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On amply strong semistar domains

AKIRA OKABE*

Introduction.

In [14] we have introduced the notion of an amply strong semistar domain (for short, ASSD) and we have showed that a Prüfer ASSD (or equivalently, an integrally closed ASSD) is a generalized Dedekind domain in [14, Proposition 58]. In this paper we continue our study of an ASSD and we prove some further properties of an ASSD. We introduce the notion of a fully stable semistar domain (for short, FSSD) as a stronger form of a stable semistar domain (for short, SSD) which was introduced in [14]. We also introduce the notion of a fully associated semistar domain (for short, FASD) and show in Proposition 50 that $D$ is an FSSD if and only if $D$ is an FASD. Moreover, we show in Proposition 42 that each integrally closed FSSD is a $v$-domain.

In Section 1, we recall some fundamental results on semistar operations and localizing systems which will be freely used later. Thereby some connections between semistar operations and localizing systems will play an important role for further investigation.

In Section 2, we first collect some basic properties of an ASSD and we turn to introducing the notion of a fully stable semistar domain (for short, FSSD). In Theorem 31, we prove that every integrally closed ASSD (or equivalently, a Prüfer ASSD) is an FSSD and in Theorem 32, we show that every localizing system on an ASSD is of overring type. Let $*$ be a star operation on $D$. Then it is easy to see that if $D$ is a conducive domain, then $*$ is stable if and only if the trivial semistar extension $*^s$ is stable. Furthermore, we introduce the notion of an endlich arithmetisch brauchbar semistar domain (for short, EABSD) and the notion of an arithmetisch brauchbar semistar domain (for short, ABSD). In Theorem 46 we show that every Prüfer ASSD is an ABSD. Hence every Prüfer ASSD is also an EABSD.

Throughout this paper the letter $D$ denotes an integral domain with quotient field $K$. We denote the integral closure of $D$ by $\bar{D}$ and we also denote the cardinality of a set $X$ by $|X|$. As usual, we denote the set of all prime ideals of $D$ by $\text{Spec}(D)$.

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* Department of Mathematics, Oyama National College of Technology, Nakakuki 771, Oyama, Tochigi, 323-0806, Japan.
1. Semistar operations and localizing systems.

Let $D$ be an integral domain with quotient field $K$ and let $K(D)$ be the set of all nonzero $D$-submodules of $K$. Then a map $E \mapsto E^*$ of $K(D)$ into $K(D)$ is called a semistar operation if the following conditions hold for all $a \in K \setminus \{0\}$ and $E, F \in K(D)$:

(S1) $(aE)^* = aE^*$;
(S2) If $E \subseteq F$, then $E^* \subseteq F^*$; and
(S3) $E \subseteq E^*$ and $(E^*)^* = E^*$.

The notion of a semistar operation was introduced by Okabe and Matsuda in [11]. For more detailed results on semistar operations, the reader is referred to [6], [11], [12], [13], [14], and [15].

Let $F(D)$ be the set of all nonzero fractional ideals of $D$, that is, all elements $E \in K(D)$ such that there exists a nonzero $d \in D$ with $dE \subseteq D$. We shall denote the set of all nonzero integral ideals of $D$ by $I(D)$ and the set of all finitely generated elements of $K(D)$ by $f(D)$. Evidently $I(D) \subseteq F(D) \subseteq K(D)$ and $f(D) \subseteq F(D)$.

A map $E \mapsto E^*$ of $F(D)$ into $F(D)$ is called a star operation on $D$, if the following conditions hold for all $a \in K \setminus \{0\}$ and $E, F \in F(D)$:

(S0) $(xD)^* = xD$ for all $x \in K \setminus \{0\}$;
(S1) $(aE)^* = aE^*$;
(S2) If $E \subseteq F$, then $E^* \subseteq F^*$; and
(S3) $E \subseteq E^*$ and $(E^*)^* = E^*$.

The reader can refer to [8, Sections 32 and 34] for basic properties of star operations.

We first recall some representative examples of semistar operations and star operations.

If we set $E^\dd = E$ for every $E \in K(D)$, then $d$ is a semistar operation on $D$ and is called the $\dd$-semistar operation or the identity semistar operation on $D$. Next, if we set $E^{-1} = (D : E) = \{x \in K \mid xE \subseteq D\}$ and $E^{\dd} = (E^{-1})^{-1}$ for every $E \in K(D)$, then $\dd$ is a semistar operation on $D$ and is called the $\dd$-semistar operation on $D$. Thirdly, if we set $E^{\dd} = K$ for every $E \in K(D)$, then $\dd$ is a semistar operation on $D$ and is called the $\dd$-semistar operation on $D$ or the trivial semistar operation on $D$.

