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Prime and semiprime acts over monoids with zero

JAVED AHSAN* AND LIU ZHONGKUI**

ABSTRACT. In this note we extend the notions of prime and semiprime ideals of a semigroup S to arbitrary S-acts and develop some of their basic properties. In particular, we characterize semigroups all of whose ideals are prime (semiprime).

1. Introduction and Preliminaries.

Let S be a monoid, that is, a semigroup with an identity element 1. A right unitary S-act M, denoted by $MS$ is a set M and a function $M \times S \to M$ such that if $ms$ denotes the image of $(m, s)$ for $m \in M$ and $s \in S$, then (i) $(ms)t = m(st)$ for $m \in M$, and $s, t \in S$; and (ii) $m1 = m$ for all $m \in M$. Left S-acts are defined similarly. An S-subact $NS$ of a right S-act $MS$ written as $NS \subseteq MS$, is a subset $N$ of M such that $ns \in N$ for all $n \in N$ and $s \in S$. Thus the subacts of the S-act $SS$ (resp. $SS$) are right (resp. left) ideal of $S$. The word ideal will mean two-sided ideal, that is, a subset of $S$ which is both a right and a left ideal of $S$. An element $d \in MS$ with $ds = d$ for all $s \in S$ is called a fixed element of M. Let $D$ denote the set of all fixed elements of M. A right S-act $M$ is called centered if $S$ is a semigroup with a two-sided zero element 0 and $\|D\| = 1$. Thus $M$ is centered if and only if there is a fixed element (necessarily unique) denoted by $\theta$ such that: (i) $\theta s = \theta$ for all $s \in S$; and (ii) $m0 = \theta$ for all $m \in M$; $\theta$ will be called the zero of $M$ (cf. [5]). If $I$ is an ideal of a semigroup $S$ then the Rees factor of $S$ modulo $I$ will be denoted by $S/I$; we recall that the equivalence classes of $S/I$ are $I$ (the zero of $S/I$) and every single element set $\{a\}$ with $a \in S - I$.

A right ideal $I$ of $S$ is called prime if for $a, b \in S$, the inclusion $aSb \subseteq I$ implies that either $a \in I$ or $b \in I$. Equivalently, $I$ is prime if and only if for any right ideals $A$ and $B$ of $S$, the set inclusion $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$; $I$ is called semiprime if for $a \in S$, $aSa \subseteq I$ implies that $a \in I$. Equivalently $A^2 \subseteq I$ implies that $A \subseteq I$ for all right ideals $A$ of $S$. Prime and semiprime ideals are useful tools in semigroup theory (cf. [3]). We extend these notions to arbitrary S-acts, where $S$ is a monoid with a two-sided zero element, analogous to the notions of prime and semiprime (ring) modules introduced by Dauns [4]. In what follows
S will denote a monoid with a two-sided zero 0 and all S-acts are centered right unitary as defined above.

2. Results.

We begin with some preliminary definitions.

**Definition 1.** A right ideal I of S is called irreducible if I = H ∩ K implies that I = H or I = K for right ideals H and K of S. More generally, an S-subact K of a right S-act M is irreducible if K = A ∩ B implies that K = A or K = B for any S-subacts A and B of M.

**Definition 2.** Let M be a right S-act. The ideal \( J(\theta) = J = \{ s \in S | Ms = \{ \theta \} \} \) of S is called the annihilator of M in S; M is called faithful if \( J = \{ 0 \} \). If K is an S-subact of a right S-act M, the set \( \{ s \in S | Ms \subseteq K \} \) is an ideal of S, called the associated ideal of K. This ideal will be denoted by \( J_K \).

**Definition 3.** An S-subact K of a right S-act M is a prime S-subact of M if for any \( v \in M \) and \( a \in S \), the inclusion \( vSa \subseteq K \) implies either \( v \in K \) or \( a \in J_K \). K is a semiprime S-subact of M if for any \( a \in M \) and any \( x \in S \), the inclusion \( axSx \subseteq K \) implies \( ax \in K \). The right S-act M itself is called prime (resp. semiprime) if the zero subact (\( \emptyset \)) of M is prime (resp. semiprime). In particular, the monoid S is prime (resp. semiprime) if the zero ideal (\( \{ 0 \} \)) of S is prime (resp. semiprime) as an S-subact of \( SS \).

**Proposition 1.** A right ideal I of S is prime (resp. semiprime) if and only if I is prime (resp. semiprime) as an S-subact of \( SS \).

