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On the fine spectrum of the Cesàro operator in $c_0$

ALİ M. AKHMEDEOV* AND FEYZİ BAŞAR**

ABSTRACT. In the present paper, the fine spectrum of the Cesàro operator in the sequence space $c_0$ has been examined. Although the discussion is made for determination of the spectrum of the Cesàro operator in the sequence space $c_0$ by Reade [14] and the others, our consequences are more refinement and include a remark concerning with the previous works. Further, a Mercerian theorem has also been given. Finally, the fine spectrum of the Cesàro operator in the sequence space $c$ has been given, without proof.

1. Preliminaries, Background and Notation.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ also be a linear operator. With $T$, we associate the operator

$$T_\alpha = T - \alpha I,$$

where $\alpha$ is a complex number and $I$ is the identity operator on $D(T)$. If $T_\alpha$ has an inverse, which is linear, we denote it by $R_\alpha(T)$, that is

$$R_\alpha(T) = T_\alpha^{-1} = (T - \alpha I)^{-1}$$

and call it the resolvent operator of $T$. We also write simply $R_\alpha$ instead of $R_\alpha(T)$ if it is clear to what operator $T$ we refer in a specific discussion.

The name resolvent is appropriate, since $R_\alpha(T)$ helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y = R_\alpha(T) y$ provided $R_\alpha(T)$ exists. More important, the investigation of properties of $R_\alpha$ will be basic for an understanding of the
operator $T$ itself. Naturally, many properties of $T_\alpha$ and $R_\alpha$ depend on $\alpha$, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all $\alpha$ in the complex plane such that $R_\alpha$ exists. Bound-

ness of $R_\alpha$ is another property that will be essential. We shall also ask for what $\alpha$’s the domain of $R_\alpha$ is dense in $X$, to name just a few aspects. For our investigation of $T$, $T_\alpha$ and $R_\alpha$, we need some basic concepts in spectral theory which are given as follows (see [9, pp. 370-371]):

**Definition 1.1.** Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ also be a linear operator with domain $D(T) \subset X$. A regular value $\alpha$ of $T$ is a complex number such that

$(R1)$ $R_\alpha(T)$ exists,
$(R2)$ $R_\alpha(T)$ is bounded,
$(R3)$ $R_\alpha(T)$ is defined on a set which is dense in $X$.

The resolvent set $\rho(T, X)$ of $T$ is the set of all regular values $\alpha$ of $T$. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the spectrum of $T$. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The point (discrete) spectrum $\sigma_p(T, X)$ is the set such that $R_\alpha(T, X)$ does not exist. A $\alpha \in \sigma_p(T, X)$ is called an eigenvalue of $T$.

The continuous spectrum $\sigma_c(T, X)$ is the set such that $R_\alpha(T, X)$ exists and satisfies (R3) but not (R2), that is, $R_\alpha(T, X)$ is unbounded.

The residual spectrum $\sigma_r(T, X)$ is the set such that $R_\alpha(T, X)$ exists (and may be bounded or not) but not satisfy (R3), that is, the domain of $R_\alpha(T, X)$ is not dense in $X$.

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T, X) = \sigma_r(T, X) = \emptyset$ and the spectrum $\sigma(T, X)$ consists of only the set $\sigma_p(T, X)$ in the finite dimensional case.

By $w$, we shall denote the space of all real or complex valued sequences. Any vector subspace of $w$ is called as a sequence space. We shall write $\ell_\infty$, $c$, $c_0$ and $bv$ for the sequence spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by $bs$, $cs$, $\ell_1$ and $\ell_p$, we denote the spaces of all bounded, convergent, absolutely and $p$-absolutely convergent series, respectively. Let $\lambda$, $\mu$ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers $a_{nk}$, where $n, k \in \mathbb{N} = \{0, 1, 2, \ldots \}$. Then, we write $Ax = ((Ax)_n)$, the $A$-transform of $x$, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. If $x \in \lambda$ implies that $Ax \in \mu$ then we say that $A$ defines a matrix mapping from $\lambda$ into $\mu$ and denote it by $A : \lambda \to \mu$. By $(\lambda : \mu)$, we mean the class of all infinite matrices such that $A : \lambda \to \mu$.

