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Sharp remainder terms of Hardy-Sobolev inequalities

ALNAR DETALLA*, TOSHIO HORIUCHI** AND HIROSHI ANDO**

ABSTRACT. In this paper we shall prove the existence of sharp remainder terms involving singular weight $(\log \frac{R}{|x|})^{-2}$ for Hardy-Sobolev inequalities of the following type:

$$\int_{\Omega} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{|u(x)|^2}{|x|^2} dx$$

for any $u \in W_0^{1,2}(\Omega)$, Ω is a bounded domain in \mathbb{R}^n , $n > 2$, with $0 \in \Omega$. Here the number of remainder terms depends on the choice of R .

1. Introduction.

In this paper, we shall study the Hardy-Sobolev inequalities of the following type:

$$(1.1) \quad \int_{\Omega} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{|u(x)|^2}{|x|^2} dx$$

for any $u \in W_0^{1,2}(\Omega)$, Ω is a bounded domain in \mathbb{R}^n , $n > 2$, with $0 \in \Omega$. By $W_0^{1,2}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{1,2,\Omega} := (\int_{\Omega} |u(x)|^2 dx + \int_{\Omega} |\nabla u|^2 dx)^{\frac{1}{2}}$. It is known that there is no function $u \in W_0^{1,2}(\Omega)$ for which the best constant $\Psi_{n,2} := (\frac{n-2}{2})^2$ is achieved. Hence it is natural to consider that there exist "missing terms" in the right hand side of (1.1). In view of this, we shall investigate the Hardy-Sobolev inequalities (1.1) and find out the remaining terms involving singular weight $(\log \frac{R}{|x|})^{-2}$. The number of remaining terms depends on the choice of R .

For the inequality involving one remaining term, such Hardy-Sobolev inequalities are known. For example, Adimurthi, Chaudhuri and Ramaswamy

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[1] have proved that there exists a constant $C > 0$, depending on $n > 2$ and $R > \sup_{\Omega}(|x|e)$ such that for $u \in W_0^{1,2}(\Omega)$

$$\int_{\Omega} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{|u(x)|^2}{|x|^2} dx + C \int_{\Omega} \frac{|u(x)|^2}{|x|^2} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx$$

where $\gamma \geq 2$.

2. Main results.

In this section we will introduce our result about the existence of *finitely many sharp missing terms* of Hardy-Sobolev inequality (1.1).

Before stating our main results let us introduce the following notations: For $t > 0$ and $k \geq 2$,

$$A_1(t) := \log \frac{R}{t}, \quad A_k(t) := \log A_{k-1}(t), \quad e_1 := e, \quad e_k := e^{e^{k-1}}.$$

Our main results are as follows:

Theorem 2.1. *Let $n \geq 2$, $k \geq 1$ and $R \geq e_k \sup_{\Omega} |x|$. For any $u \in W_0^{1,2}(\Omega)$, there exist sharp remainder terms such that*

$$(2.1) \quad \int_{\Omega} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{u(x)^2}{|x|^2} dx + \frac{1}{4} \int_{\Omega} \frac{u(x)^2}{|x|^2} \left[A_1(|x|)^{-2} + \left(A_1(|x|)A_2(|x|) \right)^{-2} + \cdots + \left(A_1(|x|)A_2(|x|) \dots A_k(|x|) \right)^{-2} \right] dx.$$

Remark 2.1. In inequality (2.2), $\frac{1}{4}$ is best constant for all k -missing terms and the exponent 2 of the weight function is sharp.

3. Key Lemmas.

In this section we will introduce lemmas which are needed in the proof of the main result.

Lemma 3.1. *Assume $u \in C_0^2(B_1)$ is radial satisfying $u(r) > 0$ where $r = |x|$. Set $v_1(r) = u(r)r^{\frac{n-2}{2}}A_1(r)^{-\frac{1}{2}}$ and $v_k(r) = v_{k-1}(r)A_k(r)^{-\frac{1}{2}}$ for $k \geq 2$. If $R \geq e_k$, then*

$$(3.1) \quad \int_{B_1} |\nabla u(x)|^2 dx = \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_k(r)^2 A_1(r)A_2(r) \dots A_k(r) \frac{dr}{r} + \frac{\omega_n}{4} \int_0^1 v_k(r)^2 A_1(r)A_2(r) \dots A_k(r) \times \left[A_1(r)^{-2} + \left(A_1(r)A_2(r) \right)^{-2} + \cdots + \left(A_1(r)A_2(r) \dots A_k(r) \right)^{-2} \right] \frac{dr}{r} + \omega_n \int_0^1 v'_k(r)^2 A_1(r)A_2(r) \dots A_k(r) r dr$$

for all $k \geq 1$.

