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Solution of an infinite system of differential equations of the analytic type with isolated singularity.

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An infinite system of differential equations:

(1) $xy_j'=f_j(x, y_1, y_2,\cdots) = f_j^{(1)}+f_j^{(2)}+\cdots,\ (j=1, 2,\cdots,\infty)$

will be considered here. F. R. Moulton treated an infinite system of differential equations in [1]. The system considered in [1] was

(m) $y_j'=f_j(x, y_1, y_2,\cdots) = a_j+f_j^{(1)}+f_j^{(2)}+\cdots,\ (j=1, 2,\cdots, \infty)$,

where the variables are complex and $f_j^{(k)}$ is the totality of terms (homogeneous in $x, y_1, y_2,\cdots$ of degree k) of f_j. Our present existence theorem will be proved by thes imilar method to that used in [1].

In the first place, we shall state the following property-N of normal determinant (N-determinant) and an theorem on which we base our argument.

Normal-determinant [2]. An infinite determinant

$$|(A)| = |(\delta_{jk}+a_{jk})| \quad (j, k=1, 2,\cdots,\infty)$$

where δ_{jk} is the Kronecker symbol, is called a normal or an N-determinant if $S=\sum_{j,k}|a_{jk}|$ converges. The fundamental theorem on the solution of an infinite system of linear equations reads as follows:

Property-N : In the infinite system of linear equations

$$\sum_{k=1}^{\infty}(\delta_{jk}+a_{jk})x_k=b_j,\ (j=1, 2,\cdots),$$

suppose that the determinant $|(A)|$ is normal and distinct from zero, and that $|b_k| < b(0<b<\infty;\ j=1, 2,\cdots)$. Then among all bounded sequences of numbers $(x_1, x_2,\cdots)$ there exists one and only one solution given by

$$x_j=\sum_k b_k|(D_{kj})|/|(A)| \quad (j=1, 2,\cdots)$$

where $|(D_{kj})|$ is the co-factor of $\delta_{kj}+a_{kj}$ in $|(A)|$ for every k.

THEOREM. Hypotheses: (i) Let $f_j(x, y_1, y_2,\cdots)$ be as described above, and let $f_j^{(1)}$ be $a_{j0}^{(1)}x+\sum_{k=1}^{\infty}a_{jk}^{(1)}y_k$. (ii) The initial condition $y_j(0)=0$.

(iii) There exist real positive numbers $(c_0, c_1, c_2,\cdots)$, $(r_0, r_1, r_2,\cdots)$ and A, satisfying the following conditions: $c_0x+c_1y_1+\cdots=$ s converges for the point $(x, y_1, y_2, \cdots)$ in the region of complex values

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(2) $|x| \leq r_0, \; |y_j| \leq r_j$

and $Ar_j s^k$ domimates $f_j^{(k)}$. (iv) Let $C = \sum_{k=1}^{\infty} c_k r_k$ and let AC be not any positive integer. (v) $|(\delta_{jk} - a_{jk}^{(1)})|$ is normal, $|(\delta_{jk} - a_{jk}^{(1)}/n)| > 0$ $(j, k, n = 1, 2, \cdots)$ and the co-factor of each element $(\delta_{jk} - a_{jk}^{(1)}/n)$ of $|(\delta_{jk} - a_{jk}^{(1)}/n)|$ is positive.

Conclution. There exists one and only one analytic solution of (1) satisfying the initial condition $y_j(0) = 0$.