If we set $E^\dd = E$ for all $E \in F(D)$, then $d$ is a star operation on $D$ and is called the $d$-operation on $D$. If we set $E_v = (E^{-1})^{-1}$ for all $E \in F(D)$, then $v$ is a star operation on $D$ and is called the $v$-operation on $D$.

We shall denote the set of all semistar operations (respectively, all star operations) on $D$ by $\text{SStar}(D)$ (respectively, $\text{Star}(D)$) as in [6].

As in [12], a semistar operation $*$ is said to be weak if $D^* = D$ and is said to be strong if $D^* \neq D$. The set of all weak semistar operations on $D$ is denoted by $\text{SStar}_w(D)$ as in [14]. Moreover we denote the set of all strong semistar operations on $D$ by $\text{SStar}_s(D)$. 
Remark 1. It is easy to see that the $\tilde{d}$-semistar operation and the $\bar{v}$-semistar operation are both weak and the $\tilde{e}$-semistar operation is strong.

Proposition 2 ([11, Proposition 17]). Let $\ast$ be a star operation on $D$. For each $E \in K(D)$, we set:

$$E^\ast = \begin{cases} E, & \text{for } E \in F(D) \\ K, & \text{for } E \in K(D) \setminus F(D) \end{cases}$$

Then the map $E \mapsto E^\ast$ is a semistar operation on $D$.

This semistar operation $\ast^e$ is called the trivial semistar extension of $\ast$. Evidently $\ast^e$ is always weak for each $\ast \in \text{Star}(D)$.

Remark 3. It is easy to see that the $\bar{v}$-semistar operation is the trivial semistar extension of the $\bar{v}$-operation. In fact, for an element $E \in K(D), E \notin F(D)$ if and only if $E^{-1} = \{0\}$, that is, $E^\bar{v} = K$. Hence we have

$$E^\bar{v} = \begin{cases} E^\bar{v}, & \text{for } E \in F(D) \\ K, & \text{for } E \in K(D) \setminus F(D) \end{cases}$$

Thus the $\bar{v}$-semistar operation is the trivial semistar extension of the $\bar{v}$-operation.

Proposition 4 (cf. [11, Lemma 45]). Let $T$ be an overring of $D$. Then

1. For each $\ast \in \text{SStar}(T)$, we set $E^\delta_{T/D}(\ast) = (ET)^\ast$ for each $E \in K(D)$. Then $\delta_{T/D}(\ast) \in \text{SStar}(D)$.

2. For each $\ast \in \text{SStar}(D)$, we set $E_{\alpha_{T/D}(\ast)} = E^\ast$ for each $E \in K(T) \subseteq K(D)$. Then $\alpha_{T/D}(\ast) \in \text{SStar}(T)$.

3. $\alpha_{T/D} \circ \delta_{T/D}$ is the identity map of $\text{SStar}(T)$ and so $\delta_{T/D}$ is an injective map.

The map $\delta_{T/D}$ (respectively, $\alpha_{T/D}$) is called the descent map (respectively, the ascent map).

Let $T$ be an overring of $D$. If we set $E^*(T) = ET$ for each $E \in K(D)$, then $\ast(T)$ is a semistar operation on $D$ and is called the semistar operation on $D$ defined by $T$. If $T \neq D$, then $\ast(T)$ is a strong semistar operation on $D$. Each semistar operation $\ast(T)$ for an overring $T$ of $D$ is said to be of overring type as in [14]. In this paper an overring $T$ of $D$ is called a proper overring in case $T \neq D$ and $T \neq K$. We shall denote the set of all proper overrings of $D$ by $\mathcal{P}(D)$.

Proposition 5. (1) For each overring $T$ of $D$, $\ast(T) = \delta_{T/D}(\bar{d}_T)$, where $\bar{d}_T$ is the $\bar{d}$-semistar operation on $T$.

(2) For overrings $S \subseteq T$ of $D$, $\ast(T) = \delta_{S/D}(\delta_{T/S}(\bar{d}_T))$.

(3) For each overring $T$ of $D$, $\delta_{T/D}(\bar{e}_T) = \bar{e}$, where $\bar{e}_T$ is the $\bar{e}$-semistar operation on $T$.

