Tables for Numerical Calculations of Complex Numbers

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I. Introductory.

With rapid developments of electrical engineering, Complex numbers are used so frequent as real numbers in the elementary arithmetics, in its various branches. In most text books treating complex numbers are given some fundamental formula, and it is proved that their calculations are reduced to those of real numbers, with due care of $i = \sqrt{-1}$. This is exactly right only under the conditions that sufficient time and labor are paid to their calculations. On this principle, most of students manage sometimes ten or more figures to extract only a square root, But some of them give up the calculations on the way, ar are lead to the erroneous results.

The present tables are intended to simplify the the numerical calculations of complex numbers. The idea is found in the Jahnke-Eemde Funktionentafeln. But the tables in this book are so gross that they can hardly be applied to the practical cases, because the step of variable x is 0.01 for $0 < x < 0.1$ and 0.1 for $0.1 < x < 1$. The auther has extended them to the tables having the step of variable x to be 0.0001 throughout for $0 < x < 1$. The students who are much troubled with the complicated calculations of complex numbers, will find a great simplicity, often even a great pleasure by using the tables.

II. Explanatory.

2.1 Definitions.

As the standard forms of complex numbers we take:

$$1+ix,\quad x+i\ ;\quad (0<x<1,\ i=\sqrt{-1})\ \text{-------- } (1)$$

Let the absolute values be m and arguments be μ and ν respectively, then:

$$1+ix=me^{i\mu},\quad x+i=me^{i\nu}\ \cdots\cdots\cdots\cdots\ (2)$$

$$\mu+\nu=90°,\quad (0<\mu<45°,\ 0<\nu<45°)\ \cdots\cdots\cdots\cdots\ (3)$$

Now we have:

$$m=\sqrt{1+x^2}=\sec\mu,\quad \mu=\frac{180°}{\pi}\tan^{-1}x\ \cdots\cdots\cdots\cdots\ (4)$$

$$\cos\mu=\frac{1}{m},\quad \sin\mu=\frac{x}{m}\ \cdots\cdots\cdots\cdots\ (5)$$

Next the reciprocals of (1) be put:

$$\frac{1}{1+ix}=g-ih,\quad \frac{1}{x+i}=h-ig\ \cdots\cdots\cdots\cdots\ (6)$$

then we have:

$$g=\frac{1}{1+x^2}=\cos^2\mu,\quad h=\frac{x}{1+x^2}=\cos\mu\sin\mu\ \cdots\cdots\cdots\cdots\ (7)$$

Thus we have seven quantities, each of which is a function of x:

$$m,\ \mu,\ \nu,\ \cos\mu=\sin\nu,\ \sin\mu=\cos\nu,\ g,\ h.$$

The present tables give the values of these seven quantities with respect to x, which is varied at the step of 0.0001.

2.2 series.

The following series can be easily deduced:

$$\left.\begin{aligned}
m&=1+\frac{1}{2}x^2-\frac{1}{8}x^4+\\
\mu&=\frac{180°}{\pi}(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\\
\cos\mu&=1-\frac{1}{2}x^2+\frac{3}{8}x^4-\\
\sin\mu&=x-\frac{1}{2}x^3+\frac{3}{8}x^5-\\
g&=1-x^2+x^4-x^6+\\
h&=x-x^3+x^5-x^7+
\end{aligned}\right\}\ \cdots\cdots\cdots\cdots\ (8)$$

where μ and ν are measured in degree. The other useful series can be deduced by Taylor's theorem:

$$\left.\begin{array}{l} m_{x+\varepsilon} \doteqdot m_x + \dfrac{x}{m_x}\varepsilon, \quad \mu_{x+\varepsilon} \doteqdot \mu_x + \dfrac{180^\circ}{\pi}\dfrac{1}{1+x^2}\varepsilon \\ \cos\mu_{x+\varepsilon} \doteqdot \cos\mu_x - \dfrac{x}{m_x^3}\varepsilon, \quad \sin\mu_{+\varepsilon} \doteqdot \sin\mu_x + \dfrac{1}{m_x^3}\varepsilon \\ g_{x+\varepsilon} \doteqdot g_x - \dfrac{2x}{m_x^4}\varepsilon, \quad h_{x+\varepsilon} \doteqdot h_x + \dfrac{1-x^2}{m_x^4}\varepsilon \end{array}\right\} \cdots\cdots (9)$$

where $\varepsilon \ll x$.