Proof. Since $R \geq e_k$, A_i is defined for all $1 \leq i \leq k$. Then direct calculation gives

$$|u'(r)|^2 = v_k(r)^2 r^{-n} A_1(r) A_2(r) \dots A_k(r) \left| \frac{n-2}{2} + C \right|^2,$$

where

$$C = \frac{1}{2} A_1(r)^{-1} + \dots + \frac{1}{2} \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} - \frac{v'_k(r)}{v_k(r)} r.$$

Then

$$\begin{aligned} \int_{B_1} |\nabla u(x)|^2 dx &= \omega_n \int_0^1 |u'(r)|^2 r^{n-1} dr \\ &= \omega_n \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \left| \frac{n-2}{2} + C \right|^2 \frac{dr}{r} \\ &= \left(\frac{n-2}{2} \right)^2 \omega_n \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \frac{dr}{r} \\ &\quad - \left(\frac{n-2}{2} \right) \omega_n \int_0^1 (v_k(r)^2)' A_1(r) A_2(r) \dots A_k(r) dr \\ &\quad + \left(\frac{n-2}{2} \right) \omega_n \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \times \\ &\quad \left[A_1(r)^{-1} + \left(A_1(r) A_2(r) \right)^{-1} + \dots \right. \\ &\quad \left. + \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} \right] \frac{dr}{r} \\ &\quad + \omega_n \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \times \\ &\quad \left[\frac{1}{2} A_1(r)^{-1} + \frac{1}{2} \left(A_1(r) A_2(r) \right)^{-1} + \dots \right. \\ &\quad \left. + \frac{1}{2} \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} - \frac{v'_k(r)}{v_k(r)} r \right]^2 \frac{dr}{r}. \end{aligned} \tag{3.2}$$

The second and third terms in the right hand side of the above equation cancels after applying integration by parts to second term.

Also after expanding the fourth term we get

(3.3)

$$\begin{aligned}
\int_{B_1} |\nabla u(x)|^2 dx &= \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \frac{dr}{r} \\
&+ \frac{\omega_n}{4} \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \times \\
&\quad \left[A_1(r)^{-2} + \left(A_1(r) A_2(r) \right)^{-2} + \dots + \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-2} \right] \frac{dr}{r} \\
&+ \frac{\omega_n}{2} \int_0^1 v_k(r)^2 \left[A_1(r)^{-1} \left\{ A_3(r) A_4(r) \dots A_k(r) + A_4(r) A_5(r) \dots A_k(r) + \right. \right. \\
&\quad \left. \left. \dots + A_k(r) + 1 \right\} + \left(A_1(r) A_2(r) \right)^{-1} \times \right. \\
&\quad \left. \left\{ A_4(r) A_5(r) \dots A_k(r) + A_5(r) A_6(r) \dots A_k(r) + \dots + A_k(r) + 1 \right\} + \right. \\
&\quad \left. \dots + \left(A_1(r) A_2(r) \dots A_{k-1}(r) \right)^{-1} \right] \frac{dr}{r} \\
&- \frac{\omega_n}{2} \int_0^1 (v_k(r)^2)' \left[A_2(r) A_3(r) \dots A_k(r) + \right. \\
&\quad \left. A_3(r) A_4(r) \dots A_k(r) + \dots + A_k(r) + 1 \right] dr \\
&+ \omega_n \int_0^1 v_k'(r)^2 A_1(r) A_2(r) \dots A_k(r) r dr.
\end{aligned}$$

The third and fourth terms in the right hand side of (3.3) cancels after applying integration by parts to fourth term. Hence we get inequality (3.1). By inductive argument we will show the validity of (3.1) for all $k \geq 1$. For $k = 1$, it is easy to verify by similar calculation that

$$\begin{aligned}
\int_{B_1} |\nabla u(x)|^2 dx &= \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_1(r)^2 A_1(r) \frac{dr}{r} \\
(3.4) \quad &+ \frac{\omega_n}{4} \int_0^1 v_1(r)^2 A_1(r)^{-1} \frac{dr}{r} + \omega_n \int_0^1 v_1'(r)^2 A_1(r) r dr.
\end{aligned}$$

