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On Characterizations of Divided Monoids

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1. Introduction.

Let G be a torsion-free abelian additive group, and let S be a subsemigroup of G which contains 0. Then S is called a *grading monoid* (cf.[8]). In this paper,unless otherwise indicated, we will call a grading monoid as a *monoid* . A non-empty subset I of S is called an *ideal* of S if $S+I \subseteq I$. In this paper an ideal of S is assumed to be *proper*, that is, $I \neq S$. If $s_1+s_2 \in I$ with $s_1, s_2 \in S$ implies $s_1 \in I$ or $s_2 \in I$, then I is called a *prime ideal* of S. For each $x \in S$, the set $x+S$ is an ideal of S and is called a *principal ideal* of S. An ideal $x+S$ is denoted by (x). For any two ideals I and J of S, we set $I+J = \{i+j \mid i \in I \text{ and } j \in J\}$. Then $I+J$ is also an ideal of S and $I+J \subseteq I \bigcap J$ holds. A nonempty subset T of a monoid S is called an *additive system* if $t, t' \in T$, then $t+t' \in T$. For each an additive system T of S, the set $S_T = \{s-t \mid s \in S \text{ and } t \in T\}$ is also a monoid and is called the *quotient monoid* of S with respect to T. Especially, if $T = S$, then the quotient monoid $S_S = \{s_1 - s_2 \mid s_1, s_2 \in S\}$ is called the *quotient group* of S and is denoted by $q(S)$. Note that $q(S)$ is evidently an abelian additive group. Each monoid which lies between S and $q(S)$ is called an *overmonoid* of S.

Before discussing divided monoids, let us recall some fundamental results and notions in monoid theory for the convenience of the reader.

Remark 1. Let P be an ideal of S. Then P is a prime ideal of S if and only if $T = S \backslash P$ is an additive system of S. In particular, if P is a prime ideal of S and $T = S\backslash P$, then the quotient monoid S_T is usualy denoted by S_P.

An element x of S is called a *unit* of S if $x+y=0$ for some $y \in S$. If U is the set of units of S, then the set $M = S\backslash U$ is an ideal of S that contains all the proper ideals of S and is called the *maximal ideal* of S. We shall often denote the maximal ideal of S by $M(S)$.

Remark 2. If P is a prime ideal of S, then PS_P is the maximal ideal of S_P. In fact, if $x \in S_P \backslash PS_P$, then $x = a-b$ with $a, b \in S\backslash P$. Hence $-x = b-a \in S_P$ which means that x is a unit of S_P.

Remark 3. If P is a prime ideal of S, then $PS_P \bigcap S = P$.

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Remark 4. Let T be an overmonoid of a monoid S. If P is a prime ideal of T, then $P \bigcap S$ is a prime ideal of S.

Definition 1. A nonempty subset A of $q(S)$ is called a *fractional ideal* of S, if $S + A \subseteq A$ and $x + A \subseteq S$ for some element $x \in S$.

Remark 5. For each element $x \in q(S)$, the set $x + S$ is a fractional ideal of S. In fact, if we set $x = a - b$ with $a, b \in S$, then $S + (x + S) = x + S$ and $b + (x + S) = a + S \subseteq S$.

2. Divided Monoids.

Definition 2. A prime ideal P of S is called a *comparable ideal* of S if $P \subseteq (x)$ or $(x) \subseteq P$ for any element $x \in S$. By using the notion of a comparable ideal, we shall introduce the notion of a divided monoid as follows: A monoid S is called a *divided monoid* if every prime ideal P of S is a comparable ideal.

Definition 3. If I is an ideal of a monoid S, then the set $I : I = \{x \in q(S) \mid x + I \subseteq I\}$ is also a monoid and is called a *conductor overmonoid.* Evidently I is an ideal of $I : I$.

Definition 4. Let P be a prime ideal of S and T be an overmonoid of S. Then P is said to be *T-strong* if $a + b \in P$ with $a, b \in T$ implies $a \in P$ or $b \in P$.

Lemma 1. *Let P be a prime ideal of a monoid S. Then the following conditions are equivalent.*

(1) P is a comparable ideal of S.

(2) $PS_P = P$.

Proof. (1) $\Rightarrow$ (2). Choose any element $x \in PS_P$. Then $x = a - b$ with $a \in P$ and $b \in S \backslash P$. Since $P \subset (b)$, $a = b + s$ with some $s \in S$, and then $x = (b + s) - b = s \in S$. Hence, we have $x \in S \bigcap PS_P = P$.