PROOF. Since, if the relation (2) is satisfied, the series $s = c_0 x + c_1 y_1 + \cdots$ converges, there exists a finite constant $M (\geq 1)$ such as

(3) $S = c_0 \dfrac{r_0}{M} + c_1 \dfrac{r_1}{M} + \cdots\cdots < 1.$

Hence if $M \cdot |x| < \bar{r}_0$ and $M \cdot |y_j| < \bar{r}_j$ are astisfied, then

(4) $|f_j| < A\bar{r}_j (S + S^2 + \cdots) = A\bar{r}_j S/(1-S).$

If an analytic solution of (1) satisfying the initial condition (ii) exists, it will have the form

(5) $y_j = \alpha_j^{(1)} x + \alpha_j^{(2)} x^2 + \cdots, \quad (j = 1, 2, \cdots, \infty).$

Substituting these series in (1) and equating the cofficients of the corresponding powers of x, it is found that

(E1) $\alpha_j^{(1)} = a_{jo}^{(1)} + \sum_{k=1}^{\infty} a_{jk}^{(1)} \alpha_k^{(1)}$

and in general

(En) $\alpha_j^{(n)} = \sum_{k=1}^{\infty} (a_{jk}^{(1)}/n) \alpha_k^{(n)} + \{[\partial^n f_j^{(n)}] + P_j^{(n)}(\alpha_k^{(1)}, \; \alpha_k^{(2)}, \cdots, \alpha_k^{(n-1)})\}/n, \quad (n \geq 2)$. Here $[\partial^n f_j^{(n)}]$ is the coefficient of the term x^n of $f_j^{(n)}$, and $P_j^{(n)}$ is a polynomial in $\alpha_k^{(1)}$, $\alpha_k^{(2)}, \cdots, \alpha_k^{(n-1)} (k = 1, 2, \cdots)$ whose coefficients are linear functions of the coefficients of f_j with positive numerical multipliers.

Since, by (v) i. e. $|(\delta_{jk} - a_{jk}^{(1)}/n)| > 0$, $\alpha_j^{(n)}$ is defined by (E1) or (En), the formal solution of (1) is only one.

In order to prove the convergence of the formal solution (5) for values of x whose absolute value is sufficiently small, we shall consider the solution of

(6) $x \dfrac{d\eta_j}{dx} = A\bar{r}_j \sigma/(1-\sigma), \quad (j = 1, 2, \cdots, \infty),$

where

(7) $\sigma = c_0 x + c_1 \eta_1 + c_2 \eta_2 + \cdots\cdots.$

The right members of (6) dominate the respective right members of (1).

Let the formal solution of (6) be

(8) $\eta_j = \beta_j^{(1)} x + \beta_j^{(2)} x^2 + \cdots\cdots, \quad (j = 1, 2, \cdots, \infty).$

The coefficie nts of these series can be obtained by equations analogous to (E1) and (En). They are therefore real and positive; and it follows from the fact that the right members of (6) dominate the right members of (1) that

(9) $\beta_j^{(k)} > |\alpha_j^{(k)}|$, $(j, k=1, 2, \cdots, \infty)$.

Therefore if (8) converges for $|x| < \rho$, then (5) also converges for at least the same value of x. It follows from (6) that

$$\frac{1}{r_1}\frac{d\eta_1}{dx} = \frac{1}{r_2}\frac{d\eta_2}{dx} = \frac{1}{r_3}\frac{d\eta_3}{dx} = \cdots = \frac{d\eta}{dx}.$$

The initial values of η_1, $\eta_2, \cdots$ are zero; hence on taking $\eta_j(0)=0$ if follows that

(10) $\eta_j = r_j\eta \quad (j=1, 2, \cdots)$.

Therefore each of equations (6) reduces to

$$(11) \quad x\frac{d\eta}{dx} = \frac{A(c_0x + C\eta)}{1-(c_0x+C\eta)}$$

where $C=\sum_{k=1}^{\infty} c_k r_k$, which is a finite constant by hypothesis (iii).

It follows from the ordinary theory for a finite number of differential equations that (11) has an analytic solution which converges if $|x|$ is sufficiently small. Hence the formal solution (8) and (5) converge for at least the same values of x.

After all, under the hypotheses as described above, (1) has only one regular solution satisfying $y_j(0)=0$. This completes the proof.

References

〔1〕 F. R. Moulton, Solution of an infinite system of differential equations of the analytic type. Proc. Nat. Acad. Sci. 1 (1915), pp. 350-354.

〔2〕 F. Riesz, Les systèmes d'équations linéaires à une infinité d'inconnues, Paris, 1952. pp.1-29.