Since $v_{k+1}(r) = v_k(r) A_{k+1}(r)^{-\frac{1}{2}}$, direct calculation gives

$$\begin{aligned}
v_k'(r)^2 &= v_{k+1}'(r)^2 A_{k+1}(r) - \frac{1}{r} v_{k+1}(r) v_{k+1}'(r) \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} \\
&+ \frac{1}{4r^2} v_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-2} A_{k+1}(r)^{-1}.
\end{aligned}$$

Then the last term in the right hand side of (3.1) becomes

$$\begin{aligned}
(3.5) \quad & \int_0^1 v'_k(r)^2 A_1(r) A_2(r) \dots A_k(r) r dr \\
&= \int_0^1 v'_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) r dr \\
&\quad + \frac{1}{4} \int_0^1 v_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \right)^{-1} \frac{dr}{r}.
\end{aligned}$$

Hence the last term in the right hand side of (3.1) generates the new terms such that

$$\begin{aligned}
(3.6) \quad & \int_{B_1} |\nabla u(x)|^2 dx = \left(\frac{n-2}{2} \right)^2 \omega_n \int_0^1 v_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \frac{dr}{r} \\
&+ \frac{\omega_n}{4} \int_0^1 v_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \left[A_1(r)^{-2} + \left(A_1(r) A_2(r) \right)^{-2} + \right. \\
&\quad \left. \dots + \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-2} + \left(A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \right)^{-2} \right] \frac{dr}{r} \\
&+ \omega_n \int_0^1 v'_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) r dr.
\end{aligned}$$

Therefore (3.1) is valid for all $k \geq 1$. \square

Remark 3.1. Ignoring the last term in the right hand side of (3.1) and for $v_k(r)^2 = u(r)^2 r^{n-2} \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1}$, $k \geq 1$ we get

$$\begin{aligned}
(3.7) \quad & \int_{B_1} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2} \right)^2 \int_{B_1} \frac{u(x)^2}{|x|^2} dx + \frac{1}{4} \int_{B_1} \frac{u(x)^2}{|x|^2} \left[A_1(|x|)^{-2} + \right. \\
&\quad \left. \left(A_1(|x|) A_2(|x|) \right)^{-2} + \dots + \left(A_1(|x|) A_2(|x|) \dots A_k(|x|) \right)^{-2} \right] dx.
\end{aligned}$$

Lemma 3.2. *Let*

$$g_k(r) = r^{-2} \left(A_1(r) \dots A_k(r) \right)^{-2} = g_{k-1}(r) A_k(r)^{-2}, \quad g_1(r) = r^{-2} A_1(r)^{-2}.$$

If $R > re_k$ then the function $g_k(r)$ is monotone decreasing for all $k \geq 1$.

Proof. We shall prove the lemma using inductive argument. For $k = 1$ we have $g_1(r) = r^{-2} A_1(r)^{-2}$ then direct calculation gives:

$$g'_1(r) = 2r^{-3} A_1(r)^{-3} (-A_1(r) + 1)$$

Since $-A_1(r) + 1 < 0$ for $R > re$, then $g_1(r)$ is monotone decreasing for $R > re$.

By inductive argument we can assume $g_k(r)$ is decreasing function for $R > re_k$. Then direct calculation gives

$$(3.8) \quad g'_k(r) = 2r^{-3} \left(A_1(r) \dots A_k(r) \right)^{-3} \times \\ \left(-A_1(r) \dots A_k(r) + A_2(r) \dots A_k(r) + \dots + A_k(r) + 1 \right)$$

and

$$(3.9) \quad -A_1(r) \dots A_k(r) + A_2(r) \dots A_k(r) + \dots + A_k(r) + 1 < 0 \quad \text{for } R > re_k.$$

Since $g_{k+1}(r) = g_k(r)A_{k+1}(r)^{-2}$, then using (3.8), we get

$$(3.10) \quad g'_{k+1}(r) = g'_k(r)A_{k+1}(r)^{-2} + \frac{2}{r} \left(A_1(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{-3} g_k(r) \\ = g'_k(r)A_{k+1}(r)^{-2} + 2r^{-3} \left(A_1(r) \dots A_k(r) \right)^{-3} A_{k+1}(r)^{-3} \\ = 2r^{-3} A_{k+1}(r)^{-2} \left(A_1(r) \dots A_k(r) \right)^{-3} \times \\ \left(-A_1(r) \dots A_k(r) + A_2(r) \dots A_k(r) + \dots + A_k(r) + 2 + A_{k+1}(r)^{-1} - 1 \right)$$

We look for estimate of R such that $g'_{k+1}(r) < 0$.

