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The index and co-index of the twisted tangent bundle over projective spaces

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ABSTRACT. We determine the index and co-index of the twisted tangent bundle of projective spaces. We also discuss the stabilty of them, and determine the set of integers that can be realized as the stable co-index of a vector bundle over the projective space.

1. Introduction.

Let α be a finite-dimensional real vector bundle over a CW complex B, and let $S(\alpha)$ be its sphere bundle with respect to some metric on α. We regard $S(\alpha)$ as a $\mathbb{Z}/2$-space by the antipodal map on each fibre. Then, the index of α, denoted $\operatorname{ind}\alpha$, is defined to be the largest integer k for which there exists a $\mathbb{Z}/2$-map from S^{k-1} to $S(\alpha)$. Here, S^{k-1} also is equipped with a $\mathbb{Z}/2$ -action by the antipodal map. As a dual notion to the index, the co-index of α, denoted co-ind α, is defined to be the smallest integer k for which there exists a $\mathbb{Z}/2$-map from $S(\alpha)$ to S^{k-1}. If there exist no such maps, we define co-ind α as ∞. When B is a finite complex, co-ind α is finite for any α. By the Borsuk-Ulam theorem, $\operatorname{ind}\alpha$ is not greater than co-ind α.

In this paper, we calculate the index and co-index of the twisted tangent bundle of the real projective space $\mathbb{R}P^n$. Let τ denote the tangent bundle of $\mathbb{R}P^n$, and let ξ denote the canonical line bundle over $\mathbb{R}P^n$. We call $\tau \otimes \xi$ the twisted tangent bundle of $\mathbb{R}P^n$. For the tangent bundle τ, we already know that $\operatorname{ind}\tau = n$, since the index of the tangent bundle of any closed manifold is equal to its dimension [CF1, Theorem 6.11]. As for the co-index of τ, it is known that co-ind τ is equal to $l(n)$ $(n \neq 1,3,7)$ or $l(n)-1$ $(n = 1,3,7)$, where $l(n)$ denotes the smallest integer m such that $\mathbb{R}P^n$ can be immersed in $\mathbb{R}^m$ [AGJ, T1].

We shall prove the following results in which the k-dimensional trivial bundle is denoted simply by k.

Theorem 1.1. *For $\tau \otimes \xi$, the twisted tangent bundle of $\mathbb{R}P^n$, we have* co-ind $(\tau \otimes \xi + k) = n + 1 + k$ *for any $k \geq 0$.*

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Theorem 1.2. *For $\tau \otimes \xi$, the twisted tangent bundle of $\mathbb{R}P^n$, we have the following.*

(1) *If n is odd, then* $\operatorname{ind}(\tau \otimes \xi) = n+1$ *and* $\operatorname{ind}(\tau \otimes \xi + k) = n+k$ *for* $k \geq 1$.

(2) *If n is even, then* $\operatorname{ind}(\tau \otimes \xi + k) = n+k$ *for any* $k \geq 0$.

For a vector bundle α, in general, we call α to be *CI-stable* if the equality $\operatorname{co-ind}(\alpha + k) = \operatorname{co-ind} \alpha + k$ holds for any positive integer k. Likewise we call α to be *I-stable* if $\operatorname{ind}(\alpha + k) = \operatorname{ind} \alpha + k$ for any positive integer k. We have the following corollaries.

Corollary 1.3. *$\tau \otimes \xi$ is CI-stable.*

Corollary 1.4.

(1) *If n is odd, then $\tau \otimes \xi$ is not I-stable but $\tau \otimes \xi + 1$ turns to be I-stable.*

(2) *If n is even, then $\tau \otimes \xi$ is I-stable.*

As for τ, the tangent bundle of $\mathbb{R}P^n$, it is not known whether τ is CI-stable, though the equality $\operatorname{co-ind}(\tau + 1) = \operatorname{co-ind} \tau + 1$ holds as shown in [T1, Theorem 3.7]. On the other hand, τ is I-stable, since the tangent bundle of any closed manifold is known to be I-stable [T2, Theorem 4.6]. Also, it is to be noted that ξ is not I-stable, but CI-stable [T1, Theorem 4.1].

