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http://www.lib.ibaraki.ac.jp/toiawase/toiawase.html
A note on denominator ideals of linear fractional transforms of an anti-integral element over an integral domain

JUNRO SATO*, KIYOSHI BABA** AND KEN-ICHI YOSHIDA***

ABSTRACT. Let \( \alpha \) be an anti-integral element of degree \( t \) over an integral domain \( R \) and \( \varphi_\alpha(X) \) the minimal polynomial of \( \alpha \) over the quotient field of \( R \). Let \( \beta \) be a linear fractional transform of \( \alpha \), that is,

\[
\beta = \frac{ca - d}{a' - b} \quad (a, b, c, d \in R, ad - bc \in R^*)
\]

where \( R^* \) is the group of units of \( R \). First we describe \( I[\beta] \), the denominator ideal of \( \beta \), in terms of \( I[\alpha] \) and \( \varphi_\alpha(a, b) \) where \( \varphi_\alpha(X, Y) = X^t \varphi_\alpha(Y/X) \). Next we introduce the ideal \( I[\alpha] \) concerning integral property of \( \alpha \) and \( \alpha^{-1} \). Then we describe \( I[\beta] \) by using \( I[\alpha] \), \( \varphi_\alpha(a, b) \) and \( \varphi_\alpha(c, d) \).

Let \( R \) be an integral domain with quotient field \( K \) and \( R[X] \) a polynomial ring over \( R \) in an indeterminate \( X \). Let \( \alpha \) be an element of an algebraic field extension of \( K \) and \( \varphi_\alpha(X) \) the monic minimal polynomial of \( \alpha \) over \( K \) with \( \deg \varphi_\alpha = t \), and write \( \varphi_\alpha(X) = X^t + \eta_1X^{t-1} + \cdots + \eta_t, (\eta_1, \ldots, \eta_t \in K) \). We define \( I[\alpha] := \bigcap_i \{ R : \mathfrak{R} \} \) and \( J[\alpha] := I[\alpha](1, \eta_1, \ldots, \eta_t) \) where \( \{ R : \mathfrak{R} \} = \{ c \in R ; \mathfrak{R} \mathfrak{R} \in R \} \) and \( (1, \eta_1, \ldots, \eta_t) \) is the \( R \)-module generated by \( 1, \eta_1, \ldots, \eta_t \). An element \( \alpha \) is called an anti-integral element of degree \( t \) over \( R \) if \( \ker \pi = I[\alpha]\varphi_\alpha(X)R[X] \).

Set \( \varphi_\alpha(X, Y) = X^t \varphi_\alpha(Y/X) \). Since \( \alpha = (b\beta - d)/(a\beta - c) \), it is easily verified that

\[
\varphi_\beta(X) = \varphi_\alpha(a, b)^{-1} \varphi_\alpha(aX - c, bX - d).
\]

Our notation is standard and our general reference for unexplained terms is [2].

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Theorem 1. Let $R$ be an integral domain and $\alpha$ an algebraic element of degree $t$ over the quotient field of $R$. Let $\beta$ be a linear fractional transform of $\alpha$, that is,

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R, ad - bc \in R^*, a\alpha - b \neq 0).$$

Then $I[\beta] = \varphi(\alpha, b)I[\alpha]$.

Proof. Since $I[\beta] = R[X] : R \varphi(X)$, we see that $\varphi(\alpha, b)I[\alpha] \subseteq I[\beta]$ by equality (1). Similarly, we have $\varphi(\alpha, c)I[\beta] \subseteq I[\alpha]$ because $\beta = (c\alpha - d)/(a\alpha - b)$. Set $u = bc - ad$. Then by equality (1) we get $\varphi(\alpha, c) = a^t \varphi(c/a) = u^t \varphi(\alpha, b)^{-1}$. Hence $I[\beta] \subseteq u^{-t} \varphi(\alpha, b)I[\alpha] = \varphi(\alpha, b)I[\alpha]$. Therefore $I[\beta] = \varphi(\alpha, b)I[\alpha]$. Q.E.D.

Let $\alpha$ be an anti-integral element of degree $t$ over an integral domain $R$. Then $\alpha$ is integral over $R$ if and only if $I[\alpha] = R$ by [3, Theorem 2.2]. Note that $\beta$ is also an anti-integral element of degree $t$ over $R$ by [1, Theorem 4].