(2) $\Rightarrow$ (1). Assume that $x \in S \backslash P$ with some $x \in S$. Then $a - x \in PS_P = P$ for any $a \in P$. Hence $a \in x + P \subseteq x + S = (x)$ and so $P \subset (x)$. □

We shall denote the set of prime ideals of S by *Spec*(S).

Lemma 2 ([6,Proposition 1(1)]). *If S is a divided monoid, then Spec(S) is a linearly ordered set.*

Proof. Let $P \not\subseteq Q$ be prime ideals of S. If we choose an element $x \in P \backslash Q$, then $Q \subset (x) \subseteq P$. Thus *Spec*($S$) is linearly ordered. □

We shall now prove the main result of this paper. This is a semigroup version of [9, Theorem 2.2]. For Theorem 3 (1) $\Longrightarrow$ (7),see the Proposition 2 in [6].

Theorem 3. *Let S be a monoid with quotient group G. Then the following conditions are equivalent:*

(1) S is a divided monoid.

(2) For each prime ideal P of S, $P : P = PS_P : PS_P$.

(3) For each prime ideal P of S, $S_P \subseteq P : P$.

(4) For each prime ideal P of S, P is S_P-strong.

(5) For each prime ideal P of S, $-x \in P : P$ whenever $x \in S \backslash P$.

(6) For each prime ideal P of S, $-x \in P : P$ whenever $x \in S_P \backslash P$.

(7) For each ideal I and prime ideal P of S, I and P are comparable.

(8) For each nonmaximal prime ideal P of S, $P = S :_S S_P$.

(9) For any two ideals I and J of S, $I \subseteq J$ or $J \subseteq \sqrt{I}$.

(10) If P is a prime ideal of S, then the maximal ideal M of $P : P$ satisfies $M \bigcap S = P$.

(11) If P is a prime ideal of S, then for each proper ideal I of $P : P$, we have $I \bigcap S \subseteq P$.

Proof. (1) $\Rightarrow$ (2). This follows from Lemma 1.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Since $S_P \subseteq P : P$, $PS_P \subseteq P(P : P) \subseteq P$, and hence $PS_P = P$. Then our assertion follows from Lemma 1.

(1) $\Rightarrow$ (4). Let $a + b \in P$ with $a, b \in S_P$. Then $a - s, b - t \in S$ for some $s, t \in S \backslash P$. Hence $(a - s) + (b - t) = a + b - (s + t) \in PS_P = P$. Then, by (1), $a - s \in P$ or $b - t \in P$. Hence $a = (a - s) + s \in P + S \subseteq P$ or $b = (b - t) + t \in P + S \subseteq P$. Thus P is S_P-strong.

(4) $\Rightarrow$ (1). Choose $x \in PS_P$. Then $x = a - s$ with $a \in P$ and $s \in S \backslash P$ and then $x + s = a \in P$. Then, by (4), we have $x \in P$ and therefore $PS_P = P$.

(1) $\Rightarrow$ (6). If $x \in S_P \backslash P$, then $x \in S_P \backslash PS_P$, and so $-x \in S_P \subseteq P : P$.

(6) $\Rightarrow$ (5). This is trivial.

(5) $\Rightarrow$ (3). Choose $a - s \in S_P$ with $a \in S$ and $s \in S \backslash P$. Then, by hypothesis, we have $-s \in P : P$ and so $a - s \in S + P : P = P : P$.

(1) $\Longleftrightarrow$ (7). This is trivial.

(1) $\Rightarrow$ (8). Let P be a nonmaximal prime ideal of S. $P \subseteq S :_S S_P$ is clear. Now suppose that $S :_S S_P \not\subseteq P$. If we choose $x \in (S :_S S_P) \backslash P$, then we have $x + S_P \subseteq S$ and $-x \in S_P$. Hence $S_P = x + S_P - x = x + S_P \subseteq S$. Thus we get $S_P = S$ and therefore $P = PS_P$ is the maximal ideal of $S_P = S$, a contradiction. Hence $S :_S S_P \subseteq P$ also holds.

(8) $\Rightarrow$ (1). If P is a nonmaximal prime ideal of S, then we get $PS_P \subseteq S$, because $P = S :_S S_P$. Hence we have $P = PS_P \bigcap S = PS_P$.

(7) $\Rightarrow$ (9). Let I and J be two ideals in S and suppose J is not contained in $\sqrt{I}$. Then there exists a prime ideal P of S such that $I \subseteq P$ but J is not contained in P. Then, by (7), we have $I \subseteq P \subset J$.

