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An Inequality for Twice Differentiable Convex Functions and Applications for the Shannon and Rényi’s Entropies

S. S. DRAGOMIR*

Abstract. A new analytic inequality for twice differentiable convex functions and applications for the Shannon and Rényi’s entropies are given.

1. Introduction.

The following converse of Jensen’s discrete inequality for convex functions of a real variable was proved in 1994 by Dragomir and Ionescu in [13].

Theorem 1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on the interval $I$, $x_i \in I (I$ is the interior of $I$), $p_i \geq 0$ ($i = 1, \ldots, n$) and $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality:

$$0 \leq \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i x_i f'(x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(x_i).$$

If $f$ is strictly convex on $I$ and $p_i > 0$ ($i = 0, \ldots, n$), then the equality holds in either inequality of (1.1) if and only if $x_1 = \ldots = x_n$.

They also pointed out some natural applications of (1.1) in connection to the arithmetic mean - geometric mean inequality, the generalized triangle inequality, etc. For other results on Jensen’s inequality see the book [1], the papers [2]-[14] and the Ph.D Dissertation [17].

A generalization of (1.1) for differentiable convex functions of several variables was obtained in 1996 by S.S. Dragomir and Goh [14]. They also considered the following analytic inequality for the logarithmic function $\log_b (\cdot)$.

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Theorem 2. Let $\xi_i, p_i > 0 \ (i = 1, \ldots, n)$ with $\sum_{i=1}^{n} p_i = 1$ and $b > 1$. Then

\begin{equation}
0 \leq \log_b \left( \sum_{i=1}^{n} p_i \xi_i \right) - \log_b \left( \sum_{i=1}^{n} p_i \log_b \xi_i \right)
\leq \frac{1}{\ln b} \left[ \sum_{i=1}^{n} \frac{p_i}{\xi_i} \sum_{i=1}^{n} p_i \xi_i - 1