Corollary 2. Let $R, \alpha, \beta$ be the same as in Theorem 1 and assume that $\alpha$ is an anti-integral element over $R$. Then the following conditions are equivalent:

(i) $\beta$ is integral over $R$.

(ii) $\varphi(\alpha, b)I[\alpha] = R$.

The proof is immediate from Theorem 1.

Note that, if $I[\alpha]$ is not a principal ideal of $R$, then $\beta$ is not integral over $R$ by Corollary 2.

Let $\alpha$ be an algebraic element of degree $t$ over a Noetherian domain $R$. We say that $\alpha$ is a super-primitive element if $J[\alpha] \subseteq \mathfrak{p}$ for every element $\mathfrak{p}$ of $Dp_1(R)$ where $Dp_1(R) = \{\mathfrak{p} \in \text{Spec}R; \text{depth}R_\mathfrak{p} = 1\}$. Super-primitive elements are anti-integral elements by [3, Theorem 1.12].

Corollary 3. Let $R$ be a Noetherian domain and $\alpha$ a super-primitive element over $R$. Let $\beta$ be the same as in Theorem 1. Then the following conditions are equivalent:

(i) $I[\beta] = I[\alpha]$.

(ii) $\varphi(\alpha, b)$ is a unit of $R$.

Proof. (ii) $\Rightarrow$ (i). The assertion is obvious from Theorem 1.

(i) $\Rightarrow$ (ii). By Theorem 1, we get $\varphi(\alpha, b)I[\alpha] = I[\alpha]$. By the definition of $I[\alpha]$, we see that $I[\alpha] \neq (0)$. By the assumption, $\alpha$ is a super-primitive element. Hence, for every element $\mathfrak{p}$ of $Dp_1(R)$, there exists a non-zero element $z$ of $I[\alpha]$ such that $I[\alpha]R_\mathfrak{p} = zR_\mathfrak{p}$ by [3, Theorem 2.11]. Therefore $\varphi(\alpha, b)R_\mathfrak{p} = R_\mathfrak{p}$ for every element $\mathfrak{p}$ of $Dp_1(R)$. Since $\bigcap_{\mathfrak{p} \in Dp_1(R)} R_\mathfrak{p} = R$, we see that $\varphi(\alpha, b)$ is in $R$. If $\varphi(\alpha, b)$ is not a unit of $R$, there exists an element $q$ of $Dp_1(R)$ such that $\varphi(\alpha, b) \subseteq q$ because every prime divisor of a principal ideal is of depth one. This is absurd. Hence $\varphi(\alpha, b)$ is a unit of $R$. Q.E.D.

Let $\alpha$ be a non-zero algebraic element of degree $t$ over an integral domain $R$. Then we define the ideal $I[\alpha]$ of $R$ by $I[\alpha] + I[\alpha^{-1}]$. Note that $\tilde{I}[\alpha] = (1, \eta)I[\alpha]$ by Theorem 1.
Remark 4. Let $\alpha$ be a non-zero algebraic element of degree $t$ over an integral domain $R$. Let $p$ be an element of $\text{Spec}(R)$. If $p \not\in \tilde{I}_[\alpha]$, then $\alpha$ is integral over $R_p$ or $\alpha^{-1}$ is integral over $R_p$.

Proof. We see that $p \not\in I_{[\alpha]}$ or $p \not\in I_{[\alpha^{-1}]}$. Then $I_{[\alpha]}R_p = R_p$ or $I_{[\alpha^{-1}]}R_p = R_p$. Hence $\alpha$ is integral over $R_p$ or $\alpha^{-1}$ is integral over $R_p$. Q.E.D.

Corollary 5. Let $R, \alpha, \beta$ be the same as in Theorem 1. Then $\tilde{I}_{[\beta]} = (\varphi_\alpha(a, b), \varphi_\alpha(c, d))I_{[\alpha]}$.

Proof. Theorem 1 implies that $I_{[\beta]} = \varphi_\alpha(a, b)I_{[\alpha]}$ and $I_{[\beta^{-1}]} = \varphi_\alpha(c, d)I_{[\alpha]}$. Hence we obtain $\tilde{I}_{[\beta]} = (\varphi_\alpha(a, b), \varphi_\alpha(c, d))I_{[\alpha]}$. Q.E.D.

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