(4) For each $T \in \mathcal{P}(D)$, $\delta_{T/D}(\text{SStar}(T)) \cap \text{SStar}_w(D) = \emptyset$. In particular, $\bar{d}$ and $\bar{v}$ are not contained in $\delta_{T/D}(\text{SStar}(T))$.

Proof. (1) For each $E \in K(D)$, $E^\delta_{T/D}(\bar{d}_T) = (ET)^{\bar{d}} = ET = E^*(T)$ and hence $\ast(T) = \delta_{T/D}(\bar{d}_T)$. 
(2) By definition, 
\[ E^{\delta_{S/D}(\delta_{T/S}(d_T)))} = (ES)^{\delta_{T/S}(d_T)} = (ET)^{d_T} = ET = E^*(T) \]
for all \( E \in K(D) \) and so \( \delta_{S/D}(\delta_{T/S}(d_T))) = \delta_T(T)) \). 

(3) This is trivial.

(4) For each \( * \in S\text{Star}(T) \), we have \( D^{\delta_{T/D}(*)} = (DT)^* = T^* \supseteq T \neq D \) and hence \( \delta_{T/D}(*) \notin S\text{Star}_w(D) \). The "in particular" statement is straightforward. \( \square \)

For each overring \( T \) of \( D \), we set \( S\text{Star}^T(D) = \{ * \in S\text{Star}(D) \mid D^* = T \} \). In case \( T = D \), it follows that \( S\text{Star}^D(D) = S\text{Star}_w(D) \).

**Proposition 6.** Let \( T \) be a proper overring of \( D \). Then

1. \( |S\text{Star}(D)| \geq |S\text{Star}_w(D)| + |S\text{Star}(T)| \).
2. \( |S\text{Star}^T(D)| = |S\text{Star}_w(T)| \).

**Proof.** (1) follows from Proposition 5 (4) and the injectivity of \( \delta_{T/D} \).

(2) This follows from the fact that \( \delta_{T/D} \) induces a bijective map from \( S\text{Star}_w(T) \) onto \( S\text{Star}^T(D) \) by [13, Lemma 32]. \( \square \)

**Remark 7 ([12, Remark 20 (2)])**. Let \( * \) be a semistar operation on \( D \). If \( D^* = K \), then \( * = \bar{e} \). In fact, for each \( E \in K(D) \), \( E^* = (DE)^* = (D^*E)^* = (KE)^* = K^* = K \) and so \( * = \bar{e} \). Thus \( S\text{Star}_K(D) = \{ \bar{e} \} \).

In [7] the notion of a localizing (or topologizing) system of ideals was introduced by P. Gabriel. A set \( F \) of ideals of \( D \) is called a localizing system of ideals (for short, localizing system) if the following conditions are satisfied:

1. (LS1) If \( I \in F \) and \( I \subseteq J \) with \( J \in \mathcal{I}(D) \), then \( J \in F \).
2. (LS2) If \( I \in F \) and \( J \in \mathcal{I}(D) \) such that \( J :_D iD \in F \) for all \( i \in I \), then \( J \in F \).

If \( F \) is a localizing system of \( D \) and \( I, J \in F \), then \( IJ \in F \) (see, [5, Proposition 5.1.1]) and so \( I \cap J \in F \) by (LS1). Thus every localizing system is a generalized multiplicative system.

A localizing system \( F \) on \( D \) is said to be of finite type if for each \( I \in F \), there exists a finitely generated ideal \( J \in F \) such that \( J \subseteq I \).

Let \( T \) be an overring of \( D \). If we set \( F(T) = \{ I \in \mathcal{I}(D) \mid IT = T \} \), then \( F(T) \) is a localizing system [16, Proposition 1.2 (i)]. A localizing system \( F \) is said to be of overring type if \( F = F(T) \) for some overring \( T \) of \( D \). It is easy to see that \( F(T) \) is a localizing system of finite type.

We shall denote the set of all localizing systems of \( D \) by \( LS(D) \) and the set of all localizing systems of finite type of \( D \) by \( LS_f(D) \).

In [16], N. Popescu defined a Prüfer domain \( D \) to be a generalized Dedekind domain in case for each overring \( T \) of \( D \), there exists a unique localizing system \( F \) of \( D \) such that \( T = D_F \), where \( D_F = \{ x \in K \mid D :_D x \in F \} \) (see [16, p. 778]). Note that for each localizing system \( F \) of \( D \), \( D_F \) is an overring of \( D \) and it can also be represented as the union of the form \( \bigcup \{ D :_K I \mid I \in F \} \). To know other interesting characterizations of a generalized Dedekind domain, see [14, Proposition 35 and Theorem 70].