For α, a vector bundle over a finite complex B, we can define the stable co-index of α, denoted $\operatorname{co-ind}^s(\alpha)$, as

$$\operatorname{co-ind}^s(\alpha) = \varinjlim_k \{\operatorname{co-ind}(\alpha + k) - \dim(\alpha + k)\}.$$

This limit always exists since $0 \leq \operatorname{co-ind}(\alpha+1) - \dim(\alpha+1) \leq \operatorname{co-ind} \alpha - \dim \alpha$ (see [T1, Lemma 3.1]). Likewise we define the stable index $\operatorname{ind}^s(\alpha)$. For example, if α is stably trivial, then, $\operatorname{co-ind}^s(\alpha) = \operatorname{ind}^s(\alpha) = 0$. Also, we know that $\operatorname{co-ind}^s(\xi) = n$ and $\operatorname{ind}^s(\xi) = 0$ from [T1, Theorem 4.1]. The above theorems imply $\operatorname{co-ind}^s(\tau \otimes \xi) = 1$ and $\operatorname{ind}^s(\tau \otimes \xi) = 0$. For the m-fold sum of the twisted tangent bundle $\tau \otimes \xi$, we shall prove the following theorem.

Theorem 1.5. *If $m \leq n$, then* $\operatorname{co-ind}^s(m \cdot \tau \otimes \xi) = m$.

If α is stably equivalent to α', it is clear that $\operatorname{co-ind}^s(\alpha) = \operatorname{co-ind}^s(\alpha')$. So, $\operatorname{co-ind}^s$ can be considered as a function from $\widetilde{KO}(B)$ to $\mathbb{Z}^+$, the set of non-negative integers. Since $\operatorname{co-ind} \alpha \leq \dim \alpha + \dim B$ ([T1, Lemma 3.1]), we have $\operatorname{co-ind}^s(\alpha) \leq \dim B$ in general. Thus, $\dim B$ is an upper bound of the image of the function $\operatorname{co-ind}^s$. From Theorem 1.5, we obtain the following corollary.

Corollary 1.6. *The image of the function* $\operatorname{co-ind}^s : \widetilde{KO}(\mathbb{R}P^n) \longrightarrow \mathbb{Z}^+$ *is equal to the set* $\{0, 1, 2, \cdots, n\}$.

Thus, over $\mathbb{R}P^n$, any non-negative integer not greater than $\dim \mathbb{R}P^n$ can be realized as the stable co-index of a vector bundle.

2. Proof of the theorems.

Let B be a finite complex and α be a real vector bundle over B. We define the ν-dimension of α to be the smallest integer k for which there is a k-dimensional vector bundle β with $\alpha+\beta$ trivial, and we denote it by ν-dim α. The ν-dimension of α is closely related to the geometric dimension of $-\alpha$, but they slightly differ in general (see examples in [T2]). The inequalities g-dim$(-\alpha) \leq \nu$-dim $\alpha \leq \dim B$ hold generally. We easily see that co-ind $\alpha \leq \dim \alpha + \nu$-dim α.

For a virtual vector bundle α, we denote by ω-dim α the largest integer k for which the kth Stiefel-Whitney class $w_k(\alpha)$ is not zero. Then it is known that co-ind $\alpha \geq \dim \alpha + \omega$-dim$(-\alpha)$ and the following theorem holds.

Theorem 2.1([T2, Theorem 2.5]). *Let α be a real vector bundle over a finite complex B, and suppose that the equality ω-dim$(-\alpha) = \nu$-dim α holds. Then,*
(1) co-ind $\alpha = \dim \alpha + \nu$-dim α.
(2) *α is CI-stable.*

Now, we prove Theorem 1.1 and Theorem 1.5.

Proof of Theorem 1.1 and 1.5.
Since $\tau + 1 \cong (n+1)\xi$, we have $\tau \otimes \xi + \xi \cong n+1$. Hence, ν-dim$(\tau \otimes \xi) \leq 1$. We also have $W(-\tau \otimes \xi) = W(\xi) = 1 + t$ where W is the total Stiefel-Whitney class and t is the generator of $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. Hence, ω-dim$(-\tau \otimes \xi) = 1$. Since ω-dim$(-\alpha) \leq \nu$-dim α in general, we have ω-dim$(-\tau \otimes \xi) = \nu$-dim$(\tau \otimes \xi) = 1$. Therefore, from Theorem 2.1, we find that co-ind$(\tau \otimes \xi) = n+1$, and also, $\tau \otimes \xi$ is CI-stable. Thus Theorem 1.1 follows. Our argument goes as well for the m-fold sum of the twisted tangent bundle $\tau \otimes \xi$. In fact, since $m \cdot \tau \otimes \xi + m\xi \cong m(n+1)$, we have ν-dim$(m \cdot \tau \otimes \xi) \leq m$. On the other hand, since $W(-m \cdot \tau \otimes \xi) = W(m\xi) = (1+t)^m$, we have ω-dim$(-m \cdot \tau \otimes \xi) = m$ if $m \leq n$. Thus, if $m \leq n$, we see that ω-dim$(-m \cdot \tau \otimes \xi) = \nu$-dim$(m \cdot \tau \otimes \xi) = m$, and, from Theorem 2.1, it follows that $m \cdot \tau \otimes \xi$ is CI-stable and co-ind$^s(m \cdot \tau \otimes \xi) = m$. □

