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On generalized Dedekind domains

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Introduction.

Throughout this paper the letter $D$ denote an integral domain with quotient field $K$. An ideal of $D$ means an integral ideal of $D$ and the set of all integral ideals of $D$ is denoted by $I(D)$.

According to [30, Theorem 4], an integral domain $D$ is a Prüfer domain if and only if every overring $T$ of $D$ is flat over $D$. In [28], N. Popescu defined a Prüfer domain $D$ to be a generalized Dedekind domain in case for each overring $T$ of $D$, there exists a unique localizing system $\mathcal{F}$ on $D$ such that $T = D_{\mathcal{F}}$ and he also gave some characterizations of a generalized Dedekind domain in terms of localizing systems. Generalized Dedekind domains were subsequently studied by A. Facchini, M. Fontana, J. A. Huckaba, I. J. Papick, S. Gabelli, E. L. Popescu and N. Popescu and many characterizations of a generalized Dedekind domain have been given. We shall collect these characterizations in Proposition 35.

In [24], the notion of a semistar operation was introduced as a generalization of a star operation. Recently M. Fontana and J. A. Huckaba found several intimate relations between localizing systems and semistar operations and they obtained many important results in [7].

In this paper we aim to investigate the connection between localizing systems and semistar operations and to obtain some new characterizations of a generalized Dedekind domain. The main results of this paper are Proposition 58, Theorem 60, Corollary 63, Lemma 68, Theorems 69 and 70. In Theorem 69, some new equivalent conditions for a Prüfer domain to be a generalized Dedekind domain will be given and in Theorem 70, some equivalent conditions for a finite dimensional valuation domain to be a generalized Dedekind domain will also be given.

In Section 1, we mainly study the connection between localizing systems and semistar operations on $D$ and we will show some fundamental properties of localizing systems and semistar operations which will be used in sequel. In this section we also introduce the notion of the stable semistar closure of a semistar operation.
In Section 2, we introduce the notion of an amply strong semistar domain (for short, ASSD) (cf. Definition 50). In Proposition 54, we give some characterizations of an ASSD. In Proposition 58, we prove that each Prüfer ASSD is a generalized Dedekind domain. But, the converse is not necessarily true. In fact, in Remark 71, we shall show that there exists a generalized Dedekind domain which is not a Prüfer ASSD. Furthermore, we introduce the notion of a stable semistar domain (for short, SSD) (cf. Definition 65) and in Theorem 69, we shall show that a Prüfer domain $D$ is a generalized Dedekind domain if and only if $D$ is an SSD. In Theorem 60, we shall give a necessary and sufficient condition for an integral domain to be a conducive domain. In Example 72, we shall show that there exists an example of a non-integrally closed ASSD.

In this paper the set of prime ideals (respectively, maximal ideals) of $D$ is denoted by $\text{Spec}(D)$ (respectively, $\text{Max}(D)$) and the cardinality of a set $X$ is denoted by $|X|$. Any other unexplained terminology is standard as in [17].

1. Localizing systems and semistar operations.

In this paper, we shall denote the set of all non-zero $D$-submodules of $K$ by $K(D)$ and we shall call each element of $K(D)$ a Kaplansky fractional ideal (for short, $K$-fractional ideal) of $D$ as in [25]. Let $F(D)$ be the set of all nonzero fractional ideals of $D$, that is, all elements $E \in K(D)$ such that there exists a nonzero element $d \in D$ with $dE \subseteq D$. The set of finitely generated $K$-fractional ideals of $D$ is denoted by $f(D)$. It is evident that $f(D) \subseteq F(D) \subseteq K(D)$. The set of all integral ideals of $D$ is denoted by $I(D)$.

In [24], we introduced the notion of a semistar operation on $D$ as a generalization of a star operation:

A map $E \mapsto E^*$ of $K(D)$ into $K(D)$ is called a semistar operation if the following conditions hold for all $a \in K \setminus \{0\}$ and $E, F \in K(D)$:

(S1) $(aE)^* = aE^*$;
(S2) If $E \subseteq F$, then $E^* \subseteq F^*$; and
(S3) $E \leq E^*$ and $(E^*)^* = E^*$.

As in [7], we shall denote the set of all semistar operations on $D$ by $S\text{Star}(D)$. A semistar operation $*$ on $D$ is said to be of finite type if $E^* = \bigcup\{F^* \mid F \subseteq E$ and $F \in f(D)\}$ for each $E \in K(D)$. The set of all semistar operations of finite type on $D$ is denoted by $S\text{Star}_f(D)$.

A semistar operation $*$ on $D$ is said to be weak if $D^* = D$. We shall denote the set of all weak semistar operations on $D$ by $S\text{Star}_w(D)$.

For every $E, F \in K(D)$, the set $\{x \in K \mid xE \subseteq F\}$ is denoted by $F :_K E$, or simply by $F : E$. Evidently $F : E$ is also a $K$-fractional ideal of $D$. If we set $F :_D E = (F :_K E) \cap D$, then $F :_D E$ is an integral ideal of $D$.

For each $E \in K(D)$, we set $E^{-1} = (D : E)$ and $E^g = (E^{-1})^{-1}$. Then the map $E \mapsto E^g$ is a semistar operation on $D$ and is called the $g$-operation on $D$. If we set $E^t = \bigcup\{F^t \mid F \subseteq E \text{ and } F \in f(D)\}$ for each $E \in K(D)$, then the map $E \mapsto E^t$ is a semistar operation on $D$ and is called the $t$-operation on $D$. The $t$-operation is of finite type.
In [13], the notion of a localizing (or topologizing) system of ideals was introduced by Gabriel. A set $\mathcal{F}$ of ideals of $D$ is called a localizing system of ideals (for short, localizing system) if the following conditions are satisfied:

(LS1) If $I \in \mathcal{F}$ and $I \subseteq J \subseteq I(D)$, then $J \in \mathcal{F}$;
(LS2) If $I \in \mathcal{F}$ and $J$ is an ideal of $D$ such that $J :_D iD \in \mathcal{F}$ for all $i \in I$, then $J \in \mathcal{F}$.

A general reference for localizing systems of ideals is [27, Section 4.9].

A localizing system $\mathcal{F}$ on $D$ is said to be of finite type if for each $I \in \mathcal{F}$, there exists a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$.

If $I, J \in \mathcal{F}$, then $IJ \subseteq \mathcal{F}$ and hence $I \cap J \subseteq \mathcal{F}$ by (LS1) (see [6, Proposition 5.1.1]). Thus every localizing system becomes a generalized multiplicative system.

We shall denote the set of localizing systems on $D$ by $\mathcal{LS}(D)$ and the set of localizing systems of finite type on $D$ by $\mathcal{LS}_f(D)$.

Let $F$ be a localizing system on $D$. If we set $DF = \{x \in K \mid D :_D x \in \mathcal{F}\}$, then $DF$ is a subring of $K$ and is called the quotient ring of $D$ relative to $F$ (see [28, p. 778]). It is easy to see that $DF = \mathcal{F}(D)$ [6, p. 126].

Let $T$ be an overring of $D$. If we set $\mathcal{F}(T) = \{I \in \mathcal{F} \mid IT = T\}$, then $\mathcal{F}(T)$ is a localizing system [28, Proposition 1.2(i)]. It is easy to see that $\mathcal{F}(T)$ is a localizing system of finite type. Here we note that $\mathcal{F}(T)$ is denoted by $\mathcal{F}_1(T)$ in [4] and $\mathcal{F}_0(T)$ in [6].

It is easily seen that if $F_1 \subseteq F_2$ are localizing systems on $D$, then $DF_1 \subseteq DF_2$ and if $T_1 \subseteq T_2$ are overrings of $D$, then $\mathcal{F}(T_1) \subseteq \mathcal{F}(T_2)$.

To begin with we state some basic but useful results concerning both localizing systems and semistar operations.

**Lemma 1.** (1) For each overring $T$ of $D$, $DF(T) \subseteq T$.
(2) If $\mathcal{F}$ is a localizing system on $D$, then $\mathcal{F}(DF) \subseteq \mathcal{F}$.

**Proof.** (1) If we choose $x \in DF(T)$, then $x \in D :_K I$ for some $I \in \mathcal{F}(T)$. Hence $xI \subseteq D$ and then $xIT \subseteq T$, and so $x \in T$ because $IT = T$.

(2) Choose $I \in \mathcal{F}(DF)$. Then $ID_F = DF$ and then $1 = \sum_{k=1}^{n} a_k x_k$ with $a_k \in I, x_k \in DF$. Since $x_k \in DF$, there exists $I_k \in \mathcal{F}$ such that $x_k I_k \subseteq D$ for each $k = 1, 2, \ldots, n$. Hence if we set $J = I_1 I_2 \cdots I_n$, then $J \in \mathcal{F}$ and $J = \sum_{k=1}^{n} a_k (x_k J) \subseteq I$, and therefore $I \in \mathcal{F}$.

