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<th>Title</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>Mathematical Journal of Ibaraki University, 37: 35-38</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10109/3104">http://hdl.handle.net/10109/3104</a></td>
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The index and co-index of the twisted tangent bundle over projective spaces

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ABSTRACT. We determine the index and co-index of the twisted tangent bundle of projective spaces. We also discuss the stability of them, and determine the set of integers that can be realized as the stable co-index of a vector bundle over the projective space.

1. Introduction.

Let \( \alpha \) be a finite-dimensional real vector bundle over a CW complex \( B \), and let \( S(\alpha) \) be its sphere bundle with respect to some metric on \( \alpha \). We regard \( S(\alpha) \) as a \( \mathbb{Z}/2 \)-space by the antipodal map on each fibre. Then, the index of \( \alpha \), denoted \( \text{ind} \alpha \), is defined to be the largest integer \( k \) for which there exists a \( \mathbb{Z}/2 \)-map from \( S^{k-1} \) to \( S(\alpha) \). Here, \( S^{k-1} \) also is equipped with a \( \mathbb{Z}/2 \)-action by the antipodal map. As a dual notion to the index, the co-index of \( \alpha \), denoted \( \text{co-ind} \alpha \), is defined to be the smallest integer \( k \) for which there exists a \( \mathbb{Z}/2 \)-map from \( S(\alpha) \) to \( S^{k-1} \). If there exist no such maps, we define \( \text{co-ind} \alpha \) as \( \infty \). When \( B \) is a finite complex, \( \text{co-ind} \alpha \) is finite for any \( \alpha \). By the Borsuk-Ulam theorem, \( \text{ind} \alpha \) is not greater than \( \text{co-ind} \alpha \).

In this paper, we calculate the index and co-index of the twisted tangent bundle of the real projective space \( \mathbb{RP}^n \). Let \( \tau \) denote the tangent bundle of \( \mathbb{RP}^n \), and let \( \xi \) denote the canonical line bundle over \( \mathbb{RP}^n \). We call \( \tau \otimes \xi \) the twisted tangent bundle of \( \mathbb{RP}^n \). For the tangent bundle \( \tau \), we already know that \( \text{ind} \tau = n \), since the index of the tangent bundle of any closed manifold is equal to its dimension [CF1, Theorem 6.11]. As for the co-index of \( \tau \), it is known that \( \text{co-ind} \tau \) is equal to \( l(n) \) (\( n \neq 1, 3, 7 \)) or \( l(n) - 1 \) (\( n = 1, 3, 7 \)), where \( l(n) \) denotes the smallest integer \( m \) such that \( \mathbb{RP}^n \) can be immersed in \( \mathbb{R}^m \) [AGJ, T1].

We shall prove the following results in which the \( k \)-dimensional trivial bundle is denoted simply by \( k \).

**Theorem 1.1.** For \( \tau \otimes \xi \), the twisted tangent bundle of \( \mathbb{RP}^n \), we have \( \text{co-ind} (\tau \otimes \xi + k) = n + 1 + k \) for any \( k \geq 0 \).
Theorem 1.2. For $\tau \otimes \xi$, the twisted tangent bundle of $\mathbb{R}P^n$, we have the following.

1. If $n$ is odd, then $\text{ind}(\tau \otimes \xi) = n + 1$ and $\text{ind}(\tau \otimes \xi + k) = n + k$ for $k \geq 1$.
2. If $n$ is even, then $\text{ind}(\tau \otimes \xi + k) = n + k$ for any $k \geq 0$.

For a vector bundle $\alpha$, in general, we call $\alpha$ to be CI-stable if the equality $\text{co-ind}(\alpha + k) = \text{co-ind} \alpha + k$ holds for any positive integer $k$. Likewise, we call $\alpha$ to be I-stable if $\text{ind}(\alpha + k) = \text{ind} \alpha + k$ for any positive integer $k$. We have the following corollaries.

Corollary 1.3. $\tau \otimes \xi$ is CI-stable.

Corollary 1.4.

1. If $n$ is odd, then $\tau \otimes \xi$ is not I-stable but $\tau \otimes \xi + 1$ turns to be I-stable.
2. If $n$ is even, then $\tau \otimes \xi$ is I-stable.

As for $\tau$, the tangent bundle of $\mathbb{R}P^n$, it is not known whether $\tau$ is CI-stable, though the equality $\text{co-ind}(\tau + 1) = \text{co-ind} \tau + 1$ holds as shown in [T1, Theorem 3.7]. On the other hand, $\tau$ is I-stable, since the tangent bundle of any closed manifold is known to be I-stable [T2, Theorem 4.6]. Also, it is to be noted that $\xi$ is not I-stable, but CI-stable [T1, Theorem 4.1].