Hence from (3.10) we look for an estimate of R such that

$$(3.11) \quad -A_1(r) \dots A_k(r) + A_2(r) \dots A_k(r) + \dots + A_k(r) + 2 < 0 \quad \text{and} \\ A_{k+1}(r)^{-1} - 1 < 0.$$

But from (3.11) we have

$$(3.12) \quad -A_1(r) \dots A_{k-1}(r) + A_2(r) \dots A_{k-1}(r) + \dots + A_{k-1}(r) + 3 \\ + 2A_k(r)^{-1} - 2 < 0.$$

Hence from (3.11) and (3.12) we need to show that

$$(3.13) \quad -A_1(r) \dots A_{k-1}(r) + A_2(r) \dots A_{k-1}(r) + \dots + A_{k-1}(r) + 3 < 0 \quad \text{and} \\ 2A_k(r)^{-1} - 2 < 0.$$

Continuing this process $k + 1$ -times, we have for $g'_{k+1}(r) < 0$, we need to show the following:

$$A_{k+1}(r)^{-1} - 1 < 0, \quad 2A_k(r)^{-1} - 2 < 0, \dots, kA_2(r)^{-1} - k < 0, \\ \text{and} \quad -A_1(r) + (k + 1) < 0.$$

Hence we have the following estimate for R respectively:

$$R > re_{k+1}, \quad R > re_k, \dots, R > re_2, \quad \text{and} \quad R > re^{k+1}.$$

Since $e_{k+1} > e_k > \dots > e_2$ and $e_{k+1} > e^{k+1}$ then we have for $R > re_{k+1}$, $g'_{k+1}(r) < 0$. Thus $g_{k+1}(r)$ is a decreasing function. Hence the lemma follows. \square

4. Proof of the Theorem 2.1.

In this section, we are going to give the proof of Theorem 2.1. We organize the proof in the following way: First we shall prove by using inductive argument inequality (2.1), then again by inductive argument we will also show the sharpness of $\frac{1}{4}$, and optimality of the exponent 2 of the weight function on (2.1).

We shall first prove inequality (2.1) for smooth positive radially nonincreasing function defined on unit ball B_1 , centered at the origin. Then for $u \in C_0^2(B_1)$, $u > 0$, radially nonincreasing, inequality (2.1) follows from Remark 3.1.

Now by density arguments, inequality (2.1) is valid for any $u \in W_0^{1,2}(B)$, $u \geq 0$, and radially nonincreasing. For a general domain Ω , we use symmetrization arguments. Let Ω^* be a ball having the same volume as Ω and let $|u|^*$ be the symmetric decreasing rearrangement of the function $|u|$. Now observe that, for any $u \in W_0^{1,2}(\Omega)$, $|u|^* \in W_0^{1,2}(\Omega^*)$ and $|u|^* > 0$ and radially nonincreasing and hence inequality (2.1) holds for $|u|^*$. It is well known that the symmetrization does not change the L^p -norm, decreases gradient norm and increases the integrals

$$\int_{\Omega} \frac{|u(x)|^2}{|x|^2} dx, \quad \int_{\Omega} \frac{|u(x)|^2}{|x|^2} A_1(|x|)^{-2} dx, \quad \int_{\Omega} \frac{|u(x)|^2}{|x|^2} (A_1(|x|)A_2(|x|))^{-2} dx, \dots, \\ \int_{\Omega} \frac{|u(x)|^2}{|x|^2} (A_1(|x|)A_2(|x|) \dots A_k(|x|))^{-2} dx$$

since the singular weights are decreasing functions of $|x|$ under our assumption on R (see Lemma 3.2). Hence inequality (2.1) also holds for any $u \in W_0^{1,2}(\Omega)$.