**Remark 2.** (1) Let $T$ be an overring of $D$. If $T = DF$ for some localizing system $\mathcal{F}$ on $D$, then $\mathcal{F}(T) \subseteq \mathcal{F}$. In fact, by Lemma 1 (2), $\mathcal{F}(T) = \mathcal{F}(DF) \subseteq \mathcal{F}$.
(2) Let $T$ be an overring of $D$. If $T = DF(T)$, then $\mathcal{F}(T) = \mathcal{F}(DF(T))$.

**Remark 3** (cf. [6, Remark 5.1.11 (b)]). Let $T$ be an overring of $D$. Then the following statements are equivalent.
(1) $T$ is flat over $D$;
(2) There exists a generalized multiplicative system $S$ of $D$ such that $T = DS$ and $IT = T$ for every $I \in S$;
(3) There exists a localizing system $\mathcal{F}$ of finite type on $D$ such that $T = DF$ and $IT = T$ for every $I \in \mathcal{F}$;
(4) \( T = D_{F(T)} \).

Let \( F \) be a localizing system on \( D \). Then it follows immediately from Remark 3 that \( D_{F} = D_{F(D_{F})} \) if and only if \( D_{F} \) is flat over \( D \). Hence, if \( D \) is a Prüfer domain, then \( D_{F} = D_{F(D_{F})} \) for each \( F \in LS(D) \).

**Proposition 4**([7, Proposition 2.8]). If \( \ast \) is a semistar operation on \( D \), then \( F^{\ast} = \{ I \in I(D) | I^\ast = D^\ast \} \) is a localizing system on \( D \).

\( F^{\ast} \) is called the localizing system associated to a semistar operation \( \ast \). It is easy to see that if \( \ast_{1} \leq \ast_{2} \) in \( SStar(D) \), then \( F^{\ast_{1}} \subseteq F^{\ast_{2}} \).

**Proposition 5.** (1) Let \( T \) be an overring of \( D \). If we set \( E^\ast(T) = ET \) for each \( E \in K(D) \), then \( \ast(T) \) is a semistar operation on \( D \).

(2) \( F^{\ast(T)} = F(T) \) for every overring \( T \) of \( D \).

**Proof.** (1) This is [24, Corollary 11].

(2) Since \( I^{\ast(T)} = IT \) for each \( I \in I(D) \) and \( D^{\ast(T)} = T \), \( IT = T \) if and only if \( I^{\ast(T)} = D^{\ast(T)} \) and therefore \( F^{\ast(T)} = F(T) \). \( \square \)

In [7] a semistar operation \( \ast \) on \( D \) is said to be stable if \( (E \cap F)^\ast = E^\ast \cap F^\ast \) for all \( E, F \in K(D) \). In [24] this property is said to be quotient. We shall denote the set of stable semistar operations on \( D \) by \( SStar_{\sigma}(D) \).

Here we recall two trivial examples of stable semistar operations. We set \( E^d = E \) (respectively, \( E^e = K \)) for each \( E \in K(D) \). Then \( d \) (respectively, \( e \)) is a semistar operation on \( D \) and is called the \( d \)-operation (respectively, the \( e \)-operation) on \( D \).

It is easy to see that \( d \) and \( e \) are stable semistar operations on \( D \) and that \( d = \ast_{(D)} \) and \( e = \ast_{(K)} \).

Non-trivial examples of stable semistar operations were given in [24, Theorem 20] and furthermore, in [24, Theorem 28], a necessary and sufficient condition for a semistar operation of finite type to be stable was given.

**Lemma 6** (cf. [6, proof of Proposition 5.1.10]). Let \( F \) be a localizing system on \( D \). Then

(1) For each overring \( T \) of \( D \), \( \alpha_T(F) = \{ I' \in I(T) | I' \supseteq JT \text{ for some } J \in F \} \) is a localizing system on \( T \).

(2) If \( F \) is of finite type, then \( \alpha_T(F) \) is also of finite type.

**Proof.** The proof is straightforward and is omitted. \( \square \)

It would be worthwhile to note that if \( I' \in \alpha_T(F) \), then \( I' \cap D \in F \).

**Lemma 7** (cf. [28, Corollary 2.3]). Let \( F \) be a localizing system on an overring \( T \) of \( D \). Then

(1) \( \delta_D(F) = \{ I \in I(D) | IT \in F \} \) is a localizing system on \( D \) and \( D_{\delta_D(F)} \subseteq T_F \).

(2) If \( F \) is of finite type, then \( \delta_D(F) \) is also of finite type.

**Proof.** (1) The first statement is clear. For the second statement, let \( x \in D_{\delta_D(F)} \). Then \( xI \subseteq D \) for some \( I \in \delta_D(F) \) and so \( xIT \subseteq T \). Therefore \( x \in T :_K IT \subseteq T_F \) because \( IT \in F \).
(2) Let \( I \in \delta_D(\mathcal{F}) \). Then \( IT \in \mathcal{F} \) and then there exists a finitely generated ideal \( J \in \mathcal{F} \) such that \( J \subseteq IT \). Then we can choose \( a_1, \ldots, a_n \in I \) such that \( J = a_1T + \cdots + a_nT \). If we set \( I' = (a_1, \ldots, a_n) \subseteq I \), then \( I' \in \delta_D(\mathcal{F}) \) and \( I' \subseteq I \), because \( I'T = J \in \mathcal{F} \). \( \square \)

\( \alpha_T(\mathcal{F}) \) is called the extension of \( \mathcal{F} \) to \( T \) and \( \delta_D(\mathcal{F}) \) is called the restriction of \( \mathcal{F} \) to \( D \).

It would be worthwhile to note that for each overring \( T \) of \( D \) and each \( \mathcal{F} \in \mathcal{LS}(D) \), \( \mathcal{F} \subseteq \delta_D(\alpha_T(\mathcal{F})) \) always holds.

**Proposition 8** ([24, Lemma 45]). Let \( T \) be an overring of \( D \). Then

1. For each \( * \in SStar(T) \), if we set \( E^{\delta_D(*)} = (ET)^* \) for every \( E \in K(D) \), then \( \delta_D(*) \) is a semistar operation on \( D \).
2. For each \( * \in SStar(D) \), if we set \( E^{\alpha_T(*)} = E^* \) for every \( E \in K(T) (\subseteq K(D)) \), then \( \alpha_T(*) \) is a semistar operation on \( T \).
3. If we define the map \( \delta_{T/D} : SStar(T) \to SStar(D) \) by \( \delta_{T/D}(*) = \delta_D(*) \) for each \( * \in SStar(T) \) and the map \( \alpha_{T/D} : SStar(D) \to SStar(T) \) by \( \alpha_{T/D}(*) = \alpha_T(*) \) for each \( * \in SStar(D) \), then \( \alpha_{T/D} \circ \delta_{T/D} \) is the identity map of \( SStar(T) \) and hence \( \delta_{T/D} \) is an injective map.

The map \( \delta_{T/D} \) (respectively, \( \alpha_{T/D} \)) is called the descent map (respectively, the ascent map).

**Lemma 9.** Let \( T \) be an overring of \( D \) and let \( * \in SStar(D) \). Then

\[ \alpha_T(\mathcal{F}^*) = \mathcal{F}^{\alpha_T(*)} = \{ I \in I(T) \mid I^* = T^* \} \]

**Proof.** Let \( I \in \alpha_T(\mathcal{F}^*) \). Then \( I \supseteq JT \) for some \( J \in \mathcal{F}^* \). Then \( J^* = D^* \) and then \( I^* \supseteq (JT)^* = (J^*T)^* = (D^*T)^* = T^* \), that is, \( I^* = T^* \).

Conversely assume that \( I^* = T^* \) with \( I \in I(T) \). Then \( I^* \cap D = T^* \cap D = D \) and therefore \( (I \cap D)^* = (I^* \cap D)^* = (T^* \cap D)^* = (T \cap D)^* = D^* \). Hence \( I \cap D \in \mathcal{F}^* \). Since \( I \supseteq (I \cap D)T \), we have \( I \in \alpha_T(\mathcal{F}^*) \).

Next, let \( I \in I(T) \). Then \( I \in \mathcal{F}^{\alpha_T(*)} \) if and only if \( I^{\alpha_T(*)} = T^{\alpha_T(*)} \), i.e. \( I^* = T^* \). \( \square \)

**Lemma 10.** Let \( T \) be an overring of \( D \) and let \( * \in SStar(T) \). Then

\[ \delta_D(\mathcal{F}^*) = \mathcal{F}^{\delta_D(*)} = \{ I \in I(D) \mid (IT)^* = T^* \} \]

**Proof.** Let \( I \in I(D) \). First, \( I \in \mathcal{F}^{\delta_D(*)} \) if and only if \( I^{\delta_D(*)} = D^{\delta_D(*)} \), i.e. \( (IT)^* = (DT)^* = T^* \). Next, \( I \in \mathcal{F}^{\delta_D(\mathcal{F}^*)} \) if and only if \( IT \in \mathcal{F}^* \), i.e. \( (IT)^* = T^* \). Hence our assertion follows. \( \square \)

**Proposition 11** ([7, Proposition 2.4 and Theorem 2.10(A)]). Let \( \mathcal{F} \) be a localizing system on \( D \). Then

1. We set \( E_\mathcal{F} = \bigcup \{ E :_K J \mid J \in \mathcal{F} \} \) for each \( E \in K(D) \). If we define \( E^{*_\mathcal{F}} = E_\mathcal{F} \) for each \( E \in K(D) \), then \( *_\mathcal{F} \) is a stable semistar operation on \( D \).
2. If \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \) in \( \mathcal{LS}(D) \), then \( *_{\mathcal{F}_1} \leq *_{\mathcal{F}_2} \).
3. For each \( \mathcal{F} \in \mathcal{LS}(D) \), then \( \mathcal{F} = \mathcal{F}^{*_{\mathcal{F}}} \).

