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On the fine spectrum of the Cesàro operator in c_0

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ABSTRACT. In the present paper, the fine spectrum of the Cesàro operator in the sequence space c_0 has been examined. Although the discussion is made for determination of the spectrum of the Cesàro operator in the sequence space c_0 by Reade [14] and the others, our consequences are more refinement and include a remark concerning with the previous works. Further, a Mercerian theorem has also been given. Finally, the fine spectrum of the Cesàro operator in the sequence space c has been given, without proof.

1. Preliminaries, Background and Notation.

Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathcal{D}(T) \to X$ also be a linear operator. With T, we associate the operator

$$T_\alpha = T - \alpha I, \tag{1.1}$$

where α is a complex number and I is the identity operator on $\mathcal{D}(T)$. If T_α has an inverse, which is linear, we denote it by $R_\alpha(T)$, that is

$$R_\alpha(T) = T_\alpha^{-1} = (T - \alpha I)^{-1} \tag{1.2}$$

and call it the *resolvent operator* of T. We also write simply R_α instead of $R_\alpha(T)$ if it is clear to what operator T we refer in a specific discussion.

The name *resolvent* is appropriate, since $R_\alpha(T)$ helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y = R_\alpha(T) y$ provided $R_\alpha(T)$ exists. More important, the investigation of properties of R_α will be basic for an understanding of the

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operator T itself. Naturally, many properties of T_α and R_α depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α in the complex plane such that R_α exists. Boundedness of R_α is another property that will be essential. We shall also ask for what α's the domain of R_α is dense in X, to name just a few aspects. For our investigation of T, T_α and R_α, we need some basic concepts in spectral theory which are given as follows (see [[]9, pp. 370-371]):

Definition 1.1. Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathcal{D}(T) \to X$ also be a linear operator with domain $\mathcal{D}(T) \subset X$. A *regular value* α of T is a complex number such that

(R1) $R_\alpha(T)$ exists,
(R2) $R_\alpha(T)$ is bounded,
(R3) $R_\alpha(T)$ is defined on a set which is dense in X.

The *resolvent set* $\rho(T,\ X)$ of T is the set of all regular values α of T. Its complement $\sigma(T,\ X) = \mathbb{C}\backslash\rho(T,\ X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,\ X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T,\ X)$ is the set such that $R_\alpha(T,\ X)$ does not exist. A $\alpha \in \sigma_p(T,\ X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,\ X)$ is the set such that $R_\alpha(T,\ X)$ exists and satisfies (R3) but not (R2), that is, $R_\alpha(T,\ X)$ is unbounded.

The *residual spectrum* $\sigma_r(T,\ X)$ is the set such that $R_\alpha(T,\ X)$ exists (and may be bounded or not) but not satisfy (R3), that is, the domain of $R_\alpha(T,\ X)$ is not dense in X.

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,\ X) = \sigma_r(T,\ X) = \emptyset$ and the spectrum $\sigma(T,\ X)$ consists of only the set $\sigma_p(T,\ X)$ in the finite dimensional case.

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called as a *sequence space.* We shall write ℓ_∞, c, c_0 and bv for the sequence spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively. Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,\ k \in \mathbb{N} = \{0,\ 1,\ 2, \dots\}$. Then, we write $Ax = \big((Ax)_n\big)$, the A-transform of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. If $x \in \lambda$ implies that $Ax \in \mu$ then we say that A defines a matrix mapping from λ into μ and denote it by $A : \lambda \to \mu$. By $(\lambda : \mu)$, we mean the class of all infinite matrices such that $A : \lambda \to \mu$.

Now, we may give the following lemma requiring in the proof of theorems given in Section 2, below:

Lemma 1.2[6, p. 59]. *T has a dense range if and only if T^* is one to one, where T^* denotes the dual operator of that operator T.*

The Cesàro operator is represented by the matrix

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \dots & \frac{1}{n+1} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \ddots \end{bmatrix}.$$

It is known that

(i) $C_1 : c_0 \to c_0$ is a linear operator,
(ii) $\|C_1\|_{(c_0:c_0)} = 1$,
(iii) $C_1^* : \ell_1 \to \ell_1$ and is represented by the matrix

$$C_1^* = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n+1} & \dots \\ 0 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n+1} & \dots \\ 0 & 0 & \frac{1}{3} & \dots & \frac{1}{n+1} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{n+1} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \ddots \end{bmatrix},$$

where C_1^* is the dual operator of C_1.