Before proving the next property of an ASSD, we shall recall some important relations between semistar operations and localizing systems that will be found in [6].
Proposition 8 (cf. [6, Propositions 2.4 and 2.8]). (1) For each localizing system \( \mathcal{F} \) of \( D \), we set \( E_{\mathcal{F}} = \bigcup \{ E : K : J \mid J \in \mathcal{F} \} \) for each \( E \in \mathcal{K}(D) \). Then the map \( E \mapsto E^*_{\mathcal{F}} := E_{\mathcal{F}} \) is a semistar operation on \( D \).

(2) For each semistar operation \( * \) on \( D \), we set \( \mathcal{F}^* = \{ I \in \mathcal{I}(D) \mid I^* = D^* \} \). Then \( \mathcal{F}^* \) is a localizing system of \( D \).

A semistar operation \( *_{\mathcal{F}} \) defined in Proposition 8 (1) is called the semistar operation associated to a localizing system \( \mathcal{F} \) and a localizing system \( \mathcal{F}^* \) defined in Proposition 8 (2) is called the localizing system associated to a semistar operation \( * \).

2. Amply strong semistar domains.

As in [14], an integral domain \( D \) is called an amply strong semistar domain (for short, ASSD) if each semistar operation on \( D \) is of overring type. It follows that if \( D \) is an ASSD, then \( S\text{Star}_w(D) = \{ d \} \).

Proposition 9. \( D \) is an ASSD if and only if \( S\text{Star}^T(D) = \{ *(T) \} \) for every overring \( T \) of \( D \).

Proof. \((\Rightarrow)\) If \( * \in S\text{Star}^T(D) \), then \( * = *(T) \).

\((\Leftarrow)\) Let \( * \in S\text{Star}(D) \). Set \( T = D^* \). If \( T \neq K \), then \( * \in S\text{Star}^T(D) = \{ *(T) \} \) and so \( * = *(T) \). Next, if \( T = K \), then \( D^* = K \) and hence, by Remark 7, \( * = \bar{e} = *(K) \). In any case, each semistar operation \( * \) on \( D \) is of overring type. \( \square \)

Proposition 10 ([14, Proposition 51]). Let \( D \) be an ASSD. Then every proper overring \( T \) of \( D \) is also an ASSD.

Proposition 11. \( D \) is an ASSD if and only if \( | S\text{Star}(D) | = | \mathcal{P}(D) | + 2 \).

Proof. \((\Rightarrow)\) Let \( \mathcal{P}(D) = \{ R_\alpha \mid \alpha \in A \} \). Then by hypothesis, \( S\text{Star}(D) = \{ \bar{d} = *(D) \} \cup \{ \bar{e} = *(K) \} \cup \{ *(R_\alpha) \mid \alpha \in A \} \) and so \( | S\text{Star}(D) | = | \mathcal{P}(D) | + 2 \).

\((\Leftarrow)\) This is trivial. \( \square \)

Proposition 12. Let \( D \) be an ASSD. Then the following statements hold:

(1) \( S\text{Star}_w(D) = \{ \bar{d} \} \) and \( \text{Star}(D) = \{ d \} \).

(2) For each \( T \in \mathcal{P}(D) \) and for each \( *(S) \in S\text{Star}(T) \) with an overring \( S \) of \( T \), \( \delta_{T/D}(*(S)) = *(S) \in S\text{Star}(D) \), where \( *(S) \) is the semistar operation on \( T \) defined by \( S \).

(3) For each \( T \in \mathcal{P}(D) \), \( S\text{Star}_w(T) = \{ \bar{d}_T \} \) and \( \text{Star}(T) = \{ d_T \} \), where \( \bar{d}_T \) (respectively, \( d_T \)) is the \( \bar{d} \)-semistar operation (respectively, the \( d \)-operation) on \( T \).

Proof. \((1)\) This is [14, Proposition 52 (1)].

\((2)\) Let \( *(S) \in S\text{Star}(T) \) with an overring \( S \) of \( T \). Then \( D^{\delta_{T/D}(*(S))} = (DT)^*(S) = T^{*(S)} = TS = S \) and so \( \delta_{T/D}(*(S)) = *(S) \in S\text{Star}(D) \), because \( D \) is an ASSD.

