Math. J., Ibaraki Univ., Vol. 35, 2003

Torsion-free abelian semigroup rings XI

RYÛKI MATSUDA*

Recently I am publishing two books [M9] and [M10]. This is a note for them, and this is also a continuation of [M8].

Thus let G be a torsion-free abelian additive group. A subsemigroup S of G which contins 0 is called a grading monoid (or a g-monoid). Throughout the paper S denotes a g-monoid which is not $\{0\}$, R denotes a commutative ring with 1, and D denotes an integral domain. A commutative ring with 1 is also called simply a ring.

If S be a g-monoid, then the group $\{s - s' \mid s, s' \in S\}$ is called the quotient group of S, and is denoted by $\mathrm{q}(S)$. A subsemigroup of $\mathrm{q}(S)$ which contains S is called an oversemigroup of S. Let I be a non-empty subset of $\mathrm{q}(S)$. If $S+I \subset I$, and if $s + I \subset S$ for some element $s \in S$, then I is called a fractional ideal of S. Let $\mathrm{F}(S)$ be the set of fractional ideals of S, $\mathrm{F}_f(S)$ be the set of finitely generated fractional ideals of S, and let $\mathrm{F}'(S)$ be the set of non-empty subset I of $\mathrm{q}(S)$ such that $S + I \subset I$. A mapping $I \longmapsto I^*$ of $F'(S)$ to $\mathrm{F}'(S)$ is called a semistar operation on S if the following conditions hold for all elements $a \in \mathrm{q}(S)$ and $I, J \in \mathrm{F}'(S)$: $(a + I)^* = a + I^*; I \subset I^*$; If $I \subset J$, then $I^* \subset J^*$; and $(I^*)^* = I^*$. If we set $I^{d'} = I, I^e = \mathrm{q}(S)$, and $I^{v'} = (I^{-1})^{-1}$ for each $I \in \mathrm{F}'(S)$, we have three semistar operations called d$'$-operation, e-operation and v$'$-operation on S, where $I^{-1} = \{x \in \mathrm{q}(S) \mid x + I \subset S\}$ for each subset I of $\mathrm{q}(S)$ (We set $\emptyset^{-1} = \mathrm{q}(S)$). The set of semistar operations on S is denoted by $\Sigma'(S)$.

For each star operation $*$ on S, we have notions of $\mathrm{D}^*(S), D_f^*(S), C^*(S)$ and $\mathrm{C}_f^*(S)$ (cf. [M2]). Extending $\mathrm{F}(S)$ to $\mathrm{F}'(S)$, we may naturally and analogously define them for semistar operations.

Thus let $*$ be a semistar operation on S. We define as follows:

$\mathrm{div}\ {}^*(I) = \{J \in \mathrm{F}'(S) \mid J^* = I^*\}$ for $I \in \mathrm{F}'(S)$.

$\mathrm{div}\ (I) = \mathrm{div}^{v'}(I)$.

$\mathrm{div}\ {}^*(I) + \mathrm{div}^*(J) = \mathrm{div}^*(I^* + J^*)$ for $I, J \in \mathrm{F}'(S)$.

If $I^* \subset J^*$, then $\mathrm{div}\ {}^*(I) \geq \mathrm{div}^*(J)$.

$\mathrm{D}\ {}^*(S) = \{\mathrm{div}^*(I) \mid I \in \mathrm{F}'(S)\}$.

$\mathrm{D}\ (S) = \mathrm{D}^{v'}(S)$.

$\mathrm{D}\ {}_f^*(S) = \{\mathrm{div}^*(I) \mid I \in \mathrm{F}_f(S)\}$.

Received December 14, 2002.

2000 *Mathematics Subject Classification.* Primary 13A15.

Key words. commutative semigroup, commutative semigroup ring, semistar operation.

* Department of Mathematical Sciences, Ibaraki University, Mito, Ibaraki 310-8512, Japan.