**Proof.** Obvious.

**Proposition 2.** Every nonzero S-subact N of a prime S-act M is a prime S-act.

**Proof.** Suppose for \( a \in S \) and \( v \in N \), we have \( vSa = \{ \theta \} \). If \( v \neq \theta \), then since M is a prime S-act (that is, \( \{ \theta \} \) is a prime subact of M), it follows that \( a \in J(\theta) = J = \{ s \in S | Ms = \{ \theta \} \} \subseteq \{ s \in S | Ms = \{ \theta \} \} \). Hence N is prime.

**Proposition 3.** Let K be a proper subact of a right S-act M. Then the following statements are true:

(a) If K is a prime subact then \( J_K \) is a prime ideal of S.

(b) If K is a semiprime subact then \( J_K \) is a semiprime ideal of S.

**Proof.** (a) Consider the inclusion \( aSa \subseteq J_K \) for \( a,b \in S \). Assume \( a \notin J_K \). Then \( Ma \not\subseteq K \). Hence there exists \( x \in M \) such that \( xa \notin K \). Since \( aSa \subseteq J_K \), \( M(aSa) \subseteq K \). This implies that \( x(aSa) \subseteq K \). Since K is a prime subact and \( xa \notin K \), it follows that \( b \in J_K \), that is, \( J_K \) is a prime ideal.

(b) To show that \( J_K \) is a semiprime ideal, consider \( aSa \subseteq J_K \) for some \( a \in S \). Suppose \( a \notin J_K \). Then \( Ma \not\subseteq K \), and so there exists \( x \in M \) such that \( xa \notin K \). Since \( aSa \subseteq J_K \), \( M(aSa) \subseteq K \). This implies that \( x(aSa) \subseteq K \), that is, \( xaSa \subseteq K \). This contradicts that K is a semiprime subact. Hence \( a \in J_K \).
Proposition 4. Let \( M \) be a right \( S \)-act. Then for an \( S \)-subact \( K \) of \( M \) and the associated ideal \( J_K \), the following assertions are equivalent:

(a) \( K \) is a prime \( S \)-subact of \( M \).
(b) For all right \( S \)-subacts \( V \) of \( M \) and (right) ideals \( A \) of \( S \), \( VA \subseteq K \) implies that either \( V \subseteq K \) or \( A \subseteq J_K \).
(c) For all \( S \)-subacts \( K \) and \( W \) of \( M \) such that \( K \) is properly contained in \( W \), and for all ideals \( B \) of \( S \) such that \( J_K \) is properly contained in \( B \), one has \( WB \nsubseteq K \).

Proof. We can prove the proposition by analogy with the proof of [4, Theorem 1.3].

Corollary. A right \( S \)-act \( M \) is prime if and only if every nonzero subact of \( M \) has the same associated ideal.

Proposition 5. A monoid \( S \) is prime if and only if there exists a faithful prime \( S \)-act.

Proof. If monoid \( S \) is prime, then \( S_S \) is a faithful prime \( S \)-act. Conversely, let \( M \) be a faithful prime \( S \)-act. We show that \( S \) is a prime monoid, that is, \((0)\) is a prime ideal of \( S \). Suppose that \( aSb = (0) \) for some \( a, b \in S \). If \( a \neq 0 \) then \( MaS \neq (\theta) \). For if \( MaS = (\theta) \) then \( aS \subseteq \{ s \in S | Ms = (\theta) \} \). Thus \( a = 0 \), which is contrary to the assumption. Hence there exists \( x \in M \) such that \( xaS \neq (\theta) \). But \( aSb = (0) \). Hence \( xaSb = (\theta) \) is a proper \( S \)-subact of \( M \). Since \( M \) is a prime \( S \)-act and \( xaSb = (\theta) \) and also since \( xa \) is a nonzero element of \( M \), \( b \in J = \{ s \in S | Ms = (\theta) \} = (0) \). Hence \((0)\) is a prime ideal of \( S \), showing that \( S \) is a prime monoid.

Proposition 6. Let \( P \) be an ideal of \( S \). Then the following conditions are equivalent:

(a) \( P \) is a prime ideal.
(b) There exists a prime right \( S \)-act \( M \) with \( P = J_{(\theta_M)} = \{ s \in S | Ms = (\theta_M) \} \).

Proof. (a) \( \Rightarrow \) (b). Suppose that \( P \) is a prime ideal of \( S \). Then the Rees factor semigroup \( \overline{S} = S/P \) is a prime monoid with zero and thus by Proposition 5, there exists a faithful prime \( \overline{S} \)-act \( M \) such that \( \{ \overline{s} \in \overline{S} | M\overline{s} = (\theta_M) \} = (0) \). From this it follows that \( P = J_{(\theta_M)} \).