The semistar operation \( *_{\mathcal{F}} \) is called the semistar operation associated to a localizing system \( \mathcal{F} \).
We can construct a stable semistar operation \( \ast_{\mathcal{F}} \) for each semistar operation \( \ast \) on \( D \) by Proposition 11. In general, \( \ast_{\mathcal{F}} \) is a non-trivial stable semistar operation and is denoted by \( \ast \) in [7, p.181].

Now we can define a partial order \( \leq \) on \( \text{SStar}(D) \) in the following way:

\[ \ast_1 \leq \ast_2 \iff E^{\ast_1} \subseteq E^{\ast_2} \text{ for each } E \in \mathbf{K}(D). \]

For \( \ast_1, \ast_2 \in \text{SStar}(D) \), it is easily seen that \( \ast_1 \leq \ast_2 \) if and only if \( (E^{\ast_1})^{\ast_2} = E^{\ast_2} \) for all \( E \in \mathbf{K}(D) \) (see [24, p. 6]).

**Lemma 12.** Let \( T \) be an overring of \( D \) and let \( \mathcal{F} \) be a localizing system on \( D \). Then

\[ \alpha_T(\ast_{\mathcal{F}}) = \ast_{\alpha_T(\mathcal{F})} \text{ in } \text{SStar}(T). \]

**Proof.** Let \( E \in \mathbf{K}(T) (\subseteq \mathbf{K}(D)) \). Then \( E^{\alpha_T(\ast_{\mathcal{F}})} = E^{\ast_{\mathcal{F}}} = \bigcup\{E : K J | J \in \mathcal{F}\} \) and \( E^{\ast_{\alpha_T(\mathcal{F})}} = \bigcup\{E : K J' | J' \in \alpha_T(\mathcal{F})\} \). Now if \( J' \in \alpha_T(\mathcal{F}) \), then \( J' \supseteq JT \) for some \( J \in \mathcal{F} \) and moreover, \( E : K J = E : K JT \). Hence \( E^{\ast_{\alpha_T(\mathcal{F})}} = \bigcup\{E : K J' | J' \in \alpha_T(\mathcal{F})\} = \bigcup\{E : K JT | J \in \mathcal{F}\} = E^{\alpha_T(\ast_{\mathcal{F}})} \) and so \( \alpha_T(\ast_{\mathcal{F}}) = \ast_{\alpha_T(\mathcal{F})} \). \( \square \)

**Lemma 13.** Let \( T \) be an overring of \( D \) and let \( \mathcal{F} \) be a localizing system on \( T \). Then \( \ast_{\delta_D(\mathcal{F})} \leq \delta_D(\ast_{\mathcal{F}}) \) in \( \text{SStar}(D) \).

**Proof.** Let \( E \in \mathbf{K}(D) \). Then \( E^{\ast_{\delta_D(\mathcal{F})}} = \bigcup\{E : K J | J \in \delta_D(\mathcal{F})\} \) and \( E^{\delta_D(\ast_{\mathcal{F}})} = (ET)^{\ast_{\mathcal{F}}} = \bigcup\{ET : K I | I \in \mathcal{F}\} \). Now, if \( J \in \delta_D(\mathcal{F}) \), then \( JT \in \mathcal{F} \) and \( E : K J \subseteq (E : K JT)T \subseteq ET : K JT \). Hence \( E^{\ast_{\delta_D(\mathcal{F})}} \subseteq E^{\delta_D(\ast_{\mathcal{F}})} \) for every \( E \in \mathbf{K}(D) \) and so \( \ast_{\delta_D(\mathcal{F})} \leq \delta_D(\ast_{\mathcal{F}}) \). \( \square \)

**Proposition 14.** Let \( T \) be an overring of \( D \). Then

\[ \alpha_T(\ast_{(T)}(\mathcal{F})) = \ast_{\alpha_T(\mathcal{F}(T))} = \delta_T, \text{ where } \delta_T \text{ is the } d \text{-operation on } T. \]

**Proof.** Let \( E \in \mathbf{K}(T) \). Then \( E^{\alpha_T(\ast_{(T)}(\mathcal{F}))} = E^{\ast_{(T)}(\mathcal{F})} = \bigcup\{E : K I | I \in \mathcal{F}(T)\} \). Now, if \( I \in \mathcal{F}(T) \), then \( IT = T \) and hence if \( x I \subseteq E \), then \( xIT \subseteq TE = E \) and so \( x \in E \). Thus \( E^{\alpha_T(\ast_{(T)}(\mathcal{F}))} = E \). Next, \( E^{\ast_{\alpha_T(\mathcal{F}(T))}} = \bigcup\{E : K J' | J' \in \alpha_T(\mathcal{F}(T))\} = \bigcup\{E : K JT | J \in \mathcal{F}(T)\} = E : K T = E. \) \( \square \)

**Remark 15.** For every localizing system \( \mathcal{F} \) on \( D \), \( \ast_{(D, \mathcal{F})} \leq \ast_{\mathcal{F}} \). In fact, for each \( E \in \mathbf{K}(D) \), \( E^{\ast_{(D, \mathcal{F})}} = ED_{\mathcal{F}} = E(\bigcup\{D : K I | I \in \mathcal{F}\}) \subseteq \bigcup\{E : K I | I \in \mathcal{F}\} = E_{\mathcal{F}} = E^{\ast_{\mathcal{F}}} \) because \( E(D : K I) \subseteq E : K I \).

**Proposition 16([7, Proposition 2.6]).** Let \( \mathcal{F} \) be a localizing system on \( D \). Then the following are equivalent.

1. \( \ast_{(D, \mathcal{F})} = \ast_{\mathcal{F}} \);
2. \( ID_{\mathcal{F}} = I_{\mathcal{F}} \) for each \( I \in \mathfrak{I}(D) \);
3. \( D_{\mathcal{F}} \) is flat over \( D \) and \( \mathcal{F} = \mathcal{F}(D_{\mathcal{F}}) \).
Lemma 17([6, Lemma 5.1.2]). Let \( \{F_\alpha\} \) be a family of localizing systems on \( D \). Then the intersection \( \mathcal{F} = \bigcap\{F_\alpha\} \) is also a localizing system on \( D \) and \( D_{\mathcal{F}} = \bigcap\{D_{F_\alpha}\} \) holds.

Let \( \Delta \) be a subset of \( \text{Spec}(D) \). Then \( \mathcal{F}(\Delta) = \{I \mid I \in \mathcal{I}(D) \text{ and } I \not\subseteq P \text{ for all } P \in \Delta\} \) is a localizing system of \( D \). If \( \Delta = \{P\} \), then we denote simply by \( \mathcal{F}_P \) the localizing system \( \mathcal{F}(\{P\}) \). It is obvious that if \( \Delta = \{P_\alpha\} \), then \( \mathcal{F}(\Delta) = \bigcap\{\mathcal{F}_{P_\alpha} \mid P_\alpha \in \Delta\} \). It is also easy to show that \( D_{\mathcal{F}_P} = D_P \). Hence the following follows immediately from Lemma 17.

Proposition 18([6, Proposition 5.1.4]). Let \( \Delta = \{P_\alpha\} \) be a subset of \( \text{Spec}(D) \). Then \( D_{\mathcal{F}(\Delta)} = \bigcap\{D_{\mathcal{F}_{P_\alpha}} \mid P_\alpha \in \Delta\} = \bigcap\{D_{P_\alpha} \mid P_\alpha \in \Delta\} \).

Proposition 19(cf. [24, Proposition 9]). Let \( \Lambda = \{\alpha \mid \alpha \in A\} \) be a family of semistar operations on \( D \). If we set \( E^*(\Lambda) = \bigcap\{E^* \mid \alpha \in A\} \) for each \( E \in K(D) \), then \( \star(\Lambda) \) is a semistar operation on \( D \).

Proposition 20. Let \( \Lambda = \{\alpha \mid \alpha \in A\} \) be a family of stable semistar operations on \( D \). Then \( \star(\Lambda) \) is a stable semistar operation on \( D \).