$\mathrm{D}_f(S) = \mathrm{D}_f^{v'}(S)$.
$\mathrm{C}^*(S) = \mathrm{D}^*(S)/\{\mathrm{div}^*(\alpha) \mid \alpha \in G\}$, where $G = \mathrm{q}(S)$.
$\mathrm{C}(S) = \mathrm{C}^{v'}(S)$.
$\mathrm{C}_f^*(S) = \mathrm{D}_f^*(S)/\{\mathrm{div}^*(\alpha) \mid \alpha \in G\}$.
$\mathrm{C}_f(S) = \mathrm{C}_f^{v'}(S)$.

Let $*$ be a semistar operation on S. If $\mathrm{D}_f^*(S)$ is a group, then S is called a Prüfer $*$-multiplication semigroup.

Let $*'$ be a semistar operation on S. If the restriction $* = *' \mid \mathrm{F}(S)$ is a star operation on S, then $*'$ is called an extension of $*$ to a semistar operation on S. Any star operation on S has an extension to a semistar operation, and a star operation $*$ on S may have more than one extentions to semistar operations on S.

Proposition 1. *Let $*$ be a star operation on S, and let $*'$ be an extension of $*$ to a semistar operation on S.*

(1) S is a Prüfer $*$-multiplication semigroup if and only if S is a Prüfer $*'$-multiplication semigroup.

(2) If S is a Prüfer $*$-multiplication semigroup, then the group $\mathrm{D}_f^*(S)$ is canonically isomorphic onto $\mathrm{D}_f^{*'}(S)$.

(3) S is a Prüfer v-multiplication semigrouop if and only if S is a Prüfer v'-multiplication semigroup.

For each star operation $*$ on R, we have notions of $\mathrm{D}^*(R), D_f^*(R), C^*(R)$ and $\mathrm{C}_f^*(R)$ ([M1]). Similarly, we may naturally define them for semistar operations.

Thus let $*$ be a semistar operation on R, and let $\mathrm{F}'(R)$ be the set of non-zero R-submodules of $\mathrm{q}(R)$, where $\mathrm{q}(R)$ is the total quotient ring of R. A non-zerodivisor of $\mathrm{q}(R)$ is also called a regular element. A fractional ideal which contains regular elements is called regular. We define as follows:
$\mathrm{div}^*(I) = \{J \in \mathrm{F}'(R) \mid J^* = I^*\}$ for $I \in \mathrm{F}'(R)$.
$\mathrm{div}(I) = \mathrm{div}^{v'}(I)$.
$\mathrm{div}^*(I) + \mathrm{div}^*(J) = \mathrm{div}^*(I^*J^*)$ for regular $I, J \in \mathrm{F}'(R)$.
If $I^* \subset J^*$, then $\mathrm{div}^*(I) \geq \mathrm{div}^*(J)$.
$\mathrm{D}^*(R) = \{\mathrm{div}^*(I) \mid I \in \mathrm{F}'(R), I \text{ is regular }\}$.
$\mathrm{D}(R) = \mathrm{D}^{v'}(R)$.
$\mathrm{D}_f^*(R) = \{\mathrm{div}^*(I) \mid I \in \mathrm{F}_f(R), I \text{ is regular }\}$.
$\mathrm{D}_f(R) = \mathrm{D}_f^{v'}(R)$.
$\mathrm{C}^*(R) = \mathrm{D}^*(R)/\{\mathrm{div}^*(\alpha) \mid \alpha \in K, \alpha \text{ is regular }\}$, where $K = \mathrm{q}(R)$.
$\mathrm{C}(R) = \mathrm{C}^{v'}(R)$.
$\mathrm{C}_f^*(R) = \mathrm{D}_f^*(R)/\{\mathrm{div}^*(\alpha) \mid \alpha \in K, \alpha \text{ is regular }\}$.
$\mathrm{C}_f(R) = \mathrm{C}_f^{v'}(R)$.

Let $*$ be a semistar operation on R. If $\mathrm{D}_f^*(R)$ is a group, then R is called a Prüfer $*$-multiplication ring.

Let I be an R-submodule of $\mathrm{q}(R)$ such that $rI \subset R$ for some regular element $r \in R$. Then I is called a fractional ideal of R. Let $\mathrm{F}(R)$ be the set of non-zero fractional ideals of R.

Let $*'$ be a semistar operation on R. If the restriction $* = *' \mid \mathrm{F}(R)$ is a star operation on R, then $*'$ is called an extension of $*$ to a semistar operation on R. Any star operation on R has an extension to a semistar operation.

