo-valuation domain), or there is \( \min v(M) \).

**Proof.** Suppose the contrary. Let \( 0 \neq x \in M \). There are elements \( x_1, \ldots, x_n \in M \) such that \( v(x_1) > v(x_1) > \cdots > v(x_n) > 0 \). Then \( x = \frac{x_1}{x_1} \cdot \frac{x_2}{x_2} \cdots \frac{x_{n-1}}{x_{n-1}} \cdot \frac{x_n}{x_n} \in M^n \subset D \).

Hence \( D \) is a PVD; a contradiction.

Let \( D \) be a valuation domain with maximal ideal \( M \), let \( v \) be a valuation belonging to \( D \), and let \( \Gamma \) be the value group of \( v \). If there is \( \min v(M) \), then we may assume that \( M \) is the rank one convex subgroup of \( \Gamma \), and \( \min v(M) = 1 \in \bar{Z} \subset \Gamma \).

For, the rank one convex subgroup of \( \Gamma \) is isomorphic with the ordered group \( \bar{Z} \). Therefore \( \Gamma \) is order isomorphic with an ordered group \( \Gamma' \) the rank one convex subgroup of which is .
(2.2) Let $D$ be an i-local domain with maximal ideal $P$, let $M$ be the maximal ideal of $D$, and assume that $|\Sigma(D)| < \infty$. Then we have,

1. $\dim(D) < \infty$.
2. There is only a finite number of overrings of $D$.
3. $\bar{D} = V$ is a valuation domain.
4. $V$ is a finitely generated $D$-module.
5. $V/M = K$ is a simple extension field of $D/P = k$ with $[K : k] < \infty$.
6. $V, M \in F(D)$.
7. $F'(D) = F(D) \cup \{q(D)\}$.

Proof. (1) follows from (2).
(2) If $T$ is an overring of $D$, then there is a semistar operation $I \mapsto IT$ on $D$.
(3) Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the set of valuation overrings of $D$. Then we have $\bar{D} = \cap_\lambda V_\lambda$.
(4) $\bar{D}$ is a finitely generated overring of $D$.
(5) There is only a finite number of intermediate fields between $k$ and $K$.
(6) There are elements $x_1, \ldots, x_n \in V$ such that $V = \sum_1^n DX_i$ for some positive integer $n$.
(7) There is $0 \neq d \in D$ such that $dV \subset D$. Let $v$ be a valuation belonging to $V$. Let $I \in F'(D)$ so that $v(I)$ is not bounded below. Let $x \in q(D)$. There is $a \in I$ such that $v(a) < v(x)$. Then $x \in aV \subset (a/d)D \subset (1/d)I$. Hence $q(D) \subset (1/d)I$, and hence $I = q(D)$.

(2.3) Let $D$ be an i-local domain such that $\bar{D} = V$ is a valuation ring, and let $M$ be the maximal ideal of $D$. Assume that $M^n \subset D$ for some positive integer $n$. Then we have,

1. $F'(D) = F(D) \cup \{q(D)\}$.
2. Let $T$ be an overring of $D$. Then either $T \supset V$ or $T \subset V$.
3. Let $\Sigma'_1 = \{* \in \Sigma'(D) \mid D^* \supset V\}$. Then there is a canonical bijection from $\Sigma'(V)$ onto $\Sigma'_1$.
4. Let $\Sigma'_2 = \{* \in \Sigma'(D) \mid D^* \not\subset V\}$. Then $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.
5. Let $\{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings $T$ of $D$ with $T \not\supset V$. Then there is a canonical bijection from the disjoint union $\bigcup_\lambda \Sigma(T_\lambda)$ onto $\Sigma'_2$.

Proof. (1) Similar to (2.2)(7).
(2) Assume that $T \not\subset V$, and take an element $x \in T - V$. We may assume that $1/x \in M^n$. Let $a \in V$. Then $a(1/x) \in P$, hence $a \in xP \subset T$.
(3) The map $* \mapsto \delta_D(*)$ gives a bijection from $\Sigma'(V)$ onto $\Sigma'_1$.
(4) follows from (1).
(5) Similar to (3).

(2.4) Let $D$ be an i-local domain. Assume that $\bar{D} = V$ is a valuation ring with maximal ideal $M$, let $\mathcal{K}$ be a complete representative system of $V$ modulo $M$, $v$ be a valuation belonging to $V$ with value group $\Gamma$, assume that $\mathcal{K}$ is the rank one convex subgroup of $\Gamma$, and $v(\pi) = 1 \in \bar{Z}$ for some $\pi \in \mathcal{V}$. Let $x \in q(D) - \{0\}$ with $v(x) \in \bar{Z}$. Let $k$ be a positive integer with $k > v(x)$. Then $x$ can be expressed uniquely as
\[ x = \alpha_l \pi^l + \alpha_{l+1} \pi^{l+1} + \cdots + \alpha_{k-1} \pi^{k-1} + a \pi^k, \] where \( l = v(x) \) and each \( \alpha_i \in K \) with \( \alpha_l \not\equiv 0 \pmod{M} \) and \( a \in V \).

Proof. Since \( \frac{x}{\pi} \) is a unit of \( V \), we have \( \frac{x}{\pi_l} \equiv \alpha_l \pmod{M} \) for a unique \( 0 \not\equiv \alpha_l \in K \).

(2.5) Proposition Let \( D \) be an \( i \)-local domain with maximal ideal \( P \), and assume that \( D = V \) is a valuation ring with maximal ideal \( M \), \( v \) be a valuation belonging to \( V \) with the value group \( \Gamma \). Assume that \( D \supset M^3 \). Then,

1. \( D \) is either a PVD or, we may assume that \( K \) is the rank one convex subgroup of \( \Gamma \).
2. If \( D/P = V/M \), then \( D \) is an APVD.

Proof. (1) follows from (2.1).

(2) Suppose the contrary. Then we may apply (2.4), and we may assume that \( K \subset D \). Since \( D \) is not an APVD, we may choose \( x \in P - M^3 \). If \( v(x) = 1 \), then \( x^2 \in P - M^3 \) and \( x^2 \in M^2 \). Hence we may assume that \( v(x) = 2 \). We have \( x = \alpha \pi^2 + a \pi^3 \) for \( \alpha \in K \) and \( a \in V \). Since \( \alpha \in D - P \), we have \( \pi^2 \in P \), and hence \( M^2 \subset P \). Since \( D \) is not an APVD, we may choose \( x \in P - M^2 \). Then \( \pi \in P \), and hence \( M = P \); a contradiction.

References