Proof. Let \( E, F \in K(D) \). Then \( (E \cap F)^*(\Lambda) = \bigcap\{E \cap F^* \mid \alpha \in A\} = \bigcap\{E^* \cap F^* \mid \alpha \in A\} \) for each \( E \in K(D) \), then \( \star(\Lambda) \) is a stable semistar operation as required. \( \square \)

Definition 21. Let \( \star \) be a semistar operation on \( D \). We set \( \hat{\star} = \bigcap\{\alpha \mid \alpha \leq \alpha \} \text{ and } \star_\alpha \in \text{SStar}_D(D) \}. \) Then, by Proposition 20, \( \hat{\star} \) is a stable semistar operation on \( D \).

It is easy to see that \( \star \leq \hat{\star} \) and \( \hat{\star} \leq \hat{\star} \). We shall call \( \hat{\star} \) the upper stable semistar closure of \( \star \).

Proposition 22([7, Theorem 2.10(B) and Corollary 2.11]). (1) \( \hat{\star} = \star \leq \star \) for each semistar operation \( \star \) on \( D \).

(2) If \( \star_1 \leq \star_2 \) in \( \text{SStar}(D) \), then \( \hat{\star_1} \leq \hat{\star_2} \).

(3) \( \star \star \) is stable if and only if \( \star \) is a stable semistar operation.

(4) For each \( \star \in \text{SStar}(D) \), \( \hat{\star} = \hat{\star} \).

We shall call \( \hat{\star} \) the lower stable semistar closure of \( \star \).

Proposition 23(cf. [7, Proposition3.7]). (1) For each \( \star \in \text{SStar}(D) \), \( \hat{\star} \) is the largest stable semistar operation on \( D \) such that \( \star \leq \star \).

(2) For each \( \star \in \text{SStar}(D) \), \( \hat{\star} \) is the least stable semistar operation on \( D \) such that \( \star \leq \hat{\star} \).

(3) A semistar operation \( \star \) on \( D \) is stable if and only if \( \hat{\star} = \hat{\star} \).

Proposition 24. Let \( \star \) be a semistar operation on \( D \). Then

(1) \( \mathcal{F}(D^\star) \subseteq \mathcal{F}^\star \) for each \( \star \in \text{SStar}(D) \).

(2) \( \star_{\mathcal{F}(D^*)} \leq \star_{\mathcal{F}^*} \leq \star \).

Proof. (1) Let \( I \in \mathcal{F}(D^*) \). Then \( ID^* = D^* \) and then \( I^* = D^* \). Thus \( I \in \mathcal{F}^* \).

(2) This follows from Proposition 11(2). \( \square \)
Lemma 25. Let $\Delta = \{P_\alpha\}$ be a subset of $\text{Spec}(D)$. For each $E \in \mathcal{K}(D)$, we set $E^{*\Delta} = \bigcap\{EP_\alpha\ | \ P_\alpha \in \Delta\}$. Then

(1) The mapping $E \mapsto E^{*\Delta}$ of $\mathcal{K}(D)$ into $\mathcal{K}(D)$ is a stable semistar operation on $D$.

(2) For each $I \in \mathcal{I}(D)$, $I^{*\Delta} = D^{*\Delta}$ if and only if $I \not\subseteq P_\alpha$ for all $P_\alpha \in \Delta$.

(3) $ED_{P_\alpha} = E^{*\Delta}D_{P_\alpha}$ for each $E \in \mathcal{K}(D)$ and for each $P_\alpha \in \Delta$.

(4) $F^{*\Delta} = F(\Delta)$ and $\ast = \ast F^{*\Delta}$.

Proof. (1) and (2) follow from [24, Theorem 20].

(3) follows from [24, Corollary 10].

(4) First, $F^{*\Delta} = F(\Delta)$ holds by (2). Next, $\ast$ is stable by (1) and therefore $\ast_{\Delta} = \ast F^{*\Delta} = \ast F(\Delta)$ by Proposition 22(3). $\square$

Here we note that Lemma 25(4) is [7, Lemma 4.2]. We also note that the map $\Delta \mapsto \ast_{\Delta}$ is contravariant, that is, if $\Delta_1 \subseteq \Delta_2$, then $\ast_{\Delta_2} \leq \ast_{\Delta_1}$.

As in [7], a semistar operation on $D$ is said to be spectral if there exists a nonempty subset $\Delta = \{P_\alpha \ | \ \alpha \in A\}$ of $\text{Spec}(D)$ such that $*=\ast_{\Delta}$. In this case, we say that $*$ is the spectral semistar operation associated with $\Delta$.

Remark 26. (1) For every localizing system $\mathcal{F}$ on $D$, $\mathcal{F} \subseteq \bigcap\{\mathcal{F}_Q\ | \ Q \in \text{Spec}(D)$ and $Q \notin \mathcal{F}\}$ (see [6, (5.1e)]).

(2) If $\mathcal{F} = \bigcap\{\mathcal{F}_P\ | \ P \in \Delta\}$ for some subset $\Delta$ of $\text{Spec}(D)$, then $\mathcal{F} = \bigcap\{\mathcal{F}_Q\ | \ Q \in \text{Spec}(D)$ and $Q \notin \mathcal{F}\}$ (see [6, (5.1f)]).

Remark 27. For each $P \in \text{Spec}(D)$, we have $\mathcal{F}_P = \mathcal{F}(D_P) = \mathcal{F}^{*(D_P)}$. In fact, $\mathcal{F}(D_P) = \{I \in \mathcal{I}(D) \ | \ I|D_P = D_P\} = \{I \in \mathcal{I}(D) \ | \ I \not\subseteq P\} = \mathcal{F}_P$. Next, by Proposition 5(2), $\mathcal{F}(D_P) = \mathcal{F}^{*(D_P)}$. Hence we have $\mathcal{F}_P = \mathcal{F}(D_P) = \mathcal{F}^{*(D_P)}$.

Lemma 28([7, Lemma 3.1]). Let $\mathcal{F}$ be a localizing system on $D$. Then $\mathcal{F}_J = \{I \in \mathcal{F} \ | \ J \subseteq I$ for some finitely generated ideal $J \in \mathcal{F}\}$ is a localizing system of finite type on $D$.

As in [7], a localizing system $\mathcal{F}$ on $D$ is said to be spectral if there exists a nonempty subset $\Delta = \{P_\alpha \ | \ \alpha \in A\}$ of $\text{Spec}(D)$ such that $\mathcal{F} = \bigcap\{\mathcal{F}_{P_\alpha}\ | \ P_\alpha \in \Delta\}$.

Lemma 29([6, Lemma 5.1.5]). Let $\mathcal{F}$ be a localizing system of finite type on $D$. Then

(1) If $I$ is an ideal of $D$ with $I \notin \mathcal{F}$, then there exists a prime ideal $P$ of $D$ such that $I \subseteq P$ and $P \notin \mathcal{F}$.

(2) $\mathcal{F} = \bigcap\{\mathcal{F}_Q\ | \ Q \in \text{Spec}(D)$ and $Q \notin \mathcal{F}\}$ and therefore $\mathcal{F}$ is always spectral.

Corollary 30([6, Corollary 5.1.6]). In a Noetherian integral domain every localizing system is spectral.

A localizing system $\mathcal{F}$ is said to be principal, if for each $I \in \mathcal{F}$, there exists a principal ideal $I'$ in $\mathcal{F}$ such that $I' \subseteq I$.

Proposition 31([6, Proposition 5.1.7]). Let $\mathcal{F}$ be a localizing system on $D$. Then the following are equivalent:

(1) $\mathcal{F}$ is a spectral localizing system on $D$;
(2) $\mathcal{F} = \bigcap \{ \mathcal{F}_\alpha \mid \alpha \in A \}$, where $\mathcal{F}_\alpha$ is a principal localizing system on $D$ for each $\alpha \in A$;

(3) $\mathcal{F} = \bigcap \{ \mathcal{F}_\alpha \mid \alpha \in A \}$, where $\mathcal{F}_\alpha$ is a localizing system of finite type on $D$ for each $\alpha \in A$;

(4) For each ideal $I$ of $D$ with $I \not\in \mathcal{F}$, there exists a prime ideal $P$ of $D$ such that $I \subseteq P$ and $P \not\in \mathcal{F}$.

A localizing system of finite type can be characterized in the following way:

**Proposition 32** ([6, Proposition 5.1.8]). Let $\mathcal{F}$ be a localizing system on $D$. Then the following are equivalent:

(1) $\mathcal{F}$ is a localizing system of finite type;

(2) There exists a quasi-compact subspace $\Delta$ of $\text{Spec}(D)$ such that $\mathcal{F} = \bigcap \{ \mathcal{F}_P \mid P \in \Delta \}$;

(3) $\mathcal{F} = \bigcap \{ \mathcal{F}_P \mid P \in \mathcal{F} \}$, where $\mathcal{F} = \{ P \in \text{Spec}(D) \mid P \not\in \mathcal{F}$ and it is maximal with respect to this property $\}$, and $\mathcal{F}$ is a quasi-compact subspace of $\text{Spec}(D)$.

**Proposition 33** ([7, Proposition 3.2]). Let $\mathcal{F}$ be a localizing system and $*$ be a semistar operation on $D$. Then

(1) If $\mathcal{F}$ is of finite type, then $\mathcal{F}^*$ is of finite type.