\((3)\) This is [14, Proposition 52]. \( \square \)
It would be worthwhile to note that for any overrings $S \supset T$ of $D$, we always have $\delta_{T/D}(\ast^T(S)) = \ast(S)$ without the assumption that $D$ is an ASSD. In fact, for each element $E \in K(D), E^{\ast^T/D}(\ast^T(S)) = (ET)^{\ast^T(S)} = (ET)S = ES = E^\ast(S)$ and hence $\delta_{T/D}(\ast^T(S)) = \ast(S)$.

**Corollary 13.** Let $D$ be an ASSD. Then $|\text{SStar}(R)| \leq |\text{SStar}(D)| - 1$ for any proper overring $R$ of $D$.

**Proof.** This follows from Propositions 6 (1) and 12 (1). □

**Corollary 14([14, Corollary 62]).** If $D$ is an ASSD, then every nonzero ideal of $D$ is divisorial.

**Corollary 15([14, Corollary 63]).** Let $D$ be an ASSD. Then $D$ is a Prüfer domain if and only if $D$ is integrally closed.

**Corollary 16(cf. [14, Corollary 64 (1)]).** Every integrally closed ASSD is a generalized Dedekind domain. In particular, the integral closure of an ASSD is a Prüfer domain.

**Corollary 17.** Every quasilocal integrally closed ASSD is a valuation domain.

**Proof.** This follows immediately from Corollary 15. □

An integral domain $D$ is called a conducive domain if $(D :K R) = \{x \in K \mid xR \subseteq D\} \neq (0)$ for each overring $R$ of $D$ other than $K$. For characterizations of a conducive domain, the reader is referred to [13, Proposition 7].

**Remark 18.** It follows from [13, Proposition 7] that $D$ is a conducive domain if and only if the $\bar{d}$-semistar operation is the trivial semistar extension of the $d$-operation.

**Proposition 19([14, Theorem 60]).** Let $D$ be an integral domain. Then $D$ is a conducive domain if and only if the map $\chi : \text{Star}(D) \mapsto \text{SStar}_w(D)$ with $\chi(\ast) = \ast^e$ is a bijective map.

**Proposition 20([14, Corollary 61]).** Every ASSD is a conducive domain.

**Corollary 21.** If $D$ is an ASSD, then $|\text{Star}(T)| = |\text{SStar}_w(T)| = |\text{SStar}^T(D)| = 1$ for each overring $T$ of $D$ such that $T \neq K$.

**Proof.** By Propositions 9 and 12 (3), $|\text{SStar}_w(T)| = |\text{SStar}^T(D)| = 1$. By Proposition 20, $D$ is conducive and hence, by [4, Lemma 2.0 (i)], $T$ is also conducive. Then by Proposition 19, $|\text{Star}(T)| = |\text{SStar}_w(T)|$ and so $|\text{Star}(T)| = 1$. □

**Proposition 22([11, Corollary 10]).** Let $\Gamma = \{R_\alpha \mid \alpha \in A\}$ be a family of overrings of $D$. If we set $E^*(T) = \cap\{E^{\ast}(R_\alpha) \mid \alpha \in A\}$ for each $E \in K(D)$, then $\ast(T)$ is a semistar operation on $D$.

**Proposition 23.** Let $\Gamma = \{R_\alpha \mid \alpha \in A\}$ be a family of overrings of $D$. If $D$ is a Prüfer domain, then we have $\ast(T) = \ast(T)$ for some overring $T$ of $D$. But then $T = \cap\{D^{\ast}(R_\alpha) \mid \alpha \in A\} = \cap\{R_\alpha \mid \alpha \in A\}$. □
Corollary 24. Let $\Gamma_0 = \{V_\alpha \mid \alpha \in A\}$ be the set of all valuation overrings of $D$. If $D$ is an ASSD, then $\star(\Gamma_0) = \star(D)$, where $\tilde{D}$ is the integral closure of $D$.

Proof. This follows immediately from Proposition 23. \qed

In [13, Definition 26], an integral domain $D$ is called a finite semistar domain (for short, FSD) if $|\text{SStar}(D)| < \infty$.

Proposition 25. Let $D$ be an ASSD. Then $D$ is an FSD if and only if $D$ has only finitely many overrings of $D$.