Now, we may give the following lemma requiring in the proof of theorems given in Section 2, below:

**Lemma 1.2**[6, p. 59]. $T$ has a dense range if and only if $T^*$ is one to one, where $T^*$ denotes the dual operator of that operator $T$. 
The Cesàro operator is represented by the matrix

\[ C_1 = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \ddots 
\end{bmatrix}. \]

It is known that

(i) \( C_1 : c_0 \rightarrow c_0 \) is a linear operator,
(ii) \( \|C_1\|_{(c_0:c_0)} = 1 \),
(iii) \( C_1^* : \ell_1 \rightarrow \ell_1 \) and is represented by the matrix

\[ C_1^* = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} & \cdots \\
0 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} & \cdots \\
0 & 0 & \frac{1}{3} & \cdots & \frac{1}{n+1} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{n+1} & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \ddots 
\end{bmatrix}, \]

where \( C_1^* \) is the dual operator of \( C_1 \).

The fine spectrum of the Cesàro operator on the sequence space \( \ell_p \) has been studied by González [7], where \( 1 < p < \infty \). The spectrum of the Cesàro operator on the sequence spaces \( c_0 \) and \( bv \) have also been investigated by Reade [14] and Okutoyi [13], respectively. The fine spectrum of the Rhally operators on the sequence spaces \( c_0 \) and \( c \) have been examined by Yıldırım [17]. The fine spectrum of the Cesàro operator on the sequence space \( \ell_p \) have recently been studied by Akhmedov and Başar [1], independently than that of González [7], by the different way; where \( 1 < p < \infty \). More recently, the fine spectrum of the difference operator \( \Delta \) on the sequence spaces \( c_0 \) and \( c \) has been worked by Altay and Başar [4]. The spectrum of the Cesàro operator on the sequence spaces \( c_0 \) and \( c \) are studied [14, 18]. In this work, our purpose is to investigate the fine spectrum of the Cesàro operator in the sequence spaces \( c_0 \) and \( c \) which is the natural continuation of Akhmedov and Başar [1].

2. The Fine Spectrum of the Cesàro Operator in the Sequence Spaces \( c_0 \) and \( c \).

In this section, the fine spectrum of the Cesàro operator in the sequence spaces \( c_0 \) and \( c \) has been examined. Furthermore, a Mercerian theorem has also been stated and proved.

We shall begin with quoting some lemmas that the first of them is Weierstrass Criterion and is needed in the proof of Theorem 2.4, below, and the others summarize the consequences on the spectrum of the Cesàro operator in the sequence space \( c_0 \), in the existing literature.
Lemma 2.1[8, Weierstrass Criterion; pp. 398–399]. A series $\sum_{n=0}^{\infty} a_n$ of complex terms, for which

$$\frac{a_{n+1}}{a_n} = 1 - \frac{b}{n} - \frac{A_n}{n^d}$$

with $A_n$ bounded, is absolutely convergent if and only if $\text{Re}(b) > 1$; where $b$ is a complex number, $d > 1$ and $\text{Re}(b)$ denotes the real part of the complex number $b$. For $\text{Re}(b) \leq 0$ the series is invariably divergent. If $0 < \text{Re}(b) \leq 1$, both the series

$$\sum_{n=0}^{\infty} |a_n - a_{n+1}| \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n a_n$$

are convergent.

Lemma 2.2[14, Theorem 3]. $\sigma(C_1, c_0) = \{\alpha : |\alpha - \frac{1}{2}| \leq \frac{1}{2}\}$.

Lemma 2.3[16]. $\sigma_p(C_1, c_0) = \emptyset$.

Theorem 2.4. $\sigma_p(C_1^*, c_0^* \cong \ell_1) = \{\alpha \in C : |\alpha - \frac{1}{2}| < \frac{1}{2}\} \cup \{1\}$.

Proof. Suppose that $C_1^* f = \alpha f$ for $f = (f_0, f_1, f_2, \ldots) \neq \theta$ in $c_0^* \cong \ell_1$. Then, by solving the system of linear equations

$$\begin{cases} f_0 + \frac{1}{2} f_1 + \frac{1}{3} f_2 + \frac{1}{4} f_3 + \cdots = \alpha f_0 \\ \frac{1}{2} f_1 + \frac{1}{3} f_2 + \frac{1}{4} f_3 + \cdots = \alpha f_1 \\ \frac{1}{3} f_2 + \frac{1}{4} f_3 + \cdots = \alpha f_2 \\ \vdots \end{cases}$$

we obtain that

$$f_n = \left[ \prod_{k=1}^{n} \left(1 - \frac{1}{k\alpha} \right) \right] f_0 \quad (n \geq 1).$$

(i) If $m$ is the first integer for which $f_m \neq 0$, then $\alpha = 1/(m + 1)$. So, $\alpha = 1/(m + 1)$ is an eigenvalue with the corresponding eigenvector

$$f = \left[ \sum_{k=0}^{m} (-1)^{k-1} \binom{m}{k} e^{(k)} \right] f_0 \in \ell_1;$$

where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in $k^{th}$ place for each $k \in \mathbb{N}$.