2.3 Negative signs.

As $0 < x < 1$, the negative signs are interpreted as follows:

$$\left.\begin{array}{l} 1+ix = m\angle\mu, \quad 1-ix = m\angle-\mu \\ -1+ix = m\angle 180^\circ-\mu, \quad -1-ix = m\angle 180^\circ+\mu \\ x+i = m\angle\nu, \quad x-i = m\angle-\nu \\ -x+i = m\angle 180^\circ-\nu, \quad -x-i = m\angle 180^\circ+\nu \end{array}\right\} \cdots\cdots\cdots (10)$$

and,

$$\left.\begin{array}{l} \dfrac{1}{-1-ix} = \dfrac{-1}{1+ix} = -(g-ih), \quad \dfrac{1}{-1+ix} = \dfrac{-1}{1-ix} = g+ih \\ \dfrac{1}{-x-i} = \dfrac{-1}{x+i} = -(h-ig), \quad \dfrac{1}{-x+i} = \dfrac{-1}{x-i} = -(h+ig) \end{array}\right\} \cdots\cdots (11)$$

where for the convenience's sake it is written:

$$me^{i\mu} = m\angle\mu \quad \cdots\cdots\cdots\cdots \quad (12)$$

2.4 To find r, θ.

Let a given complex number be $a+ib$. The polar components r and θ can be found:

$$\left.\begin{array}{ll} \text{(i)} & a > b > 0, \quad x = \dfrac{b}{a}, \\ & a+ib = a(1+ix) = am\angle\mu = r\angle\theta. \\ \text{(ii)} & b > a > 0, \quad x = \dfrac{a}{b}, \\ & a+ib = b(x+i) = bm\angle\nu \doteqdot r\angle\theta. \end{array}\right\} \cdots\cdots\cdots (13)$$

For negative signs before a and b, the formula (10) must be referred to find θ.

Ex 1. $Z = 758 + i\,391 = \overset{a}{758}(1+i\,\overset{x}{0.51583})$

$$= \overset{a}{758} \times \overset{m}{1.1250}\angle\overset{\mu}{27.2860^\circ} = 852.90\angle 27^\circ 17' 10''.$$

2.5 To find a, b.

Let a given complex number be $r\angle\theta$. Its rectangular components a and b can be found:

(i) $0 < \theta < 45°, \quad \mu = \theta$

$$r\angle\theta = r\cos\mu + i\,r\sin\mu = a + ib.$$

(ii) $45° < \theta < 90°, \quad v = \theta$

$$r\angle\theta = r\cos v + i\,r\sin v = a + ib$$

(iii) $90° < \theta < 135°, \quad v = 180° - \theta.$

$$r\angle\theta = -r\cos v + i\,r\sin v = a + ib$$

(iv) $125° < \theta < 180°, \quad \mu = 180° - \theta.$

$$r\angle\theta = -r\cos\mu + i\,r\sin\mu = a + ib$$

If $180° < \theta < 360°$, then θ may be reduced to $0 < \theta < 180°$, by $r\angle\theta = -r\angle\theta - 180°$, and if $0 > \theta > -180°$, then θ may be reduced to $0 < \theta < 180°$ by $r\angle\theta = -r\angle\theta + 180°$.

Ex 2. $Z = 47.52\angle 83°27'59'' \doteq 47.52\angle \overset{v}{83.4664°}$

$$= 47.52(\overset{\cos v}{0.11379} + i\,\overset{\sin v}{0.99350}) = 5.4073 + i\,47.211$$

2.6 To find the reciprocal.

If a given complex number be $a+ib$, then its reciprocal can be found:

$$\left.\begin{aligned}
&\text{(i)}\quad a > b > 0, \quad x = \frac{b}{a}, \\
&\qquad \frac{1}{a+ib} = \frac{1}{a(1+ix)} = \frac{1}{a}(g - ih) \\
&\text{(ii)}\quad b > a > 0, \quad x = \frac{a}{b}, \\
&\qquad \frac{1}{a+ib} = \frac{1}{b(x+i)} = \frac{1}{b}(h - ig)
\end{aligned}\right\} \quad \cdots\cdots\cdots \quad (16)$$

To the negative signs before a and b, the formula (11) must be referred. If a given complex number be $r\angle\theta$, then its reciprocal can directly abtained:

$$\frac{1}{r\angle\theta} = \frac{1}{r}\angle -\theta \quad \cdots\cdots\cdots\cdots \quad (17)$$

Ex. 3. $$Z = \frac{1}{38.05 + i\,27.23} = \frac{1}{\underset{a}{38.05}(1 + i\,\underset{x}{0.71564})}$$

$$= \frac{1}{38.05}(\overset{g}{0.66132} - i\,\overset{h}{0.47326}) = 0.017380 - i\,0.012438$$