We construct a test function which we will use to show the sharpness of $\frac{1}{4}$ and optimality of the exponent 2 of the weight function in (2.1). For $k \geq 1$, $\alpha > 0$ we set

$$(4.1) \quad z_k(r) = \begin{cases} A_k(r)^{-\frac{\alpha}{2}}, & 0 < r \leq \frac{1}{R} \\ A_k(\frac{1}{R})^{-\frac{\alpha}{2}} \frac{\log r}{\log \frac{1}{R}}, & \frac{1}{R} < r < 1. \end{cases}$$

For $0 < r \leq \frac{1}{R}$ we let

$$(4.2) \quad w_{k,\epsilon}(r) = \frac{\alpha}{2} \left(A_1(r) \dots A_{k-1}(r) \right)^{-1} A_k(r)^{\frac{-\alpha}{2}-1} \frac{r}{(r + \epsilon)^2}$$

$$\begin{aligned}
(4.3) \quad \tilde{w}_{k,\epsilon}(r) &= \frac{\alpha}{2} \int_0^r \left(A_1(\rho) \dots A_{k-1}(\rho) \right)^{-1} A_k(\rho)^{-\frac{\alpha}{2}-1} \frac{\rho}{(\rho+\epsilon)^2} d\rho \\
&= A_k(r)^{-\frac{\alpha}{2}} \frac{r^2}{(r+\epsilon)^2} - 2 \int_0^r A_k(\rho)^{-\frac{\alpha}{2}} \frac{\rho\epsilon}{(\rho+\epsilon)^3} d\rho.
\end{aligned}$$

We note that as $\epsilon \rightarrow 0$, $w_{k,\epsilon}(r) = z'_k(r)$ and $\tilde{w}_{k,\epsilon}(r) = z_k(r)$ on $0 < r \leq \frac{1}{R}$.

Sharpness of $\frac{1}{4}$.

We shall prove using inductive argument that $\frac{1}{4}$ is sharp coefficient for every missing terms. From (3.1) for $k = 1$ we have

$$(4.4) \quad \frac{\int_{B_1} |\nabla u(x)|^2 dx - \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_1(r)^2 A_1(r) \frac{dr}{r}}{\omega_n \int_0^1 v_1(r)^2 A_1(r)^{-1} \frac{dr}{r}} = \frac{1}{4} + \frac{\int_0^1 v_1'(r)^2 A_1(r) r dr}{\int_0^1 v_1(r)^2 A_1(r)^{-1} \frac{dr}{r}}.$$

Then using the test function defined in (4.1) we get

$$(4.5) \quad \int_0^{\frac{1}{R}} z_1'(r)^2 A_1(r) r dr = \frac{\alpha^2}{4} \int_0^{\frac{1}{R}} A_1(r)^{-\alpha-1} \frac{dr}{r} = \frac{\alpha}{4} A_1\left(\frac{1}{R}\right)^{-\alpha}$$

and

$$(4.6) \quad \int_0^{\frac{1}{R}} z_1(r)^2 A_1(r)^{-1} \frac{dr}{r} = \int_0^{\frac{1}{R}} A_1(r)^{-\alpha-1} \frac{dr}{r} = \frac{1}{\alpha} A_1\left(\frac{1}{R}\right)^{-\alpha}.$$

It is easy to verify that the integrals $\int_{\frac{1}{R}}^1 z_1'(r)^2 A_1(r) r dr$ and $\int_{\frac{1}{R}}^1 z_1(r)^2 A_1(r)^{-1} \frac{dr}{r}$ are finite. Hence from (4.5) and (4.6) the ratio

$$\frac{\int_0^1 z_1'(r)^2 A_1(r) r dr}{\int_0^1 z_1(r)^2 A_1(r)^{-1} \frac{dr}{r}} \rightarrow 0$$

as $\alpha \rightarrow 0$. Thus $\frac{1}{4}$ is sharp in inequality (2.1) for $k = 1$.

By inductive argument we can assume that $\frac{1}{4}$ is sharp coefficient for every k -missing terms. Then from (3.1) for $k + 1$ we have

$$(4.7) \quad \frac{\int_{B_1} |\nabla u(x)|^2 dx - L(v_{k+1}(r)) - J(v_{k+1}(r))}{\omega_n \int_0^1 v_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \right)^{-1} \frac{dr}{r}} = \frac{1}{4} + I(v_{k+1}(r))$$

where

(4.8)

$$\begin{aligned}
L(v_{k+1}(r)) &= \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \frac{dr}{r} \\
J(v_{k+1}(r)) &= \frac{\omega_n}{4} \int_0^1 v_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \left[A_1(r)^{-2} + \right. \\
&\quad \left. \left(A_1(r) A_2(r) \right)^{-2} + \dots + \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-2} \right] \frac{dr}{r} \\
I(v_{k+1}(r)) &= \frac{\int_0^1 v'_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) r dr}{\int_0^1 v_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \right)^{-1} \frac{dr}{r}}.
\end{aligned}$$