(9) $\Rightarrow$ (1). Let P be a prime ideal of S and let x be an element of S. Then we have $P \subseteq (x)$ or $(x) \subseteq \sqrt{P} = P$ by (9).

(3) $\Rightarrow$ (10). Since P is an ideal of $P : P$, we have $P \subseteq M$ and so $P \subseteq M \bigcap S$ holds. By hypothesis, $M \supseteq M \bigcap S_P$ and so $M \bigcap S_P \neq S_P$. Hence $M \bigcap S_P \subseteq PS_P$ and therefore $M \bigcap S = (M \bigcap S_P) \bigcap S \subseteq PS_P \bigcap S = P$. Hence we have $P = M \bigcap S$, as required.

(10) $\Rightarrow$ (11). This is trivial.

(11) $\Rightarrow$ (5). Suppose that there is an element $x \in S \backslash P$ such that $-x \notin P : P$. Then x is a nonunit of $P : P$ and hence $x + P : P \subset P : P$. Then, by hypothesis, $x + S \subseteq (x + P : P) \bigcap S \subseteq P$ and so $x \in P$, a contradiction. Thus our assertion holds. $\square$

Definition 5. A monoid S is called a *Noetherian monoid* if each ideal of S is finitely generated.

Definition 6. Let T be a overmonoid of a monoid S. An element t of T is said to be *integral* over S if $nt \in S$ for some positive integer n. T is said to be *integral* over S if each element of T is integral over S. Let $\bar{S}$ be the set of elements of $q(S)$ which are integral over S. Then $\bar{S}$ is an overmonoid of S. In fact, if $x, y \in \bar{S}$, then $nx \in S$ and $my \in S$ for some integers $n > 0$ and $m > 0$ and then $mn(x + y) \in S$, which implies that $\bar{S}$ is an overmonoid of S. The overmonoid $\bar{S}$ is called the *integral closure* of S.

We give a generalization theorem of Lemma 14 of [6].

Proposition 4. *If I is a finitely generated fractional ideal of a monoid S, then $I : I$ is integral over S.*

Proof. Let $I = (a_1 + S) \bigcup (a_2 + S) \bigcup \cdots \bigcup (a_k + S)$ and let x be an element of $I : I$. Then we have $x + a_1 \in a_i + S$ for some $i = 1, 2, \cdots, k$. If $x + a_1 \in a_1 + S$, then we get $x \in S$ and our proof is over. Next, we assume that $x + a_1 \notin a_1 + S$. Then we may assume , without loss of generality, that $x + a_1 \in a_2 + S$. Then $2x+a_1 \in x+a_2+S$. If $x+a_2 \in a_1+S$, then $2x \in S$ and our proof is over. Next we assume $x+a_2 \notin (a_1 + S) \bigcup (a_2 + S)$. Then we may assume that $x+a_2 \in a_3 + S$. Now, we set $x+a_2 = a_3+s$. If $x+a_3 \in a_1+S$, then $2x+a_2 = x+a_3+s \in a_1+S$,and then $3x + a_2 \in x + a_1 + S \subseteq a_2 + S$, and therefore we get $3x \in S$. Next, if $x+a_3 \in a_2+S$, then $2x+a_3 \in x+a_2+S \subseteq a_3+S$ and therefore we get $2x \in S$. By countinuing this method, we get $kx \in S$ for some positive integer k. Hence $I : I$ is integral over S. □

Proposition 5. *Let S be a monoid. Then $\bar{S} = \bigcup\{I : I \mid I$ is a finitely generated fractional ideal of $S\}$.*

Proof. ($\supseteq$). This follows from Proposition 4.

($\subseteq$). Let $x \in q(S)$ be integral over S. Then $nx \in S$ for some integer $n > 0$. Now, let $I = S \bigcup (x + S) \bigcup (2x + S) \bigcup \cdots \bigcup ((n-1)x + S)$. Then clearly $x \in I : I$ holds. □

Proposition 6. *Let T be an overmonoid of a monoid S. Assume that T is integral over S. Then*

(1) If P is a prime ideal of S, then there is a prime ideal Q of T such that $Q \bigcap S = P$.

(2) (cf.[7, (4.2)(1)]) If M is the maximal ideal of T, then $M \bigcap S$ is the maximal ideal of S.

Proof. (1). Set $N = S \backslash P$. Then N is an additive system of T and $N \bigcap (P+T) = \emptyset$. In fact, if $t \in N \bigcap (P+T)$, then $t = p+u$ with $p \in P$ and $u \in T$. By hypothesis, $ku \in S$ for some positive integer k, and so we get $kt = kp + ku \in P + S = P$. Then $kt \in P \bigcap N$, a contradiction. Now, by Zorn's Lemma, there exists an ideal Q of T that is a maximum element of ideals I of T such that $I \bigcap N = \emptyset$. Then it is easily seen that Q is a prime ideal of T and $Q \bigcap S = P$.