Next, we turn to the index. To prove Theorem 1.2, we use the following result in [T1].

Proposition 2.2([T1, Corollary 2.5]). *Let α be a real vector bundle over $\mathbb{R}P^n$.*
(1) *If* $\dim \alpha > n$, *then* ind $\alpha = \dim \alpha$.
(2) *If* $\dim \alpha \leq n$ *and α contains ξ as a subbundle, then* ind $\alpha = n+1$.

Proof of Theorem 1.2.
In view of Proposition 2.2 (1), we have only to prove that ind$(\tau \otimes \xi)$ is equal to $n+1$ if n is odd, and n if n is even. Since ind $\alpha \leq$ co-ind α in general, we have ind$(\tau \otimes \xi) \leq n+1$ from Theorem 1.1. Also, since ind $\alpha \geq \dim \alpha$ in general, we clearly have ind$(\tau \otimes \xi) \geq n$. Now, suppose that there is a $\mathbb{Z}/2$-map $f : S^n \longrightarrow S(\tau \otimes \xi)$. Let $\tilde{f} : \mathbb{R}P^n \longrightarrow P(\tau \otimes \xi)$ be the map induced from f. Here, $P(\tau \otimes \xi)$ is the projective bundle associated to $\tau \otimes \xi$. Let $p : P(\tau \otimes \xi) \longrightarrow \mathbb{R}P^n$ be the bundle projection. Let e denote the $\mathbb{Z}/2$-Euler class of the bundle $S(\tau \otimes \xi) \longrightarrow P(\tau \otimes \xi)$. We consider the homomorphism $\tilde{f}^* : H^*(P(\tau \otimes \xi); \mathbb{Z}/2) \longrightarrow H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. From the naturality of the Euler class, we have $\tilde{f}^*(e) = t$, and so, $\tilde{f}^*(e^n) = t^n$, which is not zero. On the other hand, in $H^*(P(\tau \otimes \xi); \mathbb{Z}/2)$, we have the relation

$$e^n = w_n + w_{n-1}e + w_{n-2}e^2 + \cdots + w_1 e^{n-1},$$

where w_i's are the Stiefel-Whitney classes of $\tau \otimes \xi$ and $H^*(P(\tau \otimes \xi); \mathbb{Z}/2)$ is considered as a $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$-module in an obvious way. Since $\tau \otimes \xi + \xi \cong n+1$, we have $W(\tau \otimes \xi) = W(\xi)^{-1} = (1+t)^{-1}$. Hence, we have the relation

$$e^n = t^n + t^{n-1}e + t^{n-2}e^2 + \cdots + te^{n-1}.$$

Applying $\tilde{f}^*$, we have

$$t^n = (p \circ \tilde{f})^*(t^n) + (p \circ \tilde{f})^*(t^{n-1})t + (p \circ \tilde{f})^*(t^{n-2})t^2 + \cdots + (p \circ \tilde{f})^*(t)t^{n-1}.$$

But the homomorphism $(p \circ \tilde{f})^* : H^*(\mathbb{R}P^n; \mathbb{Z}/2) \longrightarrow H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ is either identity or zero. The latter case is impossible since t^n is not zero. Therefore, we have $t^n = nt^n$. If n is even, this contradicts $t^n \neq 0$. Thus we have proved that $\mathrm{ind}(\tau \otimes \xi)$ is equal to n if n is even.

Now, let n be odd. Then τ has a non-zero cross-section since it is orientable and its Euler class is zero. In other words, τ has a one-dimensional trivial bundle as a subbundle. Hence $\tau \otimes \xi$ has ξ as a subbundle. From Proposition 2.2 (2), it follows that $\mathrm{ind}(\tau \otimes \xi) = n+1$. This completes the proof. □

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