For $\alpha$, a vector bundle over a finite complex $B$, we can define the stable co-index of $\alpha$, denoted $\text{co-ind}^s(\alpha)$, as

$$
\text{co-ind}^s(\alpha) = \lim_{k} \{\text{co-ind}(\alpha + k) - \dim(\alpha + k)\}.
$$

This limit always exists since $0 \leq \text{co-ind}(\alpha + 1) - \dim(\alpha + 1) \leq \text{co-ind} \alpha - \dim \alpha$ (see [T1, Lemma 3.1]). Likewise, we define the stable index $\text{inds}(\alpha)$. For example, if $\alpha$ is stably trivial, then, $\text{co-ind}^s(\alpha) = \text{inds}(\alpha) = 0$. Also, we know that $\text{co-ind}^s(\xi) = n$ and $\text{inds}(\xi) = 0$ from [T1, Theorem 4.1]. The above theorems imply $\text{co-ind}^s(\tau \otimes \xi) = 1$ and $\text{inds}(\tau \otimes \xi) = 0$. For the $m$-fold sum of the twisted tangent bundle $\tau \otimes \xi$, we shall prove the following theorem.

Theorem 1.5. If $m \leq n$, then $\text{co-ind}^s(m \cdot \tau \otimes \xi) = m$.

If $\alpha$ is stably equivalent to $\alpha'$, it is clear that $\text{co-ind}^s(\alpha) = \text{co-ind}^s(\alpha')$. So, $\text{co-ind}^s$ can be considered as a function from $\mathbb{K}O(B)$ to $\mathbb{Z}^+$, the set of non-negative integers. Since $\text{co-ind} \alpha \leq \dim \alpha + \dim B$ ([T1, Lemma 3.1]), we have $\text{co-ind}^s(\alpha) \leq \dim B$ in general. Thus, $\dim B$ is an upper bound of the image of the function $\text{co-ind}^s$. From Theorem 1.5, we obtain the following corollary.

Corollary 1.6. The image of the function $\text{co-ind}^s : \mathbb{K}O(\mathbb{R}P^n) \rightarrow \mathbb{Z}^+$ is equal to the set $\{0, 1, 2, \cdots, n\}$.

Thus, over $\mathbb{R}P^n$, any non-negative integer not greater than $\dim \mathbb{R}P^n$ can be realized as the stable co-index of a vector bundle.

2. Proof of the theorems.

Let $B$ be a finite complex and $\alpha$ be a real vector bundle over $B$. We define the $\nu$-dimension of $\alpha$ to be the smallest integer $k$ for which there is a $k$-dimensional
vector bundle $\beta$ with $\alpha + \beta$ trivial, and we denote it by $\nu$-dim $\alpha$. The $\nu$-dimension of $\alpha$ is closely related to the geometric dimension of $-\alpha$, but they slightly differ in general (see examples in [T2]). The inequalities $g$-dim $(-\alpha) \leq \nu$-dim $\alpha \leq \dim B$ hold generally. We easily see that $\text{co-ind} \alpha \leq \dim \alpha + \nu$-dim $\alpha$.

For a virtual vector bundle $\alpha$, we denote by $\omega$-dim $\alpha$ the largest integer $k$ for which the $k$th Stiefel-Whitney class $w_k(\alpha)$ is not zero. Then it is known that $\text{co-ind} \alpha \geq \dim \alpha + \omega$-dim $(-\alpha)$ and the following theorem holds.

**Theorem 2.1** ([T2, Theorem 2.5]). Let $\alpha$ be a real vector bundle over a finite complex $B$, and suppose that the equality $\omega$-dim $(-\alpha) = \nu$-dim $\alpha$ holds. Then,

1. $\text{co-ind} \alpha = \dim \alpha + \nu$-dim $\alpha$.
2. $\alpha$ is CI-stable.

Now, we prove Theorem 1.1 and Theorem 1.5.

**Proof of Theorem 1.1 and 1.5.**

Since $\tau + 1 \cong (n + 1)\xi$, we have $\tau \otimes \xi + \xi \cong n + 1$. Hence, $\nu$-dim $(\tau \otimes \xi) \leq 1$. We also have $W(-\tau \otimes \xi) = W(\xi) = 1 + t$ where $W$ is the total Stiefel-Whitney class and $t$ is the generator of $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. Hence, $\omega$-dim $(-\tau \otimes \xi) = 1$. Since $\omega$-dim $(-\alpha) \leq \nu$-dim $\alpha$ in general, we have $\omega$-dim $(-\tau \otimes \xi) = \nu$-dim $(-\tau \otimes \xi) = 1$. Therefore, from Theorem 2.1, we find that $\text{co-ind}(\tau \otimes \xi) = n + 1$, and also, $\tau \otimes \xi$ is CI-stable. Thus Theorem 1.1 follows. Our argument goes as well for the $m$-fold sum of the twisted tangent bundle $\tau \otimes \xi$. In fact, since $m \cdot \tau \otimes \xi + m\xi \cong m(n + 1)$, we have $\nu$-dim $(m \cdot \tau \otimes \xi) \leq m$. On the other hand, since $W(-m \cdot \tau \otimes \xi) = W(m\xi) = (1 + t)^m$, we have $\omega$-dim $(-m \cdot \tau \otimes \xi) = m$ if $m \leq n$. Thus, if $m \leq n$, we see that $\omega$-dim $(m \cdot \tau \otimes \xi) = \nu$-dim $(m \cdot \tau \otimes \xi) = m$, and, from Theorem 2.1, it follows that $m \cdot \tau \otimes \xi$ is CI-stable and $\text{co-ind}^*(m \cdot \tau \otimes \xi) = m$. □