Proposition 2. *Let $*$ be a star operation on R, and let $*'$ be an extension of $*$ to a semistar operation on R.*

(1) R is a Prüfer $*$-multiplication ring if and only if R is a Prüfer $*'$-multiplication ring.

(2) If R is a Prüfer $*$-multiplication ring, then the group $\mathrm{D}_f^*(R)$ is canonically isomorphic onto $\mathrm{D}_f^{*'}(R)$.

(3) R is a Prüfer v-multiplication ring if and only if R is a Prüfer v'-multiplication ring.

Lemma 3([OMS]). *(1) If dim $(S) = \infty$, then $\mid \Sigma'(S) \mid = \infty$.*

(2) Assume that S is integrally closed with finite dimension n. Then S is a valuation semigroup if and only if $n + 1 \leq \mid \Sigma'(S) \mid \leq 2n + 1$.

(3) Let k and n be positive integers such that $n + 1 \leq k \leq 2n + 1$, then there exists an n-dimensional valuation semigroup S such that $\mid \Sigma'(S) \mid = k$.

(4) Let S be an integrally closed semigroup which is not a valuation semigroup, then $\mid \Sigma'(S) \mid = \infty$.

The integral closure of S in $\mathrm{q}(S)$ is simply called the integral closure of S, and is denoted by $\bar{S}$.

Proposition 4.

(1) Let n be a positive integer. Then there exists an integrally closed semigroup S with dimension n which is not a valuation semigroup.

(2) Assume that S has finite dimension n, S is not integrally closed, and that $\bar{S}$ is not a valuation semigroup. Then $\mid \Sigma'(S) \mid = \infty$.

(3) Let n be a positive integer. Then there exists a g-monoid S which has finite dimension n, S is not integrally closed, and that $\bar{S}$ is not a valuation semigroup.

(4) Let n be a positive integer. There exists a g-monoid S which has dimension n, S is not integrally closed, and $\bar{S}$ is a valuation semigroup, and that $\mid \Sigma'(S) \mid = \infty$.

(5) There exists a g-monoid S which has finite dimension, S is not integrally closed, $\bar{S}$ is a valuation semigroup, and that $\mid \Sigma'(S) \mid < \infty$.

Proof. Let Γ be a totally ordered abelian group of rank n, H a torsion-free abelian group, $G = H \oplus \Gamma$, and v the projection mapping of G to Γ. Let V be the valuation semigroup of v, and let M be the maximal ideal of V. Let K be a subgroup of H which is properly contained in H, and let $S = K \cup M$.

(1) Assume that K is integrally closed in H. Then S is a desired one.

(2) follows from Lemma 3 (4).

(3) Let K' be the integral closure of K in H. Assume that $K \subsetneqq K' \subsetneqq H$. Then S is a desired one.

(4) Let $H = \mathbf{Z}u_1 + \mathbf{Z}u_2 + \cdots$, where $u_1, u_2, \cdots$ are linearly independent over $\mathbf{Z}$. Let $K = \mathbf{Z}2u_1 + \mathbf{Z}2u_2 + \cdots$, and let $K_i = \mathbf{Z}u_1 + \cdots + \mathbf{Z}u_{i-1} + \mathbf{Z}2u_i + \mathbf{Z}2u_{i+1} + \cdots$

for each i. Each $K_i \cup M$ is an oversemigroup of S for each i. It follows that S is a desired one.

(5) Let $S = \{0, 2, 3, 4, \cdots\}$.

Let T be an extension ring of R. R is called seminormal in T, if the following condition hold: If $x \in T, x^2 \in R$ and $x^3 \in R$, then $x \in R$.

R is called t-closed in T, if the following condition hold: If $x \in T, a \in R, x^2 - ax \in R$ and $x^3 - ax^2 \in R$, then $x \in R$.

Lemma 5 *([BCM]). If R is seminormal in T, then $R[X]$ is seminormal in $T[X]$.*

Lemma 6 *([OSY]). Let E be an extension domain of D. If D is t-closed in E, then $D[X]$ is t-closed in $E[X]$.*

[M7] asserts that [OSY] proved Lemma 6 for commutative rings. Indeed, it holds for commutative rings. And its proof is a simple modification of that of Lemma 6. For convenience, we will note it.