The sharpness of $\frac{1}{4}$ for $k+1$ missing term will follow if we can show that

$$\inf_{v_{k+1} \in C_0^1((0,1))} I(v_{k+1}) = 0.$$

Then from (4.2) we have

$$\begin{aligned}
(4.9) \quad & \int_0^{\frac{1}{R}} w_{k+1,\epsilon}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) r dr \\
&= \frac{\alpha^2}{4} \int_0^{\frac{1}{R}} \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{-\alpha-1} \frac{r^3}{(r+\epsilon)^4} dr \\
&\leq \frac{\alpha^2}{4} \int_0^{\frac{1}{R}} \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{-\alpha-1} \frac{dr}{r} < \infty
\end{aligned}$$

Then applying *Lebesgue dominated convergence theorem* to (4.9) as $\epsilon \rightarrow 0$ we get

$$\begin{aligned}
(4.10) \quad & \int_0^{\frac{1}{R}} z'_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) r dr \\
&= \frac{\alpha^2}{4} \int_0^{\frac{1}{R}} \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{-\alpha-1} \frac{dr}{r} \\
&= \frac{\alpha}{4} A_{k+1}\left(\frac{1}{R}\right)^{-\alpha}.
\end{aligned}$$

Also from (4.3) we have

$$\begin{aligned}
(4.11) \quad & \int_0^{\frac{1}{R}} \tilde{w}_{k+1,\epsilon}(r)^2 \left(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) \right)^{-1} \frac{dr}{r} \\
&= \int_0^{\frac{1}{R}} \left(A_{k+1}(r)^{-\frac{\alpha}{2}} \frac{r^2}{(r+\epsilon)^2} - 2 \int_0^r A_{k+1}(\rho)^{-\frac{\alpha}{2}} \frac{\rho\epsilon}{(\rho+\epsilon)^3} d\rho \right)^2 \times \\
&\quad \left(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) \right)^{-1} \frac{dr}{r} \\
&\leq 2 \int_0^{\frac{1}{R}} \left(A_1(r)A_2(r)\dots A_k(r) \right)^{-1} A_{k+1}(r)^{-\alpha-1} \frac{r^3}{(r+\epsilon)^4} dr \\
&+ 4 \int_0^{\frac{1}{R}} \left(\int_0^r A_{k+1}(\rho)^{-\frac{\alpha}{2}} \frac{\rho\epsilon}{(\rho+\epsilon)^3} d\rho \right)^2 \left(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) \right)^{-1} \frac{dr}{r} \\
&\leq 2 \int_0^{\frac{1}{R}} \left(A_1(r)A_2(r)\dots A_k(r) \right)^{-1} A_{k+1}(r)^{-\alpha-1} \frac{dr}{r} \\
&+ 4 \int_0^{\frac{1}{R}} \left(\int_0^\infty \frac{t}{(t+1)^3} dt \right)^2 \left(A_1(r)A_2(r)\dots A_k(r) \right)^{-1} A_{k+1}(r)^{-\alpha-1} \frac{dr}{r} \\
&< \infty.
\end{aligned}$$

Again by applying *Lebesgue dominated convergence theorem* to (4.11) the second term in the right hand side of (4.11) vanish since the integral

$$\int_0^r A_{k+1}(\rho)^{-\frac{\alpha}{2}} \frac{\rho\epsilon}{(\rho+\epsilon)^3} d\rho = \int_0^{\frac{r}{\epsilon}} A_{k+1}(\epsilon t)^{-\frac{\alpha}{2}} \frac{t}{(t+1)^3} dt \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

hence we get

$$\begin{aligned}
(4.12) \quad & \int_0^{\frac{1}{R}} z_{k+1}(r)^2 \left(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) \right)^{-1} \frac{dr}{r} \\
&= 2 \int_0^{\frac{1}{R}} \left(A_1(r)A_2(r)\dots A_k(r) \right)^{-1} A_{k+1}(r)^{-\alpha-1} \frac{dr}{r} \\
&= \frac{2}{\alpha} A_{k+1}\left(\frac{1}{R}\right)^{-\alpha}.
\end{aligned}$$

Since the integrals

$$\begin{aligned}
& \int_{\frac{1}{R}}^1 z'_{k+1}(r)^2 A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r)r dr \\
& \int_{\frac{1}{R}}^1 z_{k+1}(r)^2 \left(A_1(r)A_2(r)\dots A_k(r)A_{k+1}(r) \right)^{-1} \frac{dr}{r}
\end{aligned}$$

are finite, then for a given test function $z_{k+1}(r)$ defined in (4.1), $I(z_{k+1}(r)) \rightarrow 0$ as $\alpha \rightarrow 0$.