The fine spectrum of the Cesàro operator on the sequence space ℓ_p has been studied by Gonzàlez [7], where $1 < p < \infty$. The spectrum of the Cesàro operator on the sequence spaces c_0 and bv have also been investigated by Reade [14] and Okutoyi [13], respectively. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c have been examined by Yıldırım [17]. The fine spectrum of the Cesàro operator on the sequence space ℓ_p have recently been studied by Akhmedov and Başar [1], independently than that of Gonzàlez [7], by the different way; where $1 < p < \infty$. More recently, the fine spectrum of the difference operator Δ on the sequence spaces c_0 and c has been worked by Altay and Başar [4]. The spectrum of the Cesàro operator on the sequence spaces c_0 and c are studied [14, 18]. In this work, our purpose is to investigate the fine spectrum of the Cesàro operator in the sequence spaces c_0 and c which is the natural continuation of Akhmedov and Başar [1].

2. The Fine Spectrum of the Cesàro Operator in the Sequence Spaces c_0 and c.

In this section, the fine spectrum of the Cesàro operator in the sequence spaces c_0 and c has been examined. Furthermore, a Mercerian theorem has also been stated and proved.

We shall begin with quoting some lemmas that the first of them is *Weierstrass Criterion* and is needed in the proof of Theorem 2.4, below, and the others summarize the consequences on the spectrum of the Cesàro operator in the sequence space c_0, in the existing literature.

Lemma 2.1[8, Weierstrass Criterion; pp. 398–399]. *A series* $\sum_{n=0}^{\infty} a_n$ *of complex terms, for which*

$$\frac{a_{n+1}}{a_n} = 1 - \frac{b}{n} - \frac{A_n}{n^d}$$

with A_n *bounded, is absolutely convergent if and only if* $Re(b) > 1$; *where* b *is a complex number,* $d > 1$ *and* $Re(b)$ *denotes the real part of the complex number* b. *For* $Re(b) \leq 0$ *the series is invariably divergent. If* $0 < Re(b) \leq 1$, *both the series*

$$\sum_{n=0}^{\infty} \left|a_n - a_{n+1}\right| \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n a_n$$

are convergent.

Lemma 2.2[14, Theorem 3]. $\sigma(C_1,\ c_0) = \{\alpha : |\alpha - \frac{1}{2}| \leq \frac{1}{2}\}$.

Lemma 2.3[16]. $\sigma_p(C_1,\ c_0) = \emptyset$.

Theorem 2.4. $\sigma_p(C_1^*,\ c_0^* \cong \ell_1) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right\} \cup \{1\}$.

Proof. Suppose that $C_1^* f = \alpha f$ for $f = (f_0,\ f_1,\ f_2, \dots) \neq \theta$ in $c_0^* \cong \ell_1$. Then, by solving the system of linear equations

$$\left.\begin{aligned} f_0 + \frac{1}{2}f_1 + \frac{1}{3}f_2 + \frac{1}{4}f_3 + \cdots &= \alpha f_0 \\ \frac{1}{2}f_1 + \frac{1}{3}f_2 + \frac{1}{4}f_3 + \cdots &= \alpha f_1 \\ \frac{1}{3}f_2 + \frac{1}{4}f_3 + \cdots &= \alpha f_2 \\ &\ \vdots \end{aligned}\right\}$$

we obtain that

$$f_n = \left[\prod_{k=1}^{n} \left(1 - \frac{1}{k\alpha}\right)\right] f_0 \ ; \ (n \geq 1).$$

(i) If m is the first integer for which $f_m \neq 0$, then $\alpha = 1/(m+1)$. So, $\alpha = 1/(m+1)$ is an eigenvalue with the corresponding eigenvector

$$f = \left[\sum_{k=0}^{m} (-1)^{k-1} \binom{m}{k} e^{(k)}\right] f_0 \in \ell_1;$$

where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$.