(ii) If $f_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\frac{f_{n+1}}{f_n} = 1 - \frac{1}{n} - \frac{A_n}{n^2}.$$
where $A_n = -n/\alpha(n + 1)$. Since $(A_n)$ is a bounded sequence, it follows from the Weierstrass Criterion that $f = (f_0, f_1, f_2, \ldots)$ is in $\ell_1$ if and only if $\text{Re}(1/\alpha) > 1$.

(iii) If $\alpha = 1$, then it is easy to show that the corresponding eigen vector is $(f_0, 0, 0, \ldots) \in \ell_1$.

Thus, one can easily see by combining the discussions in (i), (ii) and (iii), above, that

$$\sigma_p(C_1^*, c_0^*) = \left\{ \alpha \in \mathbb{C} : \text{Re} \left( \frac{1}{\alpha} \right) > 1 \right\} \cup \{1\}$$

$$\equiv \left\{ \alpha \in \mathbb{C} : \left| \alpha - \frac{1}{2} \right| < \frac{1}{2} \right\} \cup \{1\}$$

which is what we wished to prove. $\square$

**Theorem 2.5.** $\sigma_p(C_1, c_0) = \sigma_p(C_1^*, \ell_1)$.

**Proof.** If $\alpha \neq 1/(m+1)$ for each $m \in \mathbb{N}$, then $C_1 - \alpha I$ is a triangle matrix. Hence, $(C_1 - \alpha I)^{-1}$ exists. In the case $\alpha = 1/(m+1)$, $(C_1 - \alpha I)x = \theta$ yields that

$$x_1 = x_2 = \cdots = x_{m-1} = 0 \quad \text{and} \quad x_{m+k} = \binom{m+k}{m} x_m$$

for all $k \in \mathbb{N}$. Thus, we have

$$x = (x_m) \in c_0 \iff x_m = 0$$

which means that $x = \theta$. Hence, $C_1 - \alpha I$ is one-to-one operator on the space $c_0$. Therefore, $C_1 - \alpha I$ has an inverse. Now, it is deduced from the arguments, above, that the operator $C_1 - \alpha I$ has an inverse for $\alpha \in \sigma_p(C_1^*, \ell_1)$. Since $C_1^* - \alpha I$ is not one-to-one by Theorem 2.4, Lemma 1.2 gives the fact that the range $C_1 - \alpha I$ is not dense in $c_0$. $\square$

**Theorem 2.6.** $\sigma_c(C_1, c_0) = \{ \alpha \in \mathbb{C} : \left| \alpha - \frac{1}{2} \right| = \frac{1}{2}, \alpha \neq 1 \}$.

**Proof.** Let $\alpha \in \mathbb{C}$ such that $\left| \alpha - \frac{1}{2} \right| = \frac{1}{2}$ and $\alpha \neq 1$. Since $\alpha \neq 1/(m+1)$ for each $m \in \mathbb{N}$, $C_1 - \alpha I$ is triangle and hence has an inverse.

Consider the adjoint operator $C_1^* - \alpha I$. Then, we obtain by the linear system of homogenous equations in the matrix form $(C_1^* - \alpha I)f = \theta$ that

$$f_n = \left\{ \prod_{k=0}^{n-1} \left[ 1 - \frac{1}{(k+1)\alpha} \right] \right\} f_0 \quad \text{for} \quad n \geq 1.$$ 

Since $\text{Re}(1/\alpha) = 1$, we have

$$f = (f_0, f_1, f_2, \ldots) \in \ell_1 \iff f = \theta.$$ 

Hence, $C_1^* - \alpha I$ is one-to-one. From Lemma 1.2, the range of $C_1 - \alpha I$ is dense in $c_0$. This step completes the proof. $\square$

Combining Theorems 2.4-2.6, we have the following main theorem:
Theorem 2.7. If $C_1$ is Cesàro operator and $C_1 : c_0 \to c_0$, then

(a) $\sigma(C_1, c_0) = \{ \alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| \leq \frac{1}{2} \}$,
(b) $\sigma_p(C_1, c_0) = \emptyset$,
(c) $\sigma_c(C_1, c_0) = \{ \alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| = \frac{1}{2}, \alpha \neq 1 \}$,
(d) $\sigma_r(C_1, c_0) = \{ \alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| < \frac{1}{2} \} \cup \{1\}$.

