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On the limits of precompact linear operators

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Since Leray [6], many results on compact linear operators in Banach spaces, especially the Riesz-Schauder's theory, have been extended to the compact operators in locally convex spaces by various authors. In this short note, we are going to make an elementary observation on the limits of precompact linear operators in locally convex spaces. It is shown, among others, that, if a locally convex space E is to be such that, for any locally convex space F, the space of precompact linear operators of E into F is closed in $L_b(E,F)$, then E is normable.

1. Preliminaries. We shall collect here some well known results with their proofs so that the exposition is self-contained.

Let E be a locally convex space and E' the dual of E. We denote by E'_b, E'_s, E'_τ and E'_c respectively the space E' equipped with the strong topology, the weak* topology, the Mackey topology and the topology of uniform convergence on convex compact sets in E. For two locally convex spaces E and F, $L(E, F)$ denotes the space of continuous linear operators of E into F. We denote by $L_b(E,F)$ and $L_c(E,F)$ the space $L(E,F)$ equipped with the topology of uniform convergence on bounded sets in E and the topology of uniform convergence on convex compact sets in E, respectively. If E is the dual G' of a locally convex space G, then the space $L(G',F)$ with the topology of uniform convergence on equicontinuous sets in G' is denoted by $L_e(G',F)$. All spaces considered in what follows are supposed to be Hausdorff.

PROPOSITION 1 ([8; Exposé n° 8]). *Let E and F be locally convex spaces. Then the space $L_e((E'_b)'_c,F)$ is the subset of $L(E''_\tau,F)$ consisting of all operators which map bounded sets in E into relatively compact sets in F. The topology τ_e of $L_e((E'_b)'_c,F)$ is the same as that of uniform convergence on bounded sets in E.*

PROOF. Let $u\in L_e((E'_b)'_c,F)$ and B any bounded set in E. Since B can be regarded as an equicontinuous set in $(E'_b)'$, it is relatively compact with respect to the topology of $(E'_b)'_c$ by the Ascoli's theorem [3]. Thus the image of B under u is relatively compact in F. Since the Mackey topology $\tau(E'',E')$ is finer than the topology of $(E'_b)'_c$, we have $u\in L(E''_\tau,F)$.

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Conversely, let $u \in L(E''_\tau, F)$ be such that u maps bounded sets in E into relatively compact sets in F. Since $u \in L(E''_\tau, F)$, the transposed operator ${}^t u$ maps F' into E'. Thus it follows from our assumption that ${}^t u \in L(F'_c, E'_b)$. Any equicontinuous set in F'_c is relatively compact so that it is transformed by ${}^t u$ into a relatively compact set in E'_b. This shows that $u \in L((E'_b)'_c, F)$.

The last assertion follows readily from the fact that any equicontinuous set in $(E'_b)'$ is contained in the bipolar set of some bounded set in E. Q. E. D.

We say that a locally convex space E has the *property* (A) (*=the approximation property*) if the identity operator 1 in E is adherent to $E' \otimes E$ in $L_c(E, E)$.

PROPOSITION 2 ([8; Exposé n° 14]). *A locally convex space E has the property* (A) *if and only if, for any locally convex space F, $\overline{F \otimes E} = L_e(F'_c, E) = L_e(E'_c, F)$, where the bar denotes the closure in the latter space.*

PROOF. Suppose that E has the property (A) and $u \in L_e(F'_c, E)$. Let U be any neighborhood of 0 in E and B' any equicontinuous convex set in F'. Since B' is relatively compact and convex in F'_c, $u(B')$ is also relatively compact and convex in E. Thus, by the property (A) of E, there is a $v \in E' \otimes E$ such that $(1-v)(x) \in U$ for and $x \in u(B')$. This means that $(u - v \circ u)(B') \subset U$. Since v is of finite rank and u is continuous with respect to $\sigma(F', F)$ and $\sigma(E, E')$, $v \circ u$ is of finite rank and continuous with respect to the same topologies. It follows that $v \circ u$ is given by some element in $F \otimes E$. As U and B' are arbitrary, $\overline{F \otimes E} = L_e(F'_c, E)$.

Suppose now that $\overline{F \otimes E} = L_e(F'_c, E)$ for any F. Putting $F = E'_c$, we get $\overline{E' \otimes E} = L_e((E'_c)'_c, E)$. Clearly $(E'_c)' = E$ as abstract linear spaces. Since equicontinuous sets are relatively compact in E'_c, the identity operator 1 of $(E'_c)'_c$ onto E is continuous, so that 1 is adherent to $E' \otimes E$ in $L_e((E'_c)'_c, E)$. As any compact set in E is equicontinuous in $(E'_c)'$, we have shown that 1 is adherent to $E' \otimes E$ in $L_c(E, E)$. Hence E has the property (A).