(ii) If $f_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\frac{f_{n+1}}{f_n} = 1 - \frac{1/\alpha}{n} - \frac{A_n}{n^2} \ ,$$

where $A_n = -n/\alpha(n+1)$. Since (A_n) is a bonded sequence, it follows from the Weierstrass Criterion that $f = (f_0, f_1, f_2, \dots)$ is in ℓ_1 if and only if $Re(1/\alpha) > 1$.

(iii) If $\alpha = 1$, then it is easy to show that the corresponding eigen vector is $(f_0, 0, 0, \dots) \in \ell_1$.

Thus, one can easily see by combining the discussions in (i), (ii) and (iii), above, that

$$\begin{aligned}\sigma_p(C_1^*, c_0^*) &= \left\{\alpha \in \mathbb{C} : Re\Big(\frac{1}{\alpha}\Big) > 1\right\} \cup \{1\} \\ &\equiv \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right\} \cup \{1\}\end{aligned}$$

which is what we wished to prove. □

Theorem 2.5. $\sigma_r(C_1, c_0) = \sigma_p(C_1^*, \ell_1)$.

Proof. If $\alpha \neq 1/(m+1)$ for each $m \in \mathbb{N}$, then $C_1 - \alpha I$ is a triangle matrix. Hence, $(C_1 - \alpha I)^{-1}$ exists. In the case $\alpha = 1/(m+1)$, $(C_1 - \alpha I)x = \theta$ yields that

$$x_1 = x_2 = \cdots = x_{m-1} = 0 \quad \text{and} \quad x_{m+k} = \binom{m+k}{m} x_m$$

for all $k \in \mathbb{N}$. Thus, we have

$$x = (x_m) \in c_0 \Longleftrightarrow x_m = 0$$

which means that $x = \theta$. Hence, $C_1 - \alpha I$ is one-to-one operator on the space c_0. Therefore, $C_1 - \alpha I$ has an inverse. Now, it is deduced from the arguments, above, that the operator $C_1 - \alpha I$ has an inverse for $\alpha \in \sigma_p(C_1^*, \ell_1)$. Since $C_1^* - \alpha I$ is not one-to-one by Theorem 2.4, Lemma 1.2 gives the fact that the range $C_1 - \alpha I$ is not dense in c_0. □

Theorem 2.6. $\sigma_c(C_1, c_0) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| = \frac{1}{2}, \alpha \neq 1\right\}$.

Proof. Let $\alpha \in \mathbb{C}$ such that $\left|\alpha - \frac{1}{2}\right| = \frac{1}{2}$ and $\alpha \neq 1$. Since $\alpha \neq 1/(m+1)$ for each $m \in \mathbb{N}$, $C_1 - \alpha I$ is triangle and hence has an inverse.

Consider the adjoint operator $C_1^* - \alpha I$. Then, we obtain by the linear system of homogenous equations in the matrix form $(C_1^* - \alpha I)f = \theta$ that

$$f_n = \left\{\prod_{k=0}^{n-1}\left[1 - \frac{1}{(k+1)\alpha}\right]\right\} f_0 \quad \text{for} \quad n \geq 1.$$

Since $Re(1/\alpha) = 1$, we have

$$f = (f_0, f_1, f_2, \dots) \in \ell_1 \Longleftrightarrow f = \theta.$$

Hence, $C_1^* - \alpha I$ is one-to-one. From Lemma 1.2, the range of $C_1 - \alpha I$ is dense in c_0. This step completes the proof. □

Combining Theorems 2.4-2.6, we have the following main theorem:

Theorem 2.7. *If C_1 is Cesàro operator and $C_1 : c_0 \to c_0$, then*

(a) $\sigma(C_1,\ c_0) = \{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| \leq \frac{1}{2}\}$,
(b) $\sigma_p(C_1,\ c_0) = \emptyset$,
(c) $\sigma_c(C_1,\ c_0) = \{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| = \frac{1}{2}\ ,\ \alpha \neq 1\}$,
(d) $\sigma_r(C_1,\ c_0) = \{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| < \frac{1}{2}\} \cup \{1\}$.