(2) If $*$ is of finite type, then $\mathcal{F}^*$ is of finite type.

**Corollary 34.** Let $\mathcal{F}$ be a localizing system on $D$. Then $\mathcal{F}$ is of finite type if and only if $\mathcal{F}^*$ is of finite type.

**Proof.** This follows from Proposition 11(3) and Proposition 33. \qed

## 2. Characterizations of generalized Dedekind domains.

In this section, we denote the set of all overrings of $D$ by $\mathcal{O}(D)$ and the set of proper overrings of $D$ by $\mathcal{P}(D)$, that is, $\mathcal{P}(D) = \mathcal{O}(D) \setminus \{ D, K \}$.

Before stating the characterizations of a generalized Dedekind domain which were proved in [3], [4], [6], [10], [11], [28] and [29], we collect some related notations and definitions as follows.

**Notation.** The set of divisorial fractional ideals (respectively, invertible fractional ideals) of $D$ is denoted by $\text{Div}(D)$ (respectively, $\text{Inv}(D)$). For each ideal $I$ of $D$, the set of minimal prime ideals of $I$ is denoted by $\text{Min}(I)$.

**Definition.** (1) A Prüfer domain $D$ is **strongly discrete** if $P \not\subseteq P^2$ for each nonzero $P \in \text{Spec}(D)$ (see [4]). In [28], a strongly discrete Prüfer domain is called an **almost-generalized-Dedekind domain**.

(2) A Prüfer domain $D$ is said to be a (#)-domain provided $\bigcap \{ D_M \mid M \in \Delta_1 \} \neq \bigcap \{ D_M \mid M \in \Delta_2 \}$, where $\Delta_1$ and $\Delta_2$ are distinct subsets of $\text{Max}(D)$. $D$ is called a (##)-domain if each overring of $D$ is a (#)-domain.

(3) A Prüfer domain $D$ is said to be a (#P)-domain provided $\bigcap \{ D_P \mid P \in \Delta_1 \} \neq \bigcap \{ D_P \mid P \in \Delta_2 \}$, where $\Delta_1$ and $\Delta_2$ are distinct subsets of $\text{Spec}(D) \setminus \{(0)\}$ such that if $P_1 \in \Delta_1, P_2 \in \Delta_2$ and $P_1 \neq P_2$, then $P_1 + P_2 = D$ (see [6, p.156]). In [28], property (#P) is called property (#'). $D$ is called a (##P)-domain if each overring of $D$ is a (#P)-domain.
We say that an integral domain $D$ satisfies the radical trace property, or is an RTP domain, if $I(D:I)$ is a radical ideal for any noninvertible ideal $I$ of $D$.

An ideal $I$ of an integral domain $D$ is called SV-stable if $I$ is invertible in $I:I$.

An integral domain $D$ is said to be piecewise Noetherian if (i) the set of prime ideals of $D$ satisfies the a.c.c.; (ii) the set of $P$-primary ideals satisfies the a.c.c. for each $P \in \text{Spec}(D)$; and (iii) $\text{Min}(I)$ is a finite set for each nonzero ideal $I$ of $D$.

Proposition 35. For a Prüfer domain $D$, the following statements are equivalent:

1. $D$ is a generalized Dedekind domain;
2. For any two distinct localizing systems $\mathcal{F}_1, \mathcal{F}_2$, $D_{\mathcal{F}_1} \neq D_{\mathcal{F}_2}$;
3. For each localizing system $\mathcal{F}$ on $D$, $\mathcal{F} = \mathcal{F}(D_{\mathcal{F}})$;
4. Every localizing system $\mathcal{F}$ on $D$ is of finite type;
5. For each localizing system $\mathcal{F}$, $\mathcal{F} = \mathcal{F}(T)$ for some overring $T$ of $D$;
6. $D$ is strongly discrete and each nonzero prime ideal of $D$ is the radical of a finitely generated ideal;
7. $D$ is strongly discrete and each nonzero radical ideal of $D$ is the radical of a finitely generated ideal;
8. $D$ is strongly discrete and satisfies property ($\#P$);
9. $D$ is strongly discrete and satisfies property ($\#\#P$);
10. $D$ is strongly discrete and satisfies property ($\#\#$);
11. Each localizing system $\mathcal{F}$ on $D$ is an irredundant intersection of the form $\mathcal{F} = \bigcap \{ \mathcal{F}_P \mid P \in \Theta \}$, where $\Theta$ is a set of comaximal prime ideals of $D$ such that the map $\lambda : \Theta \to \text{Max}(D_{\mathcal{F}})$ defined by $P \mapsto PD_{\mathcal{F}}$ is bijective;
12. $D$ is strongly discrete and $\text{Spec}(D)$ is a Noetherian space;
13. $D$ is strongly discrete and for each nonzero ideal $I$ of $D$, $\text{Min}(I)$ is a finite set;
14. $D$ is strongly discrete and for each nonzero principal ideal $I$ of $D$, $\text{Min}(I)$ is a finite set;
15. Each overring of $D$ is a generalized Dedekind domain;
16. $D$ is strongly discrete and for each nonzero finitely generated ideal $I$, $\text{Min}(I)$ is a finite set;
17. $D$ is a strongly discrete RTP domain;
18. Each nonzero prime ideal of $D$ is SV-stable;
19. Each nonzero prime ideal is divisorial and each divisorial ideal of $D$ is SV-stable;
20. $\text{Div}(D) = \{ JP_1 \cdots P_n \mid J \in \text{Inv}(D) \text{ and } P_1, P_2, \ldots, P_n \text{ are pairwise comaximal prime ideals of } D \}$;
21. $D$ is strongly discrete and any product of prime ideals of $D$ is divisorial;
22. $D$ is a piecewise Noetherian Prüfer domain.

Proof. The equivalence of (1), (2), (3), and (4) is in [28, Proposition 2.1]. (6) $\iff$ (1) is in [28, Theorem 2.5] and the implication (2) $\Rightarrow$ (8) $\Rightarrow$ (6) is in [28, Proposition 3.2].
The equivalence of (1), (3), (4), (6), (7), (9), (10), (11), (12), (13), (14), and (15) is in [4, Theorem 2.7].

The equivalence (8) \iff (9) follows from [6, Proposition 5.5.1].

(6) \Rightarrow (20) follows from [11, Proposition 3.2] and (20) \Rightarrow (10) follows from [11, Theorem 3.3].

The equivalence of (6), (10), and (21) is in [11, Corollary 3.4].

The equivalence (13) \iff (16) is in [29, Theorem 3].

The equivalence of (13), (17), (18), and (19) is in [10, Theorem 8].

The equivalence (6) \iff (22) is in [3, Proposition 3.1].

(3) \Rightarrow (5). This is trivial. (5) \Rightarrow (3). Let \( \mathcal{F} \) be a localizing system on \( D \) and assume that \( \mathcal{F} = \mathcal{F}(T) \) for some overring \( T \) of \( D \). Then, by Lemma 3, \( T = D_{\mathcal{F}(T)} = D_{\mathcal{F}} \). Hence \( \mathcal{F} = \mathcal{F}(D_{\mathcal{F}}) \).

**Proposition 36.** Let \( D \) be a Prüfer domain. Then

1. For each overring \( T \) of \( D \), \( T = D_{\mathcal{F}(T)} \).
2. For each localizing system \( \mathcal{F} \) on \( D \), \( D_{\mathcal{F}} = D_{\mathcal{F}(D_{\mathcal{F}})} \).
3. For each localizing system \( \mathcal{F} \) on \( D \), \( \mathcal{F} = \mathcal{F}(D_{\mathcal{F}}) \) if and only if \( \ast_{\mathcal{F}} = \ast_{(D_{\mathcal{F}})} \).

**Proof.**

1. This follows from Remark 3, because \( T \) is flat over \( D \).
2. follows from (1) if we take \( T = D_{\mathcal{F}} \) in (1).
3. follows from Proposition 16.

**Remark 37.**

1. It follows from [19, Theorem 1] that an overring \( T \) of \( D \) is flat over \( D \) if and only if \( \ast_{(T)} \) is a stable semistar operation on \( D \). Hence \( D \) is a Prüfer domain if and only if \( \ast_{(T)} \) is stable for every overring \( T \) of \( D \).
2. If \( T \) is an overring of a generalized Dedekind domain \( D \), then, by Proposition 36(1), we can take \( \mathcal{F}(T) \) as the unique localizing system \( \mathcal{F} \) which satisfies \( T = D_{\mathcal{F}(T)} \).

**Lemma 38.** Let \( T \) be an overring of \( D \). Then \( T \) is a flat overring of \( D \) if and only if \( T \subseteq D_{\delta_{D}(\mathcal{F})} \) for all localizing systems \( \mathcal{F} \) on \( T \).

2. Let \( T \) be a faithfully flat overring of \( D \) and let \( \mathcal{F} \) be a localizing system on \( D \). Then \( \mathcal{F} = \delta_{D}(\alpha_{T}(\mathcal{F})) \).