Next, we turn to the index. To prove Theorem 1.2, we use the following result in [T1].

**Proposition 2.2** ([T1, Corollary 2.5]). Let $\alpha$ be a real vector bundle over $\mathbb{R}P^n$.

1. If $\dim \alpha > n$, then $\text{ind} \alpha = \dim \alpha$.
2. If $\dim \alpha \leq n$ and $\alpha$ contains $\xi$ as a subbundle, then $\text{ind} \alpha = n + 1$.

**Proof of Theorem 1.2.**

In view of Proposition 2.2 (1), we have only to prove that $\text{ind}(\tau \otimes \xi)$ is equal to $n + 1$ if $n + 1$ is odd, and $n$ if $n$ is even. Since $\text{ind} \alpha \leq \text{co-ind} \alpha$ in general, we have $\text{ind}(\tau \otimes \xi) \leq n + 1$ from Theorem 1.1. Also, since $\text{ind} \alpha \geq \dim \alpha$ in general, we clearly have $\text{ind}(\tau \otimes \xi) \geq n$. Now, suppose that there is a $\mathbb{Z}/2$-map $f : S^n \rightarrow S(\tau \otimes \xi)$. Let $\tilde{f} : \mathbb{R}P^n \rightarrow P(\tau \otimes \xi)$ be the map induced from $f$. Here, $P(\tau \otimes \xi)$ is the projective bundle associated to $\tau \otimes \xi$. Let $p : P(\tau \otimes \xi) \rightarrow \mathbb{R}P^n$ be the bundle projection. Let $e$ denote the $\mathbb{Z}/2$-Euler class of the bundle $S(\tau \otimes \xi) \rightarrow P(\tau \otimes \xi)$. We consider the homomorphism $\tilde{f}^* : H^*(P(\tau \otimes \xi); \mathbb{Z}/2) \rightarrow H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. From the naturality of the Euler class, we have $\tilde{f}^*(e) = t$, and so, $\tilde{f}^*(e^n) = t^n$, which is not zero. On the other hand, in $H^*(P(\tau \otimes \xi); \mathbb{Z}/2)$, we have the relation

$$e^n = w_n + w_{n-1}e + w_{n-2}e^2 + \cdots + w_1e^{n-1},$$
where $w_i$'s are the Stiefel-Whitney classes of $\tau \otimes \xi$ and $H^*(P(\tau \otimes \xi); \mathbb{Z}/2)$ is considered as a $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$-module in an obvious way. Since $\tau \otimes \xi + \xi \cong n+1$, we have $W(\tau \otimes \xi) = W(\xi)^{-1} = (1 + t)^{-1}$. Hence, we have the relation

$$e^n = t^n + t^{n-1}e + t^{n-2}e^2 + \cdots + te^{n-1}.$$  

Applying $\tilde{f}^*$, we have

$$t^n = (p \circ \tilde{f})^*(t^n) + (p \circ \tilde{f})^*(t^{n-1})t + (p \circ \tilde{f})^*(t^{n-2})t^2 + \cdots + (p \circ \tilde{f})^*(t)t^{n-1}.$$  

But the homomorphism $(p \circ \tilde{f})^*: H^*(\mathbb{R}P^n; \mathbb{Z}/2) \to H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ is either identity or zero. The latter case is impossible since $t^n$ is not zero. Therefore, we have $t^n = nt^n$. If $n$ is even, this contradicts $t^n \neq 0$. Thus we have proved that $\text{ind}(\tau \otimes \xi)$ is equal to $n$ if $n$ is even.

Now, let $n$ be odd. Then $\tau$ has a non-zero cross-section since it is orientable and its Euler class is zero. In other words, $\tau$ has a one-dimensional trivial bundle as a subbundle. Hence $\tau \otimes \xi$ has $\xi$ as a subbundle. From Proposition 2.2 (2), it follows that $\text{ind}(\tau \otimes \xi) = n + 1$. This completes the proof. □

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