(2). Suppose that $P = M \bigcap S$ is not the maximal ideal of S. Then, there exists a prime ideal q of T that lies over the maximal ideal $M(S)$ of S. Then, since $q \subseteq M$, we have $M(S) = q \bigcap S \subseteq P = M \bigcap S$ and so $P = M(S)$, a contradiction. □

Definition 7. Let S be a monoid. The supremum of the lengths n, taken over all strictly asending chains $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$ of prime ideals of S, is called the *dimension* of S and is denoted by $dim(S)$.

Theorem 7. *If S is a Noetherian divided monoid, then $dim(S) = 1$.*

Proof. Let P be a prime ideal of S. Since P is a finitely generated ideal of S, $P : P$ is integral over S by Proposition 4. If M is the maximal ideal of $P : P$, then we have $M \bigcap S = P$ by Theorem 3(10). But then, by Proposition 6(2), P must be the maximal ideal of S. Thus P is the only one prime ideal of S and hence $dim(S) = 1$. □

This Theorem 7 is a Corollary of Theorem 15 in [6].

3. Pseudo valuation monoids.

In [8], we introduced the notion of a *pseudo valuation monoid* (for short, a PVM) which is a semigroup version of a pseudo-valuation domain in [4] and studied some fundamental properties of a PVM.

Definition 8. A monoid S with quotient group G is called a *pseudo valuation monoid* (for short, a *PVM*), if each prime ideal P of S is *G-strong.* If a prime ideal P of S is G-strong, then P is also called a *strongly prime ideal* of S.

We also recall that a monoid S is called a *valuation monoid* if every element x of G satisfies $x \in S$ or $-x \in S$. In [10, Proposition 1] it was proved that every valuation monoid is a PVM. Here we rist some known characterizations of a PVM for the convenience of the reader.

Proposition 8(*cf.[10,Theorem 4 and Theorem 10]*). *Let S be a monoid with maximal ideal M and quotient group G. Then the following statements are equivalent:*

(1) S is a PVM.

(2) For any two ideals I and J of S, we have either $I \subseteq J$ or $J + M \subseteq I + M$.

(3) For any two ideals I and J of S, we have either $I \subseteq J$ or $J + M \subseteq I$.

(4) M is a strongly prime ideal of S.

(5) S has the unique valuation overmonoid V with maximal ideal M.

(6) There exists a valuation overmonoid V such that every prime ideal of S is also a prime ideal of V.

In this Section we will give some new characterizations of a PVM and we will also show that each PVM is a divided monoid.

Lemma 9. *Let S be a monoid with quotient group G. Then, for a prime ideal P of S, P is a G-strong if and only if $-x \in P : P$ whenever $x \in G \backslash S$.*

Proof. ($\Longrightarrow$) Assume that $-x \notin P : P$ for some $x \in G \backslash S$. Then there exists an element $p \in P$ such that $-x + p \notin P$. Then, since $p = (-x + p) + x \in P$, we have $x \in P \subset S$, a contradiction.

($\Longleftarrow$) Suppose that $x+y \in P$ with $x, y \in G$. If $x \in S$ and $y \in S$, then clearly $x \in P$ or $y \in P$. Assume that $x \in G \backslash S$. Then, by hypothesis, $-x \in P : P$, that is , $-x + P \subseteq P$. Then $y = -x + (x+y) \in -x + P \subseteq P$. Thus P is a strongly prime ideal of S as required. □

We now give some new characterizations of a PVM.

Theorem 10. *Let S be a monoid with maximal ideal M and quotient group G. Then the following statements are equivalent.*

(1) S is a PVM.

(2) For each prime ideal P of S, we have $-x \in P : P$ whenever $x \in G \backslash S$.

(3) We have $-x \in M : M$ whenever $x \in G \backslash S$.

Proof. (1)$\Longrightarrow$ (2). This follows from Lemma 9.

(2) $\Longrightarrow$ (3). This is trivial.

(3) $\Longrightarrow$ (1). This follows from (4) $\Longrightarrow$ (1) in Proposition 8 and Lemma 9.
□

Theorem 11(*cf.[6,Proposition 1(2)]*). *Every PVM is a divided monoid.*

Proof. This follows from (1) $\Longleftrightarrow$ (4) in Theorem 3 and Definition 8. □

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