**Proof.**

1. \((\Rightarrow)\) If \( I \in \mathcal{F}(T) \), then \( IT = T \in \mathcal{F} \) and hence \( I \in \delta_{D}(\mathcal{F}) \). Thus \( \mathcal{F}(T) \subseteq \delta_{D}(\mathcal{F}) \). Therefore, \( T = D_{\mathcal{F}(T)} \subseteq D_{\delta_{D}(\mathcal{F})} \) by Remark 3.

\((\Leftarrow)\) Let \( \mathcal{F}_{0} = \{ T \} \). Then \( \mathcal{F}_{0} \) is a localizing system on \( T \). By hypothesis, \( T \subseteq D_{\delta_{D}(\mathcal{F}_{0})} \). Now, it is easy to see that \( \delta_{D}(\mathcal{F}_{0}) = \mathcal{F}(T) \) and \( T_{\mathcal{F}_{0}} = T \). But, by Lemma 7(1), \( D_{\delta_{D}(\mathcal{F}_{0})} \subseteq T_{\mathcal{F}_{0}} = T \). Therefore we have \( T = D_{\delta_{D}(\mathcal{F}_{0})} = D_{\mathcal{F}(T)} \) and so \( T \) is flat over \( D \) by Remark 3.

2. \((\subseteq)\) This is trivial.

\((\supseteq)\) Choose \( I \in \delta_{D}(\alpha_{T}(\mathcal{F})) \). Then \( IT \in \alpha_{T}(\mathcal{F}) \). Hence \( IT \supseteq JT \) for some \( J \in \mathcal{F} \). Then, since \( T \) is faithfully flat over \( D \), \( I \supseteq J \) and so \( I \in \mathcal{F} \).

**Corollary 39.** Let \( D \) be a Prüfer domain. If each semistar operation on \( D \) is of finite type, then \( D \) is a generalized Dedekind domain.

**Proof.** This follows from Corollary 34 and Proposition 35.
Lemma 40. (1) Let \( D^{*1} = D^{*2} \) with \( *_{1}, *_{2} \in SStar(D) \). If \( *_{1} \leq *_{2} \), then \( F^{*1} \subseteq F^{*2} \).

(2) Let \( *_{1} \) and \( *_{2} \) be stable semistar operations on \( D \). If \( F^{*1} \subseteq F^{*2} \), then \( *_{1} \leq *_{2} \).

Proof. (1) Choose \( I \in F^{*1} \). Then \( I^{*1} = D^{*1} \) and hence, if \( *_{1} \leq *_{2} \), then \( I^{*2} = (I^{*1})^{*2} = (D^{*1})^{*2} = D^{*2} \) which shows that \( I \in F^{*2} \).

(2) If \( *_{1} \) and \( *_{2} \) are stable, then, by Propositions 11(2) and 22(3), \( *_{1} = *_{F^{*1}} \leq *_{F^{*2}} = *_{2} \). \( \Box \)

Lemma 41. Let \( T \) be an overring of \( D \) and let \( * \) be a semistar operation on \( D \) such that \( T = D^{*} \). Then \( DF^{*} \subseteq T \).

Proof. By definition, \( DF^{*} = \{D : K I | I \in F^{*}\} = \{D : I \in I(D) \) and \( I^{*} = D^{*} = T \}. \) If we choose \( x \in DF^{*} \), then \( xI \subseteq D \) for some \( I \in F^{*} \). Hence \( xI^{*} \subseteq D^{*} = T \). Thus \( xT \subseteq T \), and so, \( x \in T \). \( \Box \)

Proposition 42. Let \( T \) be a flat overring of \( D \) and let \( * \) be a semistar operation on \( D \) such that \( D^{*} = T \). Then \( T = DF^{*} = DF^{*(T)} = DF(T) \). \( \Box \)

Proof. If \( D^{*} = T \), then, by [24, Proposition 13], \( *(T) \leq * \). Hence, by Lemma 40, \( F^{*(T)} \subseteq F^{*} \). Furthermore, by Lemma 41, \( DF^{*} \subseteq T \). Then, by Remark 3 and Proposition 5, we have \( T = DF(T) = DF^{*(T)} \subseteq DF^{*} \subseteq T \), and hence \( T = DF(T) = DF^{*(T)} = DF^{*} \). \( \Box \)

Corollary 43. Let \( T \) be an overring of a generalized Dedekind domain \( D \). If \( * \) is a semistar operation on \( D \) such that \( D^{*} = T \), then \( F^{*} = F^{*(T)} = F(T) \).

Proof. This follows immediately from Proposition 42 and the definition of a generalized Dedekind domain. \( \Box \)

Corollary 44. Let \( T \) be an overring of a generalized Dedekind domain \( D \). Then

(1) If \( * \) is a semistar operation on \( D \) such that \( D^{*} = T \), then \( *_{F^{*}} = *_{F(T)} = *_{F^{*(T)}} = *(D^{*}) \).

(2) For any \( * \in SStar(D) \), \( *_{F^{*}} = *(D^{*}) \).

Proof. (1) Since \( T \) is flat over \( D \), \( *(T) \) is a stable semistar operation on \( D \). Hence, by Proposition 22(3) and Corollary 43, we have \( *_{F^{*}} = *_{F(T)} = *_{F^{*(T)}} = *(T) \).

(2) This follows immediately from (1). \( \Box \)

Let \( T \) be an overring of \( D \). Then we set \( SStar^{T}(D) = \{ * \in SStar(D) | D^{*} = T \} \) and \( LST^{T}(D) = \{ F \in LST(D) | DF = T \} \).

Remark 45 (cf. [9, Remark 2.1(b)] and [25, Remark 20(2)]). Let \( * \) be a semistar operation on \( D \). If \( D^{*} = K \), then \( * = \bar{e} \). In fact, for each \( E \in K(D) \), \( E^{*} = (DE)^{*} = (DE)^{*} = (KE)^{*} = K^{*} = K \) and so \( * = \bar{e} \). Thus \( SStar^{K}(D) = \{ \bar{e} \} \).

Theorem 46. Let \( T \) be an overring of a Prüfer domain \( D \). Then

(1) For each \( * \in SStar^{T}(D) \), \( F^{*} \in LST^{T}(D) \).

(2) For each \( F \in LST^{T}(D) \), \( *_{F} \in SStar^{T}(D) \).

(3) We define the map \( \alpha : SStar^{T}(D) \to LST^{T}(D) \) by setting \( \alpha(*) = F^{*} \) for each \( * \in SStar^{T}(D) \) and we define the map \( \beta : LST^{T}(D) \to SStar^{T}(D) \) by
setting \( \beta(\mathcal{F}) = \ast_{\mathcal{F}} \) for each \( \mathcal{F} \in \mathcal{LS}^T(D) \). Then the map \( \alpha \circ \beta \) is the identity map of \( \mathcal{LS}^T(D) \) and hence \( \beta \) is injective.

(4) The map \( \beta \circ \alpha \) is the identity map of \( \mathcal{SStar}_\sigma(D) \cap \mathcal{SStar}_T(D) \).

Proof. (1) This follows from Proposition 42.

(2) Let \( \mathcal{F} \in \mathcal{LS}^T(D) \). Then \( D^{*_{\mathcal{F}}} = \bigcup \{ D : K J : J \in \mathcal{F} \} = D_{\mathcal{F}} = T \), and so \( *_{\mathcal{F}} \in \mathcal{SStar}^T(D) \).

(3) For each \( \mathcal{F} \in \mathcal{LS}^T(D) \), we have \( (\alpha \circ \beta)(\mathcal{F}) = \mathcal{F}^{*_{\mathcal{F}}} = \mathcal{F} \) by Proposition 11(3). Hence \( \alpha \circ \beta \) is the identity map of \( \mathcal{LS}^T(D) \).

(4) If \( * \in \mathcal{SStar}_\sigma(D) \cap \mathcal{SStar}_T(D) \), then, by Proposition 22(3), \( (\beta \circ \alpha)(*) = *_{\mathcal{F}} = * \), and therefore \( \beta \circ \alpha \) is the identity map of \( \mathcal{SStar}_\sigma(D) \cap \mathcal{SStar}_T(D) \).

Before introducing the notion of an amply strong semistar domain, we shall recall the definition of a star operation.

A map \( E \mapsto E^* \) of \( \mathcal{F}(D) \) into \( \mathcal{F}(D) \) is called a star operation on \( D \), if the following conditions hold for all \( E \in \mathcal{F}(D) \):

\[
\begin{align*}
(S0) \quad & (x^sD)^* = xD \text{ for all } x \in K - \{0\} ; \\
(S1) \quad & (aE)^* = aE^* ; \\
(S2) \quad & E \subseteq F \text{ then } E^* \subseteq F^* ; \text{ and} \\
(S3) \quad & E \subseteq E^* \text{ and } (E^*)^* = E^* .
\end{align*}
\]

We also recall some representative examples of star operations. If we set \( \bar{E} = E \) for all \( E \in \mathcal{F}(D) \), then \( \bar{d} \) is a star operation on \( D \) and is called the \( d \)-operation. If we set \( \theta = (E^{-1})^{-1} \) for all \( E \in \mathcal{F}(D) \), then \( \theta \) is a star operation on \( D \) and is called the \( \theta \)-operation. Furthermore, if we set \( \mathcal{t} = \bigcup \{ F_{\theta} \mid F \subseteq E \text{ and } F \in \mathcal{f}(D) \} \) for each \( E \in \mathcal{F}(D) \), then \( t \) is a star operation of finite type on \( D \).