Proof. This follows from Proposition 11. \qed

By Corollary 17, every quasilocal integrally closed ASSD is a valuation domain. In the case of a finite dimensional integral domain, we can prove the following:

Proposition 26(cf. [14, Theorem 70]). Let $D$ be a finite dimensional integral domain. Then $D$ is a quasilocal integrally closed ASSD if and only if $D$ is a strongly discrete valuation domain.

Remark 27. Let $V$ be a finite dimensional non-discrete valuation domain. Then $V$ is an FSD by [13, Theorem 38]. But $V$ is not an ASSD by Proposition 26. Thus an FSD is not necessarily an ASSD.

As in [6], a semistar operation $\star$ on $D$ is said to be stable if $(E \cap F)^* = E^* \cap F^*$ for all $E, F \in K(D)$.

Proposition 28. Let $T$ be an overring of $D$. If $T$ is flat over $D$, then $\star(T)$ is a stable semistar operation.

Proof. If $T$ is flat over $D$, then $(E \cap F)T = ET \cap FT$ for all $E, F \in K(D)$ by [3, Ch.1, §2, N.6, Proposition 6] and hence $(E \cap F)^*(T) = (E \cap F)T = ET \cap FT = E^*(T) \cap F^*(T)$. \qed

It follows from Proposition 28 that $\star(D_P)$ is a stable semistar operation for each prime ideal $P$ of $D$. Moreover, if we set $E^*_\Delta = \bigcap \{EP_\alpha \mid \alpha \in A\}$ for any set $\Delta = \{P_\alpha \mid \alpha \in A\}$ of prime ideals of $D$, then $\star_\Delta$ is a stable semistar operation [11, Theorem 20].

Definition 29. An integral domain $D$ is called a fully stable semistar domain (for short, FSSD) if every semistar operation on $D$ is stable.

Example 30. Let $D$ be a one-dimensional discrete valuation domain, i.e., a DVD. Then by [11, Theorem 48], $\text{SStar}(D) = \{\bar{d}, \bar{e}\}$ and it is evident that the $\bar{d}$-operation and the $\bar{e}$-operation are both stable. Thus every DVD is an FSSD.

Theorem 31. Every integrally closed ASSD (or equivalently, a Prüfer ASSD) is an FSSD.

Proof. Let $D$ be an integrally closed ASSD and let $\star$ be an arbitrary semistar operation on $D$. Then $\star = \star(T)$ for some overring $T$ of $D$. Now, since $D$ is also a Prüfer domain, $T$ is flat over $D$. Hence by Proposition 28, $\star = \star(T)$ is a stable semistar operation. \qed
Let $P$ be a prime ideal of $D$. Then $\mathcal{F}(P) = \{ I \in \mathfrak{I}(D) \mid I \not\subseteq P \}$ is a localizing system of $D$. Hence if $\Delta$ is a nonempty subset of $\text{Spec}(D)$, then $\mathcal{F}(\Delta) = \cap \{ \mathcal{F}(P) \mid P \in \Delta \}$ is also a localizing system of $D$.

Here we recall that a localizing system $\mathcal{F}$ of $D$ is said to be spectral if there exists a nonempty subset $\Delta$ of $\text{Spec}(D)$ such that $\mathcal{F} = \mathcal{F}(\Delta)$ [5, p. 126 and Proposition 5.1.4]. It is easy to see that if $\Delta_1 \subseteq \Delta_2$ are subsets of $\text{Spec}(D)$, then $\mathcal{F}(\Delta_2) \subseteq \mathcal{F}(\Delta_1)$.

Now, if $P \in \text{Spec}(D)$, then $\mathcal{F}(P) = \mathcal{F}(DP)$, because $I \subseteq P$ if and only if $ID_P = DP$ for any $I \in \mathfrak{I}(D)$. Hence it follows that each spectral localizing system is an intersection of localizing systems of overring type.