2.7 To find the square root.

Let a given complex number be $a+ib$, then its square root can be found as follows:

$$\left.\begin{aligned}&\sqrt{1+ix}=\pm\left(\sqrt{\frac{m+1}{2}}+i\sqrt{\frac{m-1}{2}}\right),\quad \sqrt{-1+ix}=\pm\left(\sqrt{\frac{m-1}{2}}+i\sqrt{\frac{m+1}{2}}\right)\\&\sqrt{x+i}\ =\pm\left(\sqrt{\frac{m+x}{2}}+i\sqrt{\frac{m-x}{2}}\right),\quad \sqrt{-x+i}=\pm\left(\sqrt{\frac{m-x}{2}}+i\sqrt{\frac{m+x}{2}}\right)\end{aligned}\right\}\quad(18)$$

hence we have:

$$\left.\begin{aligned}&\text{(i)}\quad a>b>0,\quad x=\frac{b}{a}\\&\qquad\sqrt{a+ib}=\sqrt{a}\,\sqrt{1+ix}=\pm\sqrt{a}\left(\sqrt{\frac{m+1}{2}}+i\sqrt{\frac{m-1}{2}}\right)\\&\text{(ii)}\quad b>a>0,\quad x=\frac{a}{b}\\&\qquad\sqrt{a+ib}=\sqrt{b(x+i)}=\pm\sqrt{b}\left(\sqrt{\frac{m+x}{2}}+i\sqrt{\frac{m-x}{2}}\right)\end{aligned}\right\}\ \cdots\cdots(19)$$

If a given number is of the form $r\angle\theta$, then:

$$\sqrt{r\angle\theta}=\frac{1}{\sqrt{r}}\angle\frac{\theta}{2}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ (20)$$

In order to find the higher powers and the higher roots of a complex number, it is advised to transform the given number once into the polar form and to apply the formula:

$$\left.\begin{aligned}&(a\pm ib)^n=(r\angle\theta)^n=r^n\angle n\theta\\&(a+ib)^{\frac{1}{n}}=(r\angle\theta)^{\frac{1}{n}}=r^{\frac{1}{n}}\angle\frac{\theta}{n}\end{aligned}\right\}\ \text{-----------------}\ (21)$$

Ex. 4

$$Z=\sqrt{33.67+i\,82.55}=\sqrt{82.55(\underset{x}{0.40787}+i)}$$

$$=\underset{b}{\sqrt{82.55}}\left(\underset{\sqrt{\frac{m+x}{2}}}{\sqrt{0.74393}}+i\underset{\sqrt{\frac{m-x}{2}}}{\sqrt{0.33607}}\right)$$

$$=7.8365+i\,5.2671$$

Otherwise:

$$Z=\sqrt{82.55}\ \sqrt{\underset{m}{1.07999}\angle\underset{v}{68.8110^\circ}}$$

$$=9.4422\angle 33.9055^\circ$$

III. Acknowledgements.

The principal part of the table extends to two hundred pages one of which is given here as an example. In working the tables, I am much indebted to the Barlow's

tables which give squares, square roots, reciprocals etc. Also many thanks are due to Mr. Ryosuke Suzuki who performed much laborious computations.

x	m	μ	ν	$\cos\mu$ $=\sin\nu$	$\sin\mu$ $=\cos\nu$	g	h
0.7750	1.26516	37.7757	52.2243	0.79041	0.61257	0.62476	0.48419
1	22	793	207	37	62	70	21
2	28	829	171	33	67	64	22
3	34	864	136	30	72	58	24
4	41	900	100	26	77	52	25
5	47	936	064	22	81	46	27
6	53	972	028	18	86	39	28
7	59	37.8008	52.1992	14	91	33	30
8	65	043	957	11	96	27	31
9	71	079	921	07	0.61301	21	33
0.7760	1.26577	37.8115	52.1885	0.79003	0.61306	0.62415	0.48434
1	83	161	839	0.78999	11	09	36
2	89	186	814	95	16	03	37
3	95	222	778	92	21	0.62397	39
4	1.26601	258	742	88	26	91	40
5	08	294	706	84	31	85	42
6	04	329	671	80	36	79	44
7	20	365	635	76	41	73	45
8	26	400	600	73	46	67	47
9	32	436	564	69	51	61	48
0.7770	1.26639	37.8472	52.1528	0.78965	0.61356	0.62355	0.48450
1	45	508	492	61	61	49	52
2	51	543	457	57	66	43	53
3	57	579	421	54	71	37	55
4	64	615	385	50	76	31	56
5	70	651	349	46	80	25	58
6	76	686	314	42	85	18	59
7	82	722	278	38	90	12	61
8	88	758	242	35	95	06	62
9	94	793	207	31	0.61400	00	64
0.7780	1.26700	37.8829	52.1171	0.78927	0.61405	0.62294	0.48465
1	06	865	135	23	10	88	67
2	12	900	100	19	15	82	68
3	18	936	064	15	20	76	70
4	25	972	028	11	25	70	71
5	31	37.9008	52.0992	08	29	64	73
6	37	043	957	04	34	58	74
7	43	079	921	00	39	52	76
8	49	115	885	0.78896	44	46	77
9	55	150	850	92	49	40	79
0.7790	1.26762	37.9186	52.0814	0.78888	0.61454	0.62234	0.48480
1	68	222	778	84	59	28	82
2	74	257	743	80	64	22	83
3	80	293	707	77	69	16	85
4	87	328	672	73	74	10	86
5	93	364	636	69	78	04	88
6	99	400	600	65	83	0.62197	89
7	1.26805	435	565	61	88	91	91
8	11	471	529	58	93	85	93
9	17	506	494	54	98	79	94