The isomorphism $L_e(F'_c, E) = L_e(E'_c, F)$ is obvious. Q. E. D.

Let E and F be abstract linear spaces and suppose there is defined a bilinear form $\langle x . y \rangle$ ($x \in E$, $y \in F$) such that $\langle x, y \rangle = 0$ for all $y \in F$ implies $x = 0$ and $\langle x, y \rangle = 0$ for all $x \in E$ implies $y = 0$. Then (E, F) is called a dual pair with respect to $\langle x . y \rangle$.

PROPOSITION 3. *Let (E, F) be a dual pair with respect to $\langle x, y \rangle$ and E_s the locally convex space whose E topology is $\sigma(E, F)$. Then E_s has the property* (A).

PROOF. We note first that $(E_s)' = F$. Let U be any neighborhood of 0 in E_s. We may suppose that U is of the form $U = \{x \in E : |\langle x, y_i \rangle| < 1, i = 1, 2 \cdot, m\}$ where $\{y_i\}$ is a finite set consisting of linearly independent elements in F. Then we can find easily m elements $x_i \in E$ such that $\langle x_i, y_j \rangle = \delta_{ij}$, $i, j = 1, 2, \cdots, m$. Now we define an operator $v \in (E_s)' \otimes E = F \otimes E$ by setting

$$v(x) = \sum_{j=1}^{m} <x, y_j> x_j.$$

We have $<x-v(x), y_i> = <x-\sum <x,y_j>x_j, y_i> = 0$. Thus $(1-v)(x) \in U$ for any $x \in E$. As U is arbitrary, the operator 1 is adherent to $(E_s)' \otimes E$ in $L_c(E_s, E_s)$. Hence E_s has the property (A). Q. E. D.

2. The space of precompact linear operators. It is well known that, if E and F are normed linear spaces, then the space of precompact linear operators of E into F is closed in the space $L_b(E, F)$ of continuous linear operators (cf. [1 ; p. 96, Théorème 2]). Here a linear operator of E into F is called *precompact* if it brings some neighborhood of 0 in E into a precompact set in F. Any precompact linear operator is clearly continuous. A simple generalization is the following

THEOREM 1. *If E is a normed linear space, then the space of precompoct linear operators of E into F is closed in $L_b(E, F)$ for any locally convex space F.*

PROOF. Let $\mathfrak{F}$ be a filter consisting of precompact linear mappings of E into F and converging in $L_b(E, F)$ to some $u_0 \in L_b(E, F)$. We shall show that $u_0(B)$ is precompact in F where B is the unit ball of E. By the definition of the topology of $L_b(E, F)$, there exists, for any neighborhood V of 0 in F, a member $M \in \mathfrak{F}$ such that $u(x) - u_0(x) \in \frac{1}{2}V$ for all $x \in B$ and $u \in M$. Take any $u \in M$. Since $u(B)$ is precompact, there exist a finite number of points $\{x_i : i = 1, \cdots, n\} \subset u(B)$ such that $u(B) \subset \cup_{i=1}^{n} (x_i + \frac{1}{2}V)$ (cf. [2; p. 161, Théorèm 4]). Thus we have $u_0(B) \subset u(B) + \frac{1}{2}V \subset \cup_{i=1}^{n} (x_i + V)$. This shows that there is a finite covering of $u_0(B)$ of order V. As V is arbitrary, $u_0(B)$ is precompact by the theorem cited above. Hence u_0 is precompact. Q. E. D.

COROLLARY. *Under the same assumption as in Theorem 1, the space of precompact linear operators of E into F is complete with respect to the topology of uniform convergence on bounded sets in E.*

PROOF. Since E is a normed linear space and therefore bornological, $L_b(E,F)$ is complete. The space of precompact operators is a closed subspace of $L_b(E, F)$ by Theorem 1 and thus complete. Q. E. D.

It is of some interest that the converse of Theorem 1 is valid in the following sense. This is the main result of the present paper.

THEOREM 2. *Let E be a locally convex space and suppose that, for any locally convex space F, the space of precompact linear mappings of E into F is closed in $L_b(E,F)$. Then E is normable.*

PROOF. Set $F = (E'_b)'_s$ where the topology τ_s is $\sigma(E''; E')$. Then F has the property (A) by Proposition 3. Thus by Proposition 2

$$\overline{E' \otimes F} = L_e((E'_b)'_c, F).$$

Then Proposition 1 tells us that $L_e((E'_b)'_c, F)$ is the subspace of $L(E''_\tau, F)$ consisting of all operators which bring bounded sets in E into relatively compact sets in F. Thus, $L_e((E'_b)'_c, F)$ contains the identity operator Φ of E'' onto F $(=(E'_b)'=E'')$, since any bounded set in E is equicontinuous in $(E'_b)'$ and therefore relatively $\sigma(E'', E')$-compact in E''.