Thus sharpness of $\frac{1}{4}$ for $k+1$ missing term follow. Hence $\frac{1}{4}$ is sharp coefficient for all k -missing terms, $k \geq 1$.

Optimality of the exponent 2.

We shall also prove the optimality of the exponent 2 of the weight function using inductive argument. Assume $0 \leq \gamma < 2$. Then for $k = 1$ optimality of γ will follow if we can show that

$$\inf_{v_1(r) \in C_0^1((0,1))} I_\gamma(v_1(r)) = 0.$$

where

$$I_\gamma(v_1(r)) = \frac{\int_{B_1} |\nabla u(x)|^2 dx - \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_1(r)^2 A_1(r) \frac{dr}{r}}{\omega_n \int_0^1 v_1(r)^2 A_1(r)^{1-\gamma} \frac{dr}{r}}.$$

But from (3.1) for $k = 1$ we have

$$I_\gamma(v_1(r)) = \frac{\frac{1}{4} \int_0^1 v_1(r)^2 A_1(r)^{-1} \frac{dr}{r} + \int_0^1 v_1'(r)^2 A_1(r) r dr}{\int_0^1 v_1(r)^2 A_1(r)^{1-\gamma} \frac{dr}{r}}.$$

Then using the test function defined in (4.1) we get

$$(4.13) \quad \int_0^{\frac{1}{R}} z_1(r)^2 A_1(r)^{1-\gamma} \frac{dr}{r} = \int_0^{\frac{1}{R}} A_1(r)^{1-\gamma-\alpha} \frac{dr}{r} = \frac{1}{\alpha + \gamma - 2} A_1\left(\frac{1}{R}\right)^{-\alpha-\gamma+2}.$$

It is easy to verify that the integrals

$$\int_{\frac{1}{R}}^1 z_1'(r)^2 A_1(r) r dr, \quad \int_{\frac{1}{R}}^1 z_1(r)^2 A_1(r)^{-1} \frac{dr}{r}, \quad \text{and} \quad \int_{\frac{1}{R}}^1 z_1(r)^2 A_1(r)^{1-\gamma} \frac{dr}{r}$$

are finite. Also from (4.5) and (4.6) the integrals

$$\int_0^{\frac{1}{R}} z_1'(r)^2 A_1(r) r dr \quad \text{and} \quad \int_0^{\frac{1}{R}} z_1(r)^2 A_1(r)^{-1} \frac{dr}{r}$$

are finite respectively for $\alpha > 0$. Since $0 \leq \gamma < 2$, then from (4.13) we have for a given test function $z_1(r)$ defined in (4.1), $I_\gamma(z_1(r)) \rightarrow 0$ as $\alpha \rightarrow 2 - \gamma > 0$. Hence no inequality of type (3.7) for $k = 1$ can hold. Therefore $\gamma \geq 2$ and 2 is optimal.

Also by inductive argument we can assume that $\gamma \geq 2$ for the singular weight $A_k(|x|)^{-\gamma}$ of k -missing term. Then from (3.1), the $k+1$ -missing term is of form $\frac{\omega_n}{4} \int_0^1 v_{k+1}(r)^2 \left(A_1(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r}$. Hence from (3.1) for $k+1$ we have

$$(4.14) \quad \frac{\int_{B_1} |\nabla u(x)|^2 dx - L(v_{k+1}(r)) - J(v_{k+1}(r))}{\omega_n \int_0^1 v_{k+1}(r)^2 \left(A_1(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r}} = I_\gamma(v_{k+1}(r))$$

where $L(v_{k+1}(r))$ and $J(v_{k+1}(r))$ are defined in (4), and

$$(4.15) \quad I_\gamma(v_{k+1}(r)) = \frac{\frac{1}{4}P(v_{k+1}(r)) + Q(v_{k+1}(r))}{\omega_n \int_0^1 v_{k+1}(r)^2 \left(A_1(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r}}$$

$$(4.16) \quad \begin{aligned} P(v_{k+1}(r)) &= \int_0^1 v_{k+1}(r)^2 \left(A_1(r)A_2(r) \dots A_k(r)A_{k+1}(r) \right)^{-1} \frac{dr}{r} \\ Q(v_{k+1}(r)) &= \int_0^1 v'_{k+1}(r)^2 A_1(r)A_2(r) \dots A_k(r)A_{k+1}(r)r dr \end{aligned}$$