Proposition 47([24, Proposition 17]). Let \( * \) be a star operation on \( D \). For each \( E \in \mathcal{K}(D) \), we set:

\[
E^{*^e} = \begin{cases} 
E^*, & \text{for } E \in \mathcal{F}(D) \\
K, & \text{for } E \in \mathcal{K}(D) \setminus \mathcal{F}(D)
\end{cases}
\]

Then the map \( E \mapsto E^{*^e} \) is a semistar operation on \( D \). This semistar operation \( *^e \) is called the trivial semistar extension of \( * \). Evidently \( *^e \) is always weak for each \( * \in \mathcal{Star}(D) \).

Remark 48. It is easy to see that \( E^6 = E^\theta = E^\nu \) for all \( E \in \mathcal{F}(D) \) and \( E^6 = K \) for all \( E \in \mathcal{K}(D) \setminus \mathcal{F}(D) \). Thus the \( \nu \)-operation is the trivial semistar extension of the \( \nu \)-operation for each integral domain \( D \).

An integral domain \( D \) is called a conducive domain if \( D : R = \{ x \in K \mid xR \subseteq D \} \neq (0) \) for each overring \( R \) of \( D \) other than \( K \). It is known that \( D \) is a conducive domain if and only if \( \mathcal{K}(D) = \mathcal{F}(D) \cup \{ K \} \) (cf. [26, Proposition 7]).

Remark 49. (1) If \( D \) is a conducive domain, then \( \bar{d} \)-operation is the trivial semistar extension of the \( d \)-operation.

(2) If \( D \) is not a conducive domain, then \( \bar{d} \)-operation is not the trivial semistar extension of the \( d \)-operation. In fact, for each \( E(\neq K) \in \mathcal{K}(D) \setminus \mathcal{F}(D) \), \( E^\bar{d} = E \neq K \).
Definition 50. A semistar operation $*$ is said to be of overring type if $*=(*_T)$ for some overring $T$ of $D$. An integral domain $D$ is called an amply strong semistar domain (for short, ASSD) if each semistar operation on $D$ is of overring type.

Proposition 51. Let $D$ be an ASSD. Then every proper overring $T$ of $D$ is also an ASSD.

Proof. Let $* \in \text{SStar}(T)$. If we set $*_1 = \delta_{T/D}(*) \in \text{SStar}(D)$, then, by hypothesis, $*_1 = *_{(R)}$ for some overring $R$ of $D$. But then, $D^{\delta_{T/D}(*)} = (DT)^* = T^*$ and $D^{*_{(R)}} = R$, and therefore $R = T^* \supseteq T$. Now for all $E \in K(D)$, $E^{*_1} = E^{\delta_{T/D}(*)} = (ET)^*$ and $E^{*_{(R)}} = E^{*_{(R)}} = ER$ and so $(ET)^* = ER$. Hence for all $F \in K(T)(\subseteq K(D))$, $F^* = (FT)^* = FR = F^{*_{(R)}}$. Thus $* = *_{(R)}$ for some overring $R$ of $T$ which means that $*$ is of overring type. □

Proposition 52. (1) If $D$ is an ASSD, then $\text{SStar}_w(D) = \{d\}$ and $\text{Star}(D) = \{d\}$.

(2) If $D$ is an ASSD, then $\text{SStar}_w(T) = \{d_T\}$ for each overring $T$ of $D$, where $\{d_T\}$ is the $d$-operation on $T$.

Proof. (1) Let $* \in \text{SStar}_w(D)$. Then, by hypothesis, $* = *(T)$ for some overring $T$ of $D$ and hence $D = D^* = D^{*_{(T)}} = T$. Thus we have $\text{SStar}_w(D) = \{d\}$. Next, if $* \in \text{Star}(D)$, then $*^e = d$ and hence $* = d$. □

If $D^* = D$, then $* = *_{(D)} = d$.

(2) $T$ is also an ASSD by Proposition 51, and so $\text{SStar}_w(T) = \{d_T\}$ by (1). □

It follows from Proposition 52 that any semistar operation on an ASSD other than the $d$-operation is strong. Our terminology “amply strong” is based on this fact.

Proposition 53. If $D$ is an ASSD, then every semistar operation on $D$ is of finite type.

Proof. This follows immediately from the definition of ASSD and [23, Lemma 8]. □

We shall give some characterizations of an ASSD.

Proposition 54. For an integral domain $D$, the following statements are equivalent.

(1) $D$ is an ASSD;

(2) For every overring $T$ of $D$, $\text{SStar}_T^T(D) = \{*(T)\}$;

(3) For every overring $T$ of $D$, $\text{SStar}_w^T(T) = \{d_T\}$;

(4) $|\mathcal{O}(D)| = |\text{SStar}_f^T(D)| = |\text{SStar}(D)|$;

(5) For each $* \in \text{SStar}(D)$, $* = *(D^*)$.

Proof. (1) $\Rightarrow$ (2) If $* \in \text{SStar}_T^T(D)$, then $* = *(T)$.

(2) $\Rightarrow$ (1) Let $* \in \text{SStar}(D)$. Set $T = D^*$. If $T \neq K$, then $* \in \text{SStar}_T^T(D)$ and by hypothesis, $* = *(T)$. Next, if $T = K$, then $D^* = K$ and hence, by
Remark 45, * = \bar{e} = *_{(K)}. In any case, each semistar operation * on D is of overring type.

(1) \Rightarrow (3) This follows from Proposition 52(2).

(3) \Rightarrow (2) Let * \in \text{SStar}^T(D). Then \alpha_T(*) \in \text{SStar}_w(T) by [26, Lemma 32(2)]. Then, by hypothesis, \alpha_T(*) = \bar{d}_T. Hence * = \delta_D(\alpha_T(*)) = \delta_D(\bar{d}_T) = *_{(T)} by [26, Theorem 25(2)].

(1) \iff (4) and (2) \iff (5) Trivial. □

Lemma 55. Let \mathcal{F} be a localizing system of D. Then \ast_{\mathcal{F}} is of overring type if and only if \ast_{\mathcal{F}} = \ast_{(D_F)}.

Proof. (\Leftarrow) This is evident.

(\Rightarrow) Assume that \ast_{\mathcal{F}} = \ast_{(T)} for some overring T of D. Then T = D_{\ast_{\mathcal{F}}} = D_{\ast_{(T)}} = D_{\ast_{(D_F)}} and so \ast_{\mathcal{F}} = \ast_{(D_F)}. □

Proposition 56. Let D be a Prüfer domain. For each localizing system \mathcal{F} on D, the following are equivalent:

(1) \ast_{\mathcal{F}} = \ast_{(D_F)};
(2) I_{DF} = I_F for every I \in \mathcal{I}(D);
(3) \mathcal{F} = \mathcal{F}(D_F);
(4) \ast_{\mathcal{F}} is of overring type.

Proof. This follows from Proposition 16 and Lemma 55. □

We are now in a position to give some new characterizations of a generalized Dedekind domain.

Proposition 57. Let D be a Prüfer domain. Then the following are equivalent.

(1) D is a generalized Dedekind domain;
(2) For each localizing system \mathcal{F} on D, \ast_{\mathcal{F}} is a semistar operation of overring type;
(3) For each localizing system \mathcal{F} on D, \ast_{\mathcal{F}} = \ast_{(DF)}.

Proof. This follows from Propositions 35 and 56. □

Proposition 58. Let D be a Prüfer domain. Consider the following conditions.

(1) D is an ASSD.
(2) Each semistar operation on D is of finite type.
(3) D is a generalized Dedekind domain.

Then (1) \Rightarrow (2) \Rightarrow (3) holds.

Proof. (1) \Rightarrow (2) This follows from [23, Lemma 8] and the definition of ASSD.
(2) \Rightarrow (3) This follows from Corollary 39. □

Proposition 59. Let D be a generalized Dedekind domain and let P be a prime ideal of D. If D_P = D_F for some localizing system \mathcal{F} on D, then \mathcal{F} = \mathcal{F}_P.

Proof. Suppose that D_P = D_F. Then \ast_{(DF)} = \ast_{(D_P)}, and then, by Proposition 5, Remark 27 and Proposition 35, \mathcal{F} = \mathcal{F}(D_F) = \mathcal{F}^*(DF) = \mathcal{F}^*(DP) = \mathcal{F}_P. □

If we set \chi(*) = \ast_{\mathcal{E}} for each \ast \in \text{Star}(D), then \chi is an injective map of \text{Star}(D) into \text{SStar}(D) [26, Proposition 3]. Using this map \chi, we can give a new characterization of a conducive domain which will be used to show that every ASSD is a conducive domain.
Theorem 60. Let $D$ be an integral domain. Then $D$ is a conducive domain if and only if the map $\chi : \text{Star}(D) \to S\text{Star}_{\omega}(D)$ is a bijective map.