It follows from the identity $\overline{E'\otimes F} = L_e((E'_b)'_c, F)$ that Φ is adherent to $E'\otimes F$ in $L_e((E'_b)'_c, F)$. Since the topology τ_e of $L_e((E'_b)'_c, F)$ is that of uniform convergence on bounded sets in E, the restriction $\Phi | E$ of Φ to E is adherent to $E'\otimes F$ in $L_b(E, F)$. The elements in $E'\otimes F$ are of finite rank and à fortiori precompact. Consequently, by the assumption of the theorem, $\Phi | E$ must be precompact. Hence there exists a neighbrohood U of 0 in E such that $U=\Phi(U)$ is precompact in F. That is, U is $\sigma(E'', E')$-precompact and therefore $\sigma(E'', E')$-bounded. As $U\subset E$, U is weakly bounded in E and thus bounded by a theorem of Mackey. Hence E is normable by a theorem of Kolmogoroff.

Q.E.D.

Now we shall consider the limits of precompact operators. Ler E, F be locally convex spaces. A linear operator of E into F is called *boundedly precompact* if it is weakly continuous and brings bounded sets in E to precompact sets in F (cf. [7]). We denote for the present by $P(E, F)$ the space of all boundedly precompact continuous linear operators of E into F. Then we have

PROPOSITION 4. *The space $P(E, F)$ is closed in $L_b(E, F)$. If the completion $\hat{F}$ of F has the property* (A), *then $\overline{E'\otimes F} = P(E, F)$ where the bar denotes the closure in $L_b(E, F)$. If E is bornological, then $P(E, F)$ is complete with respect to the topology induced by $L_b(E, F)$.*

PROOF. Let $\mathfrak{F}$ be a filter in $P(E, F)$ which converges in $L_b(E, F)$ to some $u\in L_b(E, F)$. Let B be any closed convex circled subset of E and E_B the normed space which is generated by B with the norm $\|x\| = \inf_{x\in\lambda B} |\lambda|$. Then B is the unit ball of E_B. The restriction $\mathfrak{F} | E_B$ of $\mathfrak{F}$ to E_B converges in $L_b(E_B, F)$ to $u | E_B$. Since the mappings contained in $\mathfrak{F} | E_B$ are precompact, $u | E_B$ is precompact by Theorem 1. Thus $u(B)$ is precompact in F. As any bounded set in E is contained in some closed convex circled bounded subset of E, we have shown that u is boundedly precompact. Hence $P(E, F)$ is closed in $L_b(E, F)$.

Now suppose that $\hat{F}$ has the property (A). Let $u\in P(E, F)$ and φ the natural injection of F into $\hat{F}$. Then $\hat{u}=\varphi\circ u$ is a continuous linear operator of E into $\hat{F}$ which brings bounded sets in E into relatively compact sets in $\hat{F}$. Let B be any bounded convex subset of E. $\hat{u}(B)$ is then relatively compact and convex in $\hat{F}$. If U is any closed neighborhood of 0 in F, then the closure $\overline{U}$ in $\hat{F}$ of U is a neighborhood of 0 in $\hat{F}$. Since $\hat{F}$ has the property (A), there exists an element $v\in(\hat{F})'\otimes\hat{F}$ such that $y-v(y)\in \overline{U}$ for any y in $\hat{u}(B)$. It is clear that $(\hat{F})'=F'$

and F is dense in $\hat{F}$. Thus we may suppose that $v \in F' \otimes F$. Since $\hat{u}(B) = u(B) \subset F$ and $\bar{U} \cap E = U$, we have $y - v(y) \in U$ for any $y \in u(B)$. Therefore,

$$u(x) - v \circ u(x) \in U \qquad \text{for any } x \in B.$$

Since $v \in F' \otimes F$, it is easy to see that $v \circ u \in E' \otimes F$. As U and B are arbitrary, we have shown that $E' \otimes F$ is dense in $P(E, F)$ with respect to the topology induced by $L_b(E, F)$.

If E is bornological, then $L_b(E, F)$ is complete and therefore $P(E, F)$ is complete with respect to the topology induced by $L_b(E, F)$. Q. E. D.

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