Also from (4.3) we have

$$(4.17) \quad \begin{aligned} & \int_0^{\frac{1}{R}} \tilde{w}_{k+1,\epsilon}(r)^2 \left(A_1(r)A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r} \\ &= \int_0^{\frac{1}{R}} \left(A_{k+1}(r)^{-\frac{\alpha}{2}} \frac{r^2}{(r+\epsilon)^2} - 2 \int_0^r A_{k+1}(\rho)^{-\frac{\alpha}{2}} \frac{\rho\epsilon}{(\rho+\epsilon)^3} d\rho \right)^2 \times \\ & \quad \left(A_1(r)A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r} \\ &\leq 2 \int_0^{\frac{1}{R}} \left(A_1(r)A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\alpha-\gamma} \frac{r^3}{(r+\epsilon)^4} dr \\ &+ 4 \int_0^{\frac{1}{R}} \left(\int_0^r A_{k+1}(\rho)^{-\frac{\alpha}{2}} \frac{\rho\epsilon}{(\rho+\epsilon)^3} d\rho \right)^2 \left(A_1(r)A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r} \\ &\leq 2 \int_0^{\frac{1}{R}} \left(A_1(r)A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\alpha-\gamma} \frac{dr}{r} \\ &+ 4 \int_0^{\frac{1}{R}} \left(\int_0^{\frac{r}{\epsilon}} \frac{t}{(t+1)^3} dt \right)^2 \left(A_1(r)A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\alpha-\gamma} \frac{dr}{r} \\ &< \infty. \end{aligned}$$

Again by applying *Lebesgue dominated convergence theorem* to (4.17) the second term in the right hand side of (4.17) vanish since the integral

$$\int_0^r A_{k+1}(\rho)^{-\frac{\alpha}{2}} \frac{\rho \epsilon}{(\rho + \epsilon)^3} d\rho = \int_0^{\frac{r}{\epsilon}} A_{k+1}(\epsilon t)^{-\frac{\alpha}{2}} \frac{t}{(t+1)^3} dt \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

hence we get

$$\begin{aligned} (4.18) \quad & \int_0^{\frac{1}{R}} z_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r} \\ &= 2 \int_0^{\frac{1}{R}} \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\alpha-\gamma} \frac{dr}{r} \\ &= \frac{2}{\alpha + \gamma - 2} A_{k+1} \left(\frac{1}{R} \right)^{2-\alpha-\gamma}. \end{aligned}$$

By (4.10) and (4.12) we see that the integrals

$$\begin{aligned} & \int_0^{\frac{1}{R}} z'_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) r dr \\ & \int_0^{\frac{1}{R}} z_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \right)^{-1} \frac{dr}{r} \end{aligned}$$

are finite respectively for $\alpha > 0$.

It is also easy to verify that the integrals

$$\begin{aligned} & \int_{\frac{1}{R}}^1 z'_{k+1}(r)^2 A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) r dr \\ & \int_{\frac{1}{R}}^1 z_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) A_{k+1}(r) \right)^{-1} \frac{dr}{r} \\ & \int_{\frac{1}{R}}^1 z_{k+1}(r)^2 \left(A_1(r) A_2(r) \dots A_k(r) \right)^{-1} A_{k+1}(r)^{1-\gamma} \frac{dr}{r} \end{aligned}$$

are finite.

Then from (4.18) and given $0 \leq \gamma < 2$ we have $I_\gamma(z_{k+1}(r)) \rightarrow 0$ as $\alpha \rightarrow 2 - \gamma > 0$. Hence no inequality having $k + 1$ -missing terms of type (2.1) can hold. Thus $\gamma \geq 2$ and 2 is optimal for the singular weight $A_{k+1}(|x|)^{-\gamma}$ of $k + 1$ -missing term. Therefore the optimality of the exponent 2 of the weight function in inequality (2.1) follows. \square

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