(b) \( \Rightarrow \) (a). Suppose that \( M \) is a prime \( S \)-act with \( P = J_{(\theta_M)} \). Then \( M \) is a prime act over the Rees factor semigroup \( S/P \) which is faithful. Hence by Proposition 5, \( S/P \) is a prime semigroup, and so \( P \) is a prime ideal of \( S \).

Next we state the following characterization of semiprime monoids which can be proved by analogy with the proof of [6, Proposition 10.16].

Theorem 1. Let \( S \) be a nontrivial monoid with zero. Then the following assertions are equivalent:

(a) \( S \) is a semiprime monoid.
(b) The intersection of all prime ideals of \( S \) is equal to \((0)\).
(c) If \( A \) is an ideal of \( S \) with \( A^2 = (0) \), then \( A = (0) \).
(d) \( S \) has no nonzero nilpotent right or left ideals.
Lemma 1. Let $M$ be a finitely generated $S$-act over a monoid $S$. Then every proper subact of $M$ is contained in a maximal subact of $M$.

Proof. By analogy with the proof of [2, Theorem 2.8], we can prove this lemma.

Lemma 2. If $K$ is a maximal subact of a right $S$-act $M$, then $K$ is a prime $S$-subact.

Proof. For elements $v \in M$ and $a \in S$, consider the inclusion $vSa \subseteq K$ with $v \notin K$. Since $K$ is a maximal subact of $M$ and $v \notin K$, $K \cup vS = M$. Let $m$ be an arbitrary element of $M$. Then $m \in K$ or $m \in vS$. Thus $m = k$ for some $k \in K$ or $m = vs$ for some $s \in S$. Then $ma = ka \in K$ or $ma = vsa \in vSa \subseteq K$. Thus $ma \in K$, in any case. Hence $a \in J_K$, showing that $K$ is a prime subact.

Combining Lemmas 1 and 2, we obtain

Theorem 2. Let $M$ be a finitely generated act over a monoid $S$ with zero. Then every proper subact of $M$ is contained in a prime subact.

Next we prove two more lemmas.

Lemma 3. Let $K$ be a proper subact of a right $S$-act $M$. Then $K$ is the intersection of all the irreducible subacts containing it.

Proof. For $m \in M \setminus K$, let $V_m$ be any subact of $M$ maximal with respect to $K \subseteq V_m$ but $m \notin V_m$. Suppose that $V_m = A \cap B$ for subacts $A, B$ of $M$ with $A \neq V_m$ and $B \neq V_m$. The maximality of $V_m$ implies that $m \in A$ as well as $m \in B$. But then $m \in A \cap B = V_m$ is a contradiction. Thus $K = \bigcap\{V_m|m \in M - K\}$ is an intersection of irreducible subacts.

Lemma 4. If $A$ is an irreducible subact of a right $S$-act $M$, then the following conditions are equivalent:
   (a) $A$ is prime.
   (b) $A$ is semiprime.

Proof. (a) $\Rightarrow$ (b). Suppose that $A$ is prime and for $\alpha \in M$ and $x \in S$, we have $\alpha xSx \subseteq A$. Since $A$ is prime, it follows that either $\alpha x \in A$ or $x \in JA$. Suppose $\alpha x \notin A$. Since $\alpha x \in Mx$, $Mx \subseteq A$. Hence $x \notin JA$. But this contradicts that $A$ is prime. Note that (a) $\Rightarrow$ (b) does not depend upon the assumption that $A$ is irreducible.

(b) $\Rightarrow$ (a). Assume that $A$ is semiprime but $A$ is not prime. Then there exist $a \in S$ and $v \in M$ such that $vSa \subseteq A$ and $v \notin A$ and $a \notin JA$. Since $a \notin JA$, it follows that $Ma \not\subseteq A$. Hence there exists $x \in M$ such that $xa \notin A$. Let $m = xa$ then $m \in M$ and $(vS \cup A) \cap (mS \cup A) \neq A$. For, if $(vS \cup A) \cap (mS \cup A) = A$, then since $A$ is irreducible, either $vS \cup A = A$ or $mS \cup A = A$. This implies that either $vS \subseteq A$ or $mS \subseteq A$, that is, either $v \in A$ or $m \in A$. Furthermore, we note that $(vS \cup A) \cap (mS \cup A) \not\subseteq A$; for otherwise, $A \subseteq (vS \cup A) \cap (mS \cup A) \subseteq A$, which again implies that $(vS \cup A) \cap (mS \cup A) = A$. Hence there exists $t \in S$ such that $mt \in vS$ but $mt \notin A$. Now $mtSat = xatSat \subseteq (vS)Sat \subseteq (vS)a \subseteq At \subseteq A$ but $mt = xat \notin A$. This implies that $A$ is not semiprime, which proves the desired implication.
Theorem 3. Let $M$ be a right $S$-act. Then the following statements are equivalent:

(a) Each proper $S$-subact of $M$ is semiprime.