Proof. ($\Rightarrow$) This follows from [26, Corollary 11].

($\Leftarrow$) Let $E \in K(D) \setminus F(D)$. By hypothesis, $\bar{d} = *^e$ for some $* \in \text{Star}(D)$. Then $E = E^d = E^{*^e} = K$ and therefore we have $K(D) = F(D) \cup \{K\}$. Hence our assertion is valid by [26, Proposition 7].

Corollary 61. Every ASSD is a conducive domain.

Proof. This follows from Theorem 60 and Proposition 52 (1).

Corollary 62. If $D$ is an ASSD, then every non-zero ideal of $D$ is divisorial.

Proof. If $D$ is an ASSD, then, by Proposition 52 (1), $\text{Star}(D) = \{d\}$ and then $d = v$. Hence every non-zero ideal of $D$ is divisorial.

Corollary 63. Let $D$ be an ASSD. Then $D$ is a Prüfer domain if and only if $D$ is integrally closed.

Proof. ($\Rightarrow$) This is trivial.

($\Leftarrow$) This follows from Corollary 62 and [18, Theorem 5.1].

Corollary 64. Let $D$ be an integrally closed ASSD. Then

1. $D$ is a generalized Dedekind domain.
2. Every $M \in \text{Max}(D)$ is a finitely generated ideal of $D$.

Proof. (1) This follows from Proposition 58 and Corollary 63.

(2) This follows from Corollary 62 and [18, Theorem 5.1].

Definition 65. An integral domain $D$ is called a stable semistar domain (for short, SSD) if for each overring $T$ of $D$, there exists a unique stable semistar operation $*$ on $D$ such that $D^* = T$.

Remark 66. Let $T$ be an overring of $D$. If $D^* = T$ with $* \in S\text{Star}(D)$, then $*_{T^*} \in S\text{Star}_{\sigma}(D)$ and $D^*_{T^*} = D_{T^*} = D^* = T$.

Proposition 67. Let $T$ be an overring of $D$. Then, for every $* \in S\text{Star}^T(D)$, $*_{T^*} \in S\text{Star}_{\sigma}(D) \cap S\text{Star}^T(D)$.

Lemma 68. Let $D$ be a Prüfer domain. Then the following statements are equivalent.

1. $D$ is an SSD;
2. For each overring $T$ of $D$, $S\text{Star}_{\sigma}(D) \cap S\text{Star}^T(D) = \{*(T)\}$.

Proof. (2) $\Rightarrow$ (1) This is trivial.

(1) $\Rightarrow$ (2) Since $D$ is a Prüfer domain, $*(T)$ is stable and $D^{*(T)} = T$ always holds. Hence, by hypothesis, $S\text{Star}_{\sigma}(D) \cap S\text{Star}^T(D) = \{*(T)\}$. □

Using the terminology of SSD, we can give the following characterizations of a generalized Dedekind domain.
Theorem 69. Let $D$ be a Prüfer domain. Then the following statements are equivalent.

1. $D$ is a generalized Dedekind domain;
2. $D$ is an SSD;
3. For each overring $T$ of $D$, $\text{SStar}_{\sigma}(D) \cap \text{SStar}^T(D) = \{*(T)\}$.

Proof. (1) $\Rightarrow$ (2) Let $T$ be an overring of $D$. Suppose that $T = D^*$ for some $* \in \text{SStar}_{\sigma}(D)$. By Corollary 43, $\mathcal{F}^* = \mathcal{F}^*(T)$ and therefore, $* = *_{\mathcal{F}^*} = *_{\mathcal{F}^*(T)} = *(T)$ by Proposition 22(3), and hence $*$ is uniquely determined.

(2) $\Rightarrow$ (1) Let $T$ be an overring of $D$. Suppose that $T = DF$ for some localizing system $\mathcal{F}$ on $D$. Then $T = D^T$. On the other hand, $D^* = T$. Thus $D^* = D^{*T}$. But $D^T$ and $*(T)$ are both stable by Proposition 11(1) and Remark 37(1). Hence, by hypothesis, $\mathcal{F} = *(T)$ and then, $\mathcal{F} = D^{*T}$ by Proposition 11(3). Hence $\mathcal{F}$ is uniquely determined.

(2) $\Leftrightarrow$ (3) This follows from Lemma 68. □

In Proposition 58, we showed that every Prüfer ASSD is a generalized Dedekind domain. In the case of a finite dimensional valuation domain, it will be shown that the converse is also valid.

Theorem 70. Let $V$ be a valuation domain of dimension $n < \infty$. Then the following statements are equivalent.

1. $V$ is a generalized Dedekind domain;
2. Each semistar operation on $V$ is of finite type;
3. $V$ is an ASSD;
4. $V$ is a strongly discrete valuation domain.

Proof. The implication (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is in Proposition 58.

(1) $\Leftrightarrow$ (4) This follows from [28, Proposition 2.4].

(4) $\Rightarrow$ (3) Assume that $V$ is a strongly discrete valuation domain. Then, by [26, Corollary 39], $|\text{SStar}(V)| = n + 1$. Hence, if $(0) \subset P_1 \subset P_2 \subset \cdots \subset P_{n-1} \subset M$ is the maximal chain of prime ideals in $V$, then $\text{SStar}(V) = \{d = *_{V_{P_{n-1}}}, \cdots, *_{V_{P_1}}, \bar{e} = *(K)\}$ and therefore every semistar operation on $V$ is of overring type, as required. □

Here we note that there exists an example of a generalized Dedekind domain which is not an ASSD. Hence the converse of the implication (1) $\Rightarrow$ (3) in Proposition 58 is in general not valid.

Remark 71. (1) Let $D$ be a Dedekind domain which is not local. Then, $D$ is not a conducive domain. In fact, if $D$ is conducive, then $D$ is a conducive Noetherian domain and then, by [2, Corollary 2.7], $D$ is local, a contradiction.

(2) Every Dedekind domain is a generalized Dedekind domain by [28, Corollary 2.2]. Hence, if $D$ is a Dedekind domain which is not local, then $D$ is a generalized Dedekind domain but $D$ is not an ASSD, because every ASSD is a conducive domain by Corollary 61.

Lastly we show the existence of an example of a non-integrally closed ASSD for justifying Corollary 63.
Example 72(cf. [26, Example 9]). Let $k$ be a field and let $D = k[[X^2, X^3]]$ be the subring of $k[[X]]$ consisting of those power series with zero $X$ term. Then $D$ is a one-dimensional Noetherian local domain with maximal ideal $M = (X^2, X^3)$. Now it is easily seen that there are no rings contained properly between $D$ and $k[[X]]$. Moreover, $K = k((X))$ is the quotient field of $D$ and there are no rings contained properly between $k[[X]]$ and $K$, because $k[[X]]$ is a DVR. Hence $k[[X]]$ is the unique proper overring of $D$. Now, since $D : k[[X]] = X^2k[[X]] ≠ (0)$, $D$ is a conducive domain by [2, Lemma 2.0 (ii)]. Evidently $k[[X]]$ is the integral closure of $D$ and therefore $D$ is not integrally closed.

Next we shall show that $|\text{Star}(D)| = 1$, i.e., $\text{Star}(D) = \{d\}$. Since $G(D) = \text{grade } M = \text{height } M = 1$, it follows that the maximal ideal $M = (X^2, X^3)$ of a Noetherian local domain $D$ is generated by two elements. Then, by [20, Exercise 1 in §4-5], $D$ is a Gorenstein domain. But then, since $D$ is a one-dimensional Noetherian local domain, it follows from [20, Theorem 222] that $D$ is a Gorenstein domain if and only if each non-zero ideal of $D$ is a divisorial ideal. Hence we have $\text{Star}(D) = \{d\}$. Here we recall from [26, Remark 37] that $|\text{SStar}(D)| = |\text{Star}(D)| + 2$. Hence in this case we get $|\text{SStar}(D)| = 3$.

Now we shall show that $D$ is an ASSD. For simplicity, we denote $k[[X]]$ by $R$. Let $* \in \text{SStar}(D)$. If $D^* = D$, then $* \in \text{SStar}_D(D)$ and then $* = d$ by [26, Corollary 11], because $D$ is a conducive domain and $\text{Star}(D) = \{d\}$. If $D^* = R$, then $* = \delta_D(\alpha_R(*))$ by [26, Theorem 25 (2)]. Now, since $R = k[[X]]$ is a DVR, $\text{SStar}(R) = \{d, e\}$ by [24, Theorem 48]. If $\alpha_R(*) = d$, then $* = \delta_D(d) = *_{(R)}$ by [26, Theorem 25 (2)]. If $\alpha_R(*) = e$, then $* = \delta_D(e) = *_{(K)}$. Lastly if $D^* = K$, then evidently $* = *_{(K)}$. Thus it follows that $D$ is an ASSD.

REFERENCES