Now, we may give the corresponding theorems to Theorem 2.7 on the fine spectrum of the Cesàro operator in the sequence space $c$ which can be proved in the similar way to that of the space $c_0$, above. We should record here that the adjoint operator $C_1^*$ on the dual space $c^* = \mathbb{C} \oplus \ell_1$ of the sequence space $c$ is represented by the matrix of the form

$$
\begin{bmatrix}
1 & 0 \\
0 & C_1^t
\end{bmatrix},
$$

where $C_1^t$ denotes the transpose of the matrix $C_1$. So, we omit the detail.

Theorem 2.8. If $C_1$ is Cesàro operator and $C_1 : c \to c$, then

(a) $\sigma(C_1, c) = \sigma(C_1, c_0)$,
(b) $\sigma_p(C_1, c) = \{1\}$,
(c) $\sigma_c(C_1, c) = \{ \alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| = \frac{1}{2} \}$,
(d) $\sigma_r(C_1, c) = \{ \alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| < \frac{1}{2} \}$.

Since $\sigma(A, \ell_\infty) = \sigma(A, c)$ by Cartlidge [5] whenever a matrix operator $A$ is bounded on $c$, as an easy consequence of Theorem 2.8, we have:

Theorem 2.9. If $C_1$ is Cesàro operator and $C_1 : \ell_\infty \to \ell_\infty$, then $\sigma(C_1, \ell_\infty) = \sigma(C_1, c)$.

Remark. We should remark the reader from now on that the parts (a) and (b) of Theorem 2.7 are established in [14] and [16]. In contrast to [14, Theorem 2], we have showed that $\sigma_p(C_1^*, c_0^* \cong \ell_1)$ does not consist of only the points of $\alpha \in \mathbb{C}$ such that $|\alpha - \frac{1}{2}| < \frac{1}{2}$, and the point $\alpha = 1$ must also be added to them.

After giving the concept of Mercerian theorem, we state and prove a Mercerian theorem. Let $A$ be an infinite matrix and the set $c_A$ denotes the summability field of that matrix $A$. A theorem which proves that $c_A = c$ is called a Mercerian theorem, after Mercer, who proved a significant theorem of this type [11, p. 186].

Now, we may give our final theorem.

Theorem 2.10. Suppose that $\alpha$ is in the set

$$
\{ \alpha \in \mathbb{C} : |\alpha + 1| > |\alpha - 1| \} = \{ \alpha \in \mathbb{C} : \text{Re}(\alpha) > 0 \}.
$$

Then the convergence field of $A = \alpha I + (1 - \alpha)C_1$ is the space $c$.

Proof. Let us write $A = (\beta I - C_1) / (\beta - 1)$ with $\beta = \alpha / (\alpha - 1)$. Then, the inverse of the operator $A = (\beta I - C_1) / (\beta - 1)$ exists and continuous whenever $\beta \in \rho(C_1, c)$. This yields the desired fact that $|\alpha + 1| > |\alpha - 1|$ which completes the proof.
We conclude the work by expressing that the aim of the next paper is to examine the fine spectrum, in the sense of the present paper and Akhmedov and Başar [1] of the Cesaro operator in the sequence spaces \( r_0^t \), \( r_p^t \) and \( bv_p \) which are linearly isomorphic to the spaces \( c_0 \) and \( \ell_p \), and defined by

\[
\begin{align*}
\mathcal{V}_0^t &= \left\{ (x_k) : \lim_{n \to \infty} \sum_{k=0}^{n} \frac{t_k x_k}{t_0 + t_1 + \cdots + t_n} = 0 \right\}, \\
\mathcal{V}_p^t &= \left\{ (x_k) : \sum_{k=0}^{\infty} \left| \frac{t_k x_k}{t_0 + t_1 + \cdots + t_k} \right|^p < \infty \right\}, \quad (1 \leq p < \infty),
\end{align*}
\]

where \( t = (t_k) \) is a bounded sequence of positive numbers, and

\[
\begin{align*}
\mathcal{V}_b^p &= \left\{ x = (x_k) \in \mathcal{W} : \sum_{k} \left| x_k - x_{k-1} \right|^p < \infty \right\}, \quad (1 \leq p < \infty),
\end{align*}
\]

and studied by Malkowsky [12], Altay and Başar [2], and Başar and Altay [3], respectively.

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