**Theorem 32.** Each localizing system of an ASSD is of overring type.

**Proof.** Let $D$ be an ASSD and let $\mathcal{F}$ be an arbitrary localizing system of $D$. Then, by hypothesis, $*\mathcal{F} = *_{(T)}$ for some overring $T$ of $D$. But, by [14, Proposition 5 (2)], $\mathcal{F}^{*_{(T)}} = \mathcal{F}(T)$ and by [6, Theorem 2.10 (A)], $\mathcal{F}^{*_{T}} = \mathcal{F}$ for each localizing system $\mathcal{F}$. Hence $\mathcal{F} = \mathcal{F}^{*_{T}} = \mathcal{F}^{*_{(T)}} = \mathcal{F}(T)$ and so every localizing system of $D$ is of overring type. \hfill \Box

**Corollary 33.** Every Prüfer ASSD is a generalized Dedekind domain.

**Proof.** This follows from Theorem 32 and the equivalence $(1) \iff (5)$ in [14, Proposition 35].

**Proposition 34.** If $D$ is an FSSD, then every overring of $D$ is also an FSSD.

**Proof.** Let $D$ be an FSSD and let $T$ be an overring of $D$. For each $* \in \text{SSStar}(T)$, we have $* = \alpha_T \delta_{T/D}(*)$ by Proposition 4 (3). By hypothesis, $\delta_{T/D}(*)$ is stable and hence, for all $E, F \in K(T)$, $(E \cap F)^* = (E \cap F) \alpha_T \delta_{T/D}(*) = (E \cap F) \delta_{T/D}(*) = E \delta_{T/D}(*) \cap F \delta_{T/D}(*) = (ET)^* \cap (FT)^* = E^* \cap F^*$. Thus each $* \in \text{SSStar}(T)$ is stable. \hfill \Box

**Corollary 35.** If $D$ is a Prüfer FSSD, then every overring of $D$ is also a Prüfer FSSD.

Let $*$ be a star operation on $D$. Then $*$ is called a stable star operation on $D$ if $(E \cap F)^* = E^* \cap F^*$ for all $E, F \in \mathcal{F}(D)$.

**Remark 36.** Let $*$ be a star operation on $D$. Then it is easy to see that if $*^c$ is stable, then $*$ is also stable. If $D$ is a conducive domain, then $K(D) = \mathcal{F}(D) \cup \{K\}$ by [13, Proposition 7]. Hence it follows that if $*$ is stable, then $*^c$ is also stable, because $E \cap K = E$ holds for each $E \in \mathcal{F}(D)$.

An ideal $I$ of $D$ is called a divisorial ideal (or a $v$-ideal) if $I_v = I$. Here we shall denote the set of all divisorial ideals of $D$ by $\mathcal{D}(D)$. A divisorial ideal $I$ of $D$ is said to be of finite type or of $v$-finite if $I = J_v$ for some finitely generated ideal $J$ of $D$.

**Proposition 37**([8, Theorem 34.3]). Let $D$ be an integral domain with quotient field $K$. Then the following conditions are equivalent:

1. $D(D)$ is a group under the multiplication $E_v \cdot F_v = (EF)_v$.
2. $I : I = D$ for each $v$-ideal $I$ of $D$. 


(3) $E : E = D$ for each $E \in \mathcal{F}(D)$.
(4) $D$ is completely integrally closed.

Let $*$ be a semistar operation on $D$. Then $*$ is said to be endlich arithmetisch brauchbar (for short, e.a.b.) if for all $E, F,$ and $G \in \mathcal{F}(D)$, $(EF)^* \subseteq (EG)^*$ and is said to be arithmetisch brauchbar (for short, a.b.) if for all $E \in \mathcal{F}(D)$ and all $F, G \in \mathcal{K}(D)$, $(EF)^* \subseteq (EG)^*$.

Let $*$ be a star operation on $D$. Then $*$ is said to be endlich arithmetisch brauchbar (for short, e.a.b.) if for all $E, F,$ and $G \in \mathcal{F}(D)$, $(EF)^* \subseteq (EG)^*$ and is said to be arithmetisch brauchbar (for short, a.b.) if for all $E \in \mathcal{F}(D)$ and all $F, G \in \mathcal{F}(D)$, $(EF)^* \subseteq (EG)^*$.

It is evident that each a.b. semistar operation (respectively, a.b. star operation) on $D$ is an e.a.b. semistar operation (respectively, e.a.b. star operation) on $D$.

Now we recall that an integral domain $D$ is called a $v$-domain if the $v$-operation on $D$ is e.a.b. (see [8, p.418]).

**Proposition 38** ([8, Theorem 34.6]). Let $D$ be an integral domain with quotient field $K$. Then the following conditions are equivalent:

1. $D$ is a $v$-domain.
2. $I : I = D$ for each divisorial ideal $I$ of finite type.
3. Each divisorial ideal of finite type of $D$ has an inverse in $D(D)$.

**Remark 39.** (1) It easily follows from Propositions 37 and 38 that every completely integrally closed domain is a $v$-domain. An example of an integrally closed domain which is not a $v$-domain is given in [8, Exercise 2 in § 33].