Proposition 7. *If R is t-closed in the extension ring T, then $R[X]$ is t-closed in $T[X]$.*

Proof. Suppose that $R[X]$ is not t-closed in $T[X]$. There exists elements $F = \sum b_i X^i \in T[X] - R[X]$ and $f = \sum a_i X^i \in R[X]$ such that $F^2 - fF \in R[X]$ and $F^3 - fF^2 \in R[X]$. We choose the degree m of F minimal. Let $G = F - b_0$, and let $g = f - a_0$. Easily we have $b_0 \in R$.

Step 1. There exists a positive integer r such that $b_0^r F \in R[X]$.

For, we may assume that $b_0 \neq 0$. An element x of T is called good for b_0 if there erxists a positive integer r' such that $b_0^{r'} x \in R$. Suppose that $b_0, \cdots, b_{i-1}$ are good. The coefficient of $F^2 - fF$ (resp. $F^3 - fF^2$) of degree i is $2b_0 b_i - a_0 b_i + e_1$ (resp. $3b_0^2 b_i - 2a_0 b_0 b_i + e_2$), where e_1 (resp. e_2) is good. Then

$$b_0^2 b_i = 2b_0(2b_0 b_i - a_0 b_i) - (3b_0^2 b_i - 2a_0 b_0 b_i) + e_3, \text{where } e_3 \text{ is good.}$$

Hence there exists a positive integer r such that $b_0^r F \in R[X]$.

Step 2. $b_0 F, b_0 F^2 \in R[X]$.

For, let r be the minimal positive integer r such that $b_0^r F \in R[X]$. Suppose that $r \geq 2$. Then $(b_0^{r-1} F)^2, (b_0^{r-1} F)^3 \in R[X]$. By Lemma 5, we have $b_0^{r-1} F \in R[X]$; a contradiction.

Step 3. $a_0 G, a_0 G^2 \in R[X]$.

For, we may assume that $a_0 \neq 0$. Step 2 implies that $G^2 - fG \in R[X]$ and $G^3 - fG^2 \in R[X]$. Assume that $b_0, \cdots, b_{i-1}$ are good for a_0. The coefficient of $G^2 - fG$ of degree i is $a_0 b_i + e$, where e is good. Hence there exixts a positive integer r such that $a_0^r G \in R[X]$.

Let r be the minimal positive integer r such that $a_0^r G \in R[X]$. Suppose that $r \geq 2$. Then $(a_0^{r-1} G)^2, (a_0^{r-1} G)^3 \in R[X]$. By Lemma 5, we have $a_0^{r-1} G \in R[X]$; a contradiction.

Step 4. $G^2 - gG \in R[X]$ and $G^3 - gG^2 \in R[X]$. This contradicts to the choice of m.

The semigroup ring of S over R is denoted by $R[X; S]$; $R[X; S] = \{\Sigma_{finite}\ a_i X^{s_i} \mid a_i \in R, s_i \in S\}$. Proposition 7 implies the following,

Proposition 8. *If R is t-closed in the extension ring T, then $R[X;S]$ is t-closed in $T[X;S]$.*

Similarly, Lemma 5 implies the following,

Proposition 9. *If R is seminormal in the extension ring T, then $R[X;S]$ is seminormal in $T[X;S]$.*

Let T be an extension ring of R. R is called u-closed in T if the following condition holds: If $u \in T, u^2 - u \in R$ and $u^3 - u^2 \in R$, then $u \in R$.

Lemma 10([OSY]). *Let E be an extension domain of D. If D is u-closed in E, then $D[X]$ is u-closed in $E[X]$.*

[M7] also asserts that [OSY] proved Lemma 10 for commutative rings. Indeed, it holds for commutative rings. And its proof is almost the same with that of Lemma 10. For convenience, we will note it.