(b) Each proper $S$-subact of $M$ is an intersection of prime subacts of $M$.

Proof. (a) $\Rightarrow$ (b). Let $K$ be a proper $S$-subact of $M$. Then by Lemma 3, $K = \cap V_m$, where each $V_m$ is a proper irreducible subact. Also each $V_m$ is semiprime by the hypothesis. Hence by Lemma 4, each $V_m$ is prime.

(b) $\Rightarrow$ (a). Suppose $K = \bigcap_i P_i$ is any intersection of prime subacts $P_i$ of $M$. Consider the inclusion $\alpha xSx \subseteq \bigcap_i P_i$ for some $\alpha \in M$ and $x \in S$. Then $\alpha xSx \subseteq P_i$ for each $i$. Since each $P_i$ is prime, each $P_i$ is semiprime by Lemma 4. Hence $\alpha x \in P_i$ for each $i$, that is, $\alpha x \in \bigcap_i P_i = K$, showing that $K$ is semiprime.

Corollary. Every intersection of semiprime subacts of an $S$-act $M$ is a semiprime subact of $M$.

A semigroup $S$ is semisimple (cf [3, p. 76]) if and only if $I^2 = I$ for every ideal $I$ of $S$. These semigroups admit many interesting characterizations (cf. [3]). As an application of the above theorem, we obtain a characterization of semigroups in which the property $I^2 = I$ holds for every right ideal $I$ of $S$.

Theorem 4. The following conditions for $S$ are equivalent:

(a) $I = I^2$ for each right ideal $I$ of $S$.

(b) Each proper right ideal of $S$ is semiprime.

(c) Each proper right ideal of $S$ is the intersection of prime right ideals.

(d) $S$ is weakly regular (that is, $x \in xSxS$ for any $x \in S$).

(e) $S$ is right regular (that is, for any element $a \in S$, there exists $x$ in $S$ such that $a^2x = a$).

Proof. (a) $\Rightarrow$ (b). Let $I$ be a right ideal of $S$ and let $xSx \subseteq I$ for some $x \in S$. Then $(xSx)S \subseteq I$, so by the hypothesis, $xS \subseteq I$, that is, $x \in I$.

(b) $\Rightarrow$ (c). Follows from Theorem 3.

(c) $\Rightarrow$ (a). Let $I$ be a right ideal of $S$. If $I = S$ then clearly $I = I^2$. If $I \neq S$, then $I^2(\neq S)$ is the intersection of prime right ideals by the hypothesis, and hence $I^2$ is semiprime, which implies that $I^2 = I$.

(a) $\Leftrightarrow$ (d). See [1] for a proof of this equivalence.

(a) $\Leftrightarrow$ (e). See [3, Theorem 4.2, p. 122].

Next, we characterize monoids all of whose ideals are prime.

Theorem 5. The following conditions on $S$ are equivalent:

(a) Each ideal of $S$ is prime.

(b) $S$ is a semisimple semigroup and the set of ideals of $S$ is totally ordered under inclusion.

Proof. (a) $\Rightarrow$ (b). Let $I$ be an ideal of $S$. Then $I^2$, being an ideal of $S$, is prime. Hence $I \subseteq I^2$, that is, $I^2 = I$. Now let $A$ and $B$ be any ideals of $S$. We
have \( AB \subseteq A \cap B \). Since \( A \cap B \) is prime, either \( A \subseteq A \cap B \) or \( B \subseteq A \cap B \). Hence either \( A \subseteq B \) or \( B \subseteq A \).

(b) \( \Rightarrow \) (a). Let \( P \) be any ideal of \( S \) and let \( IJ \subseteq P \) for ideals \( I \) and \( J \) of \( S \). Suppose \( I \subseteq J \). Then \( I = I^2 \subseteq IJ \subseteq P \). Hence \( P \) is a prime ideal.

A right \( S \)-act \( M \) is called right noetherian if every subact of \( M \) is finitely generated. A semigroup \( S \) is called right noetherian if \( S_S \) is noetherian. This holds if and only if \( S \) satisfies the ascending chain condition for right ideals. Noetherian semigroups frequently arise in the homological classification of monoids (cf. [5]). We conclude with the following characterization of noetherian semigroups which may be of independent interest.

**Theorem 6.** Let \( S \) be a right duo monoid (that is, every right ideal of \( S \) is two-sided). Then \( S \) is right noetherian if and only if each prime right ideal of \( S \) is finitely generated.

**Proof.** Suppose that \( S \) is right noetherian. Clearly every prime right ideal of \( S \) is finitely generated. Conversely, assume that every prime right ideal of \( S \) is finitely generated. Suppose that there exists a right ideal \( I \) of \( S \) which is not finitely generated. By Zorn's lemma, we can choose a right ideal \( I_0 \) of \( S \) such that: \( I \subseteq I_0 \); and \( I_0 \) is not finitely generated; and if \( J \) is a right ideal of \( S \) and \( I_0 \subseteq J \); then either \( J = I_0 \) or else \( J \) is finitely generated. We will prove that \( I_0 \) is a prime right ideal and hence, finitely generated by the assumed hypothesis, and thus the supposition that there exists a right ideal \( I \) of \( S \) such that \( I \) is not finitely generated is impossible. Suppose that \( I_0 \) is not a prime right ideal. Then there exist right ideals \( A, B \) of \( S \) such that \( AB \subseteq I_0 \) but \( A \nsubseteq I_0 \) and \( B \nsubseteq I_0 \). Let \( a \in A \) be such that \( a \notin I_0 \). Then \( I_0 \cup aS \) contains \( I_0 \) properly. Hence \( I_0 \cup aS = x_1S \cup x_2S \cup \cdots \cup x_nS \) for some \( x_1, x_2, \ldots, x_n \in S \). Let \( J = \{ x \in S | ax \in I_0 \} \). Then \( I_0 \cup B \subseteq J \). For if \( x \in I_0 \) then \( ax \in I_0 \), since \( S \) is a right duo monoid (that is, every right ideal is two-sided). On the other hand, if \( x \in B \) then \( ax \in AB \subseteq I_0 \). This verifies \( I_0 \cup B \subseteq J \). Now since \( B \nsubseteq I_0 \), \( J \) contains \( I_0 \) properly, and hence \( J = y_1S \cup y_2S \cup \cdots \cup y_mS \) for some \( y_1, y_2, \ldots, y_m \in S \). Without loss of generality we suppose that \( x_i = as_i \) for some \( s_i \in S \) and \( i = 1, \ldots, p \), and \( x_i = b_i \) for some \( b_i \in I_0 \) and \( i = p + 1, p + 2, \ldots, n \). Clearly \( b_{p+1}S \cup \cdots \cup b_nS \cup aJ \subseteq I_0 \), for if \( x = b_is \) with some \( b_i \in I_0 \) and \( s \in S \), then \( x \in I_0 \); on the other hand, if \( x \in aJ \) then \( x = ab \) for some \( b \in J \) and hence \( x = ab \in I_0 \). We now show that \( I_0 \subseteq b_{p+1}S \cup \cdots \cup b_nS \cup aJ \). Let \( y \in I_0 \). Then \( y \in I_0 \cup aS = x_1S \cup x_2S \cup \cdots \cup x_nS \). Hence \( y = x_is \) for some \( s \in S \) and \( i = 1, 2, \ldots, n \). Thus \( y = x_is = as_is \) for some \( i \in \{ 1, 2, \ldots, p \} \) or \( y = b_is \) (\( b_i \in I_0 \)) for some \( i \in \{ p + 1, \ldots, n \} \). If \( y = b_is \) then \( y \in b_{p+1}S \cup \cdots \cup b_nS \); and if \( y = as_is \) then, since \( y \in I_0 \) it follows that \( s_is \in J \). Hence \( y \in aJ \). Thus \( y \in b_{p+1}S \cup \cdots \cup b_nS \cup aJ \), showing that \( I_0 \subseteq b_{p+1}S \cup \cdots \cup b_nS \cup aJ \), which, in turn, implies that \( I_0 = b_{p+1}S \cup \cdots \cup b_nS \cup aJ \). Since \( aJ = ay_1S \cup \cdots \cup ay_mS \), it follows that \( I_0 = b_{p+1}S \cup \cdots \cup b_nS \cup ay_1S \cup \cdots \cup ay_mS \). This implies that \( I_0 \) is finitely generated. But this is impossible. Therefore \( I_0 \) is a prime right ideal and it is finitely generated by the hypothesis. This disproves the assumption that there is a right ideal \( I \) which is not finitely generated.
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