Now, we may give the corresponding theorems to Theorem 2.7 on the fine spectrum of the Cesàro operator in the sequence space c which can be proved in the similar way to that of the space c_0, above. We should record here that the adjoint operator C_1^* on the dual space $c^* = \mathbb{C} \oplus \ell_1$ of the sequence space c is represented by the matrix of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & C_1^t \end{bmatrix},$$

where C_1^t denotes the transpose of the matrix C_1. So, we omit the detail.

Theorem 2.8. *If C_1 is Cesàro operator and $C_1 : c \to c$, then*

(a) $\sigma(C_1,\ c) = \sigma(C_1,\ c_0)$,
(b) $\sigma_p(C_1,\ c) = \{1\}$,
(c) $\sigma_c(C_1,\ c) = \{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| = \frac{1}{2}\}$,
(d) $\sigma_r(C_1,\ c) = \{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| < \frac{1}{2}\}$.

Since $\sigma(A,\ \ell_\infty) = \sigma(A,\ c)$ by Cartlidge [5] whenever a matrix operator A is bounded on c, as an easy consequence of Theorem 2.8, we have:

Theorem 2.9. *If C_1 is Cesàro operator and $C_1 : \ell_\infty \to \ell_\infty$, then $\sigma(C_1,\ \ell_\infty) = \sigma(C_1,\ c)$.*

Remark. We should remark the reader from now on that the parts (a) and (b) of Theorem 2.7 are established in [14] and [16]. In contrast to [14, Theorem 2], we have showed that $\sigma_p(C_1^*,\ c_0^* \cong \ell_1)$ does not consist of only the points of $\alpha \in \mathbb{C}$ such that $|\alpha - \frac{1}{2}| < \frac{1}{2}$, and the point $\alpha = 1$ must also be added to them.

After giving the concept of Mercerian theorem, we state and prove a Mercerian theorem. Let A be an infinite matrix and the set c_A denotes the summability field of that matrix A. A theorem which proves that $c_A = c$ is called a Mercerian theorem, after Mercer, who proved a significant theorem of this type [11, p. 186].

Now, we may give our final theorem.

Theorem 2.10. *Suppose that α is in the set*

$$\{\alpha \in \mathbb{C} : |\alpha + 1| > |\alpha - 1|\} = \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}.$$

Then the convergence field of $A = \alpha I + (1 - \alpha)C_1$ is the space c.

Proof. Let us write $A = (\beta I - C_1)/(\beta - 1)$ with $\beta = \alpha/(\alpha - 1)$. Then, the inverse of the operator $A = (\beta I - C_1)/(\beta - 1)$ exists and continuous whenever $\beta \in \rho(C_1,\ c)$. This yields the desired fact that $|\alpha + 1| > |\alpha - 1|$ which completes the proof.

We conclude the work by expressing that the aim of the next paper is to examine the fine spectrum, in the sense of the present paper and Akhmedov and Başar [1] of the Cesàro operator in the sequence spaces r_0^t, r_p^t and bv_p which are linearly isomorphic to the spaces c_0 and ℓ_p, and defined by

$$r_0^t = \left\{ (x_k) : \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t_k x_k}{t_0 + t_1 + \cdots + t_n} = 0 \right\},$$

$$r_p^t = \left\{ (x_k) : \sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} \frac{t_j x_j}{t_0 + t_1 + \cdots + t_k} \right|^p < \infty \right\}, \ (1 \le p < \infty),$$

where $t = (t_k)$ is a bounded sequence of positive numbers, and

$$bv_p = \left\{ x = (x_k) \in w : \sum_{k} \left| x_k - x_{k-1} \right|^p < \infty \right\}, \ (1 \le p < \infty),$$

and studied by Malkowsky [12], Altay and Başar [2], and Başar and Altay [3], respectively.

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