Proposition 11. *If R is u-closed in the extension ring T, then $R[X]$ is u-closed in $T[X]$.*

Proof. Let $F = \sum b_i X^i \in T[X], F^2 - F \in R[X]$, and let $F^3 - F^2 \in R[X]$. Assume that $b_0, b_1, \cdots, b_{i-1} \in R$. The coefficient of $F^2 - F$ (resp. $F^3 - F^2$) of degree i is $2b_0b_i - b_i + a_1$ (resp. $3b_0^2b_i - 2b_0b_i + a_2$), where a_1 (resp. a_2) $\in R$. Then we have

$$b_0b_i = 3b_0(2b_0b_i - b_i) - 2(3b_0^2b_i - 2b_0b_i) \in R.$$

Hence $b_i \in R$, and hence $F \in R[X]$.

Proposition 11 implies the following,

Proposition 12. *If R is u-closed in the extension ring T, then $R[X;S]$ is u-closed in $T[X;S]$.*

Let A be a non-empty set. If A satisfies the following condition, then A is called a module over S (or an S-module):

(1) If $s \in S$ and $a \in A$, then there is defined an element $s + a \in A$. (2) $0 + a = a$. (3) If $s_1, s_2 \in S$, then $(s_1 + s_2) + a = s_1 + (s_2 + a)$.

Let A be an S-module. If A satisfies the following condition, then A is called a cancellative S-module: If $s_1, s_2 \in S, a \in A$ and $s_1 + a = s_2 + a$, then $s_1 = s_2$.

If each ideal of S is finitely generated, then S is called a Noetherian semigroup.

Lemma 13([TM]). *Let S be a Noetherian semigroup with maximal ideal M, and let A be a finitely generated cancellative S-module. Then we have $\cap_n(nM + A) = \emptyset$.*

[M6] asserts that Lemma 13 holds if only A is a finitely generated S-module.

Remark 14. Let S be a Noetherian semigroup with maximal ideal M, and let A be a finitely generated S-module. Then $\cap_n(nM + A)$ need not be empty.

Example. Let A be an S-module with only one element. Then $\cap_n(nM+A) \neq \emptyset$.

Let A be an S-module. An element $s \in S$ is called a non-zerodivisor on A if the following condition hold: If $a_1, a_2 \in A$ and $s + a_1 = s + a_2$, then $a_1 = a_2$. The set of zerodivisors on A is denoted by $\mathrm{Z}(A)$. If $\mathrm{Z}(A)$ is empty, then A is called torsion-free. Let B be a submodule of A. An element $s \in S$ is called a non-zerodivisor on A modulo B if the following condition hold: If $a \in A$ and $s + a \in B$, then $a \in B$. The set of zerodivisors on A modulo B is denoted by $\mathrm{Z}(A/B)$.

The ordered sequence $x_1, \cdots, x_n$ of elements of S is called a regular sequence on A if the following conditions hold: $(x_1, \cdots, x_n) + A \subsetneqq A$, $x_1 \notin \mathrm{Z}(A), x_2 \notin \mathrm{Z}(A/((x_1) + A)), \cdots, x_n \notin \mathrm{Z}(A/((x_1, \cdots, x_{n-1}) + A))$.

Let I be an ideal of S. Let $x_1, \cdots, x_n$ be a regular sequence in I on A. If $x_1, \cdots, x_n, x$ is not a regular sequence on A for each $x \in I$, then $x_1, \cdots, x_n$ is called a maximal regular sequence in I on A. The maximum of lengths of all regular sequences in I on A is called the grade of I on A, and is denoted by $\mathrm{G}(I, A)$.

Let S be a Noetherian semigroup with maximal ideal M. If $\mathrm{G}(M, S) = \dim (S)$, then S is called a Macaulay semigroup.

Lemma 15 *([M4]). Let S be a Noetherian semigroup, and let A be a finitely generated torsion-free cancellative S-module. Let $x_1, \cdots, x_n$ be a regular sequence on A. Then any permutation of the x's is a regular sequence on A.*

Let X be an indeterminate. The g-monoid $S[X] = S + \mathbf{Z}_0 X$ is called the polynomial semigroup of X over S, where $\mathbf{Z}_0$ denotes the non-negative integers. The following Proposition appears in [M5] without proof. We will note the proof.

Proposition 16. *The polynomial semigroup $S[X]$ is a Macaulay semigroup if and only if S is a Macaulay semigroup.*

Proof. The necessity: There exists elements $x_1, \cdots, x_n$ of M which forms a regular sequence on S, where $n = \dim(S)$. Then $x_1, \cdots, x_n, X$ is a regular sequence on $S[X]$.

The sufficiency: There exist elements $f_1, \cdots, f_{n+1}$ of the maximal ideal of $S[X]$ which forms a regular sequence on $S[X]$. Assume that $f_{n+1} = g_1 + g_2$, where g_1, g_2 are non-units of $S[X]$. Then $f_1, \cdots, f_n, g_1$ is a regular sequence on $S[X]$. By Lemma 15, there exist elements $a_1, \cdots, a_n$ in M such that $a_1, \cdots, a_n, X$ forms a regular sequence on $S[X]$. Then $a_1, \cdots, a_n$ is a regular sequence on S.

Let $\{P_\lambda \mid \lambda \in \Lambda\}$ be the set of height 1 prime ideals of S. Then S is a Krull semigroup if and only if the following conditions hold: (1) Each S_{P_λ} is the valuation semigroup of a $\mathbf{Z}$-valued valuation on $G = \mathrm{q}(S)$. (2) $S = \cap_\lambda S_{P_\lambda}$. (3) Each element of S is a unit of S_{P_λ} for almost all λ.

There naturally arises the following question: Let S be a Krull semigroup, and let V be a $\mathbf{Z}$-valued valuation oversemigroup of S. Does the center of V on S have height 1?

Remark 17. Let S be a Krull semigroup, and let V be a $\mathbf{Z}$-valued valuation oversemigroup of S. Then the center of V on S need not be of height 1.

Example. Let H be a torsion-free abelian group, and let $S = H[X, Y]$ be the polynomial semigroup over H of indeterminates X and Y. Let V be the (X, Y)-adic valuation oversemigroup of S.

Similarly, we have the following,

Remark 18. Let D be a Krull domain and let V be a $\mathbf{Z}$-valued valuation overring of D. Then the center of V on D need not be of height 1.

Let G be an abelian additive group which is not necessarily torsion-free, and let p be a prime number. Then the subgroup $\{x \in G \mid p^n x = 0$ for some positive integer $n\}$ is called p-primary component of G, and is denoted by G_p.

Let Ω be the set of prime numbers p such that $p1_R$ is a non-unit of R.

If R_M is a Noetherian ring for each maximal ideal M of R, then R is called a locally Noetherian ring.

Lemma 19([M3]). *Let H be the unit group of S, and let F be a free subgroup of H such that H/F is torsion. The following conditions are equivalent.*

(1) $R[X; S]$ is locally Noetherian.

(2) R is locally Noetherian, t.f.r. $(H) < \infty, (H/F)_p$ is a finite group for each $p \in \Omega$, and S is of the form $H + \mathbf{Z}_0 s_1 + \cdots + \mathbf{Z}_0 s_n$, where t.f.r. means torsion-free rank.

If R_M is a regular local ring for each maximal ideal M of R, then R is called locally regular. If $R[X; S]$ is locally regular, then t.f.r. $(\mathrm{q}(S)) < \infty$ by Lemma 19.

Lemma 20(cf. [L]). *Let G be a torsion-free abelian group, and let F be a free subgroup of G such that G/F is torsion. The following conditions are equivalent.*

(1) $R[X; G]$ is locally regular.

(2) R is locally regular, t.f.r. $(G) < \infty$, and $(G/F)_p$ is finite for each $p \in \Omega$.

Lemma 21. *Let H be the unit group of S. Assume that $R[X; S]$ is locally regular.*

(1) $R[X; H]$ is locally regular.

(2) S is isomorphic onto a polynomial semigroup over H of a finite number of indeterminates.

Proof. (1) follows from Lemma 20.

(2) S is a Noetherian semigroup by Lemma 19. Let $p_1, \cdots, p_n$ be a complete representative system of irreducible elements of S. Let P be a maximal ideal of R, and let $D = R_P$. D is a regular local ring, and hence D is a unique factorization domain. Let M be a maximal ideal of S. Since $D[X; S]$ is locally regular, $D[X; S]_{MD[X;S]}$ is a regular local ring. It follows that each X^{p_i} is a prime element of $D[X; S]_{MD[X;S]}$. Therefore S is a unique factorization semigroup. Then S is isomorphic onto $H[X_1, \cdots, X_n]$.

Theorem 22. *Let H be the unit group of S, and let F be a free subgroup of H such that H/F is torsion. The following conditions are equivalent.*

(1) $R[X; S]$ is locally regular.

(2) R is locally regular, t.f.r. $(H) < \infty$, $(H/F)_p$ is finite for each $p \in \Omega$, and S is isomorphic onto a polynomial semigroup over H of a finite number of indeterminates.

Proof. (2) $\Longrightarrow$ (1): Then $R[X; H]$ is locally regular. Hence the polynomial ring over $R[X; H]$ of a variable is locally regular. Since $R[X; S]$ is isomorphic onto a polynomial ring over $R[X; H]$ of a finite number of indeterminates, $R[X; S]$ is locally regular.

Let S be a g-monoid with maximal ideal M, let A be a finitely generated S-module, and let B be a submodule of A such that $A \subset B \cup (M + A)$. [SM] proved the following: If, for each element a of A, a does not belong to $M + a$, then $B = A$.

Remark 23. Let S be a g-monoid with maximal ideal M, let A be a finitely generated S-module, and let B be a submodule of A such that $A \subset B \cup (M + A)$. Then B need not coincide with A.

Example. We may construct a finitely generated S-module A with a submodule B and with an element x such that $A = B \cup \{x\}$, $x \notin B$, and $S + x = \{x\}$.

REFERENCES

[BCM] J. Brewer, D. Costa and K. McCrimmon, *Seminormality and root closure in polynomial rings and algebraic curves*, J. Algebra **58** (1979), 217-226.

[L] D. Lantz, *Preservation of local properties and chain conditions in commutative group rings*, Pacific J. Math. **63** (1976), 193-199.

[M1] R. Matsuda, *On some open questions and related results in ideal theory*, Proc. 7 th Sympos. on Commutative Ring Theory, 1985, pp. 34-41.

[M2] R. Matsuda, *Torsion-free abelian semigroup rings VI*, Bull. Fac. Sci., Ibaralki Univ. **18** (1986), 23-43.

[M3] R. Matsuda, *Notes on torsion-free abelian semigroup rings*, Bull. Fac. Sci., Ibaraki Univ. **20** (1988), 51-59.

[M4] R. Matsuda, *Note on Macaulay semigroups*, Tsukuba J. Math. **23** (1999), 189-199.

[M5] R. Matsuda, *Some results on commutative semigroups and semigroup rings*, Coll. Res. Ins. Math. Sci., Kyoto Univ. **1106** (1999), 106-115.

[M6] R. Matsuda, *Note on Larsen-McCarthy's Multiplicative Theory of Ideals*, Math. J. Ibaraki Univ. **32** (2000), 1-9.

[M7] R. Matsuda, *Note on closedness of a semigroup ring*, JP J. of algebra, number theory and applications **1** (2001), 11-17.

[M8] R. Matsuda, *Torsion-free abelian semigroup rings X*, Math. J. Ibaraki Univ. **33** (2001), 1-7.

[M9] R. Matsuda, *Multiplicative Ideal Theory for Semigroups, 2nd. ed.*, Kaisei, Tokyo, 2002.

[M10] R. Matsuda, *Commutative Semigroup Rings*, Kaisei, Tokyo (to appear).

[OMS] H. Ozawa, R. Matsuda and K. Satô, *Semistar-operations on semigroups*, Memoirs of Tohoku Inst. tech. **16** (1996), 1-14.

[OSY] N. Onoda, T. Sugatani and K. Yoshida, *Local quasinormality and closedness type criteria*, Houston J. Math. **11** (1985), 247-256.

[SM] K. Satô and R. Matsuda, *Note on Narita's Ideal Theory*, Memoirs of Tohoku Inst. Tech. **19** (1999), 9-26.

[TM] T. Tanabe and R. Matsuda, *Note on Kaplansky's Commutative Rings*, Nihonkai Math. J. **10** (1999), 31-61.