(2) Every $v$-domain is integrally closed. In fact, let $D$ be a $v$-domain and let $I$ be a finitely generated ideal of $D$. If $u \in I : I$, then $u \in I_v : I_v = D$ by Proposition 37 (2). Hence $D$ is integrally closed.

**Proposition 40.** (1) If $D$ is integrally closed and the $v$-operation is stable, then $D$ is a $v$-domain.

2. If $D$ is a $v$-domain, then the $v$-operation on $D$ is stable.

**Proof.** (1) is [1, Theorem 7] and (2) is [10, Theorem 2].

**Corollary 41.** Assume that $D$ is integrally closed. Then $D$ is a $v$-domain if and only if the $v$-operation on $D$ is stable.

**Proposition 42.** Every integrally closed FSSD is a $v$-domain.

**Proof.** Since $D$ is an FSSD, the $v$-semistar operation on $D$ is stable. But, by Remark 3, the $v$-semistar operation is the trivial semistar extension of the $v$-operation and therefore the $v$-operation is also stable by Theorem 36 (1). Then, since $D$ is also integrally closed, $D$ is a $v$-domain by Corollary 41.

**Corollary 43.** Every integrally closed ASSD is a conducive $v$-domain.

**Proof.** This follows from Proposition 20, Theorem 31, and Proposition 42.
Definition 44. An integral domain $D$ is called an *endlich arithmetisch brauchbar semistar domain* (for short, EABSD) if every semistar operation on $D$ is e.a.b. and is called an *arithmetisch brauchbar semistar domain* (for short, ABSD) if every semistar operation on $D$ is a.b.

To prove that every Prüfer ASSD is an EABSD, we need the following:

Lemma 45. Let $R$ be a Prüfer overring of $D$. Then $*(R)$ is an a.b. semistar operation on $D$.

Proof. Let $E \in \mathfrak{f}(D)$ and $F, G \in K(D)$. Now assume that $(EF)^*(R) \subseteq (EG)^*(R)$. Choose an element $d \neq 0 \in D$ such that $dE \subseteq D$. Then, by hypothesis, $(dER)(FR) = d(EFR) \subseteq d(EGR) = (dER)(GR)$. Since $R$ is a Prüfer domain and $dER$ is a finitely generated ideal of $R$, $dER$ is an invertible ideal of $R$. Hence $(dER)(FR) \subseteq (dER)(GR)$ implies $F^*(R) = FR \subseteq GR = G^*(R)$. Hence $*(R)$ is an a.b. semistar operation on $D$.

Theorem 46. Every Prüfer ASSD is an ABSD.

Proof. First, note that every semistar operation $*$ on $D$ is of the form $*_R$ for some overring $R$ of $D$ by the definition of ASSD. Then $*=*_R$ is a.b. by Lemma 45.

Corollary 47. Every Prüfer ASSD is an EABSD.

Proposition 48. For each valuation overring $V$ of $D$, $*_V$ is an a.b. semistar operation on $D$.

Proof. This follows from Lemma 45 because every valuation domain is a Prüfer domain.

Definition 49. A semistar operation $*$ on $D$ is said to be of associated type if $* = *_F$ for some localizing system $F$ of $D$. An integral domain $D$ is called a *fully associated semistar domain* (for short, FASD) if every semistar operation on $D$ is of associated type.

Proposition 50. $D$ is an FSSD if and only if $D$ is an FASD.

Proof. ($\Rightarrow$). Suppose that $D$ is an FSSD and let $* \in \text{SStar}(D)$. Then, by [6, Theorem 2.10 (B)], $* = *_F$ and so $*$ is of associated type.

($\Leftarrow$). Suppose that $D$ is an FASD and let $*$ be a semistar operation on $D$. Then $* = *_F$ for some localizing system $F$ of $D$. Then $*$ is stable by [6, Proposition 2.4].

Remark 51. The $\tilde{d}$-semistar operation and the $\bar{e}$-semistar operation on $D$ are both of associated type. If we set $F = \{D\}$, then $\{D\}$ is a localizing system on $D$ and $\tilde{d} = *_D$. Next, if we set $F = \text{I}(D)$, then $\text{I}(D)$ is a localizing system of $D$ and $\bar{e} = *_{\text{I}(D)}$ as shown in [15, Remark 35 (3)].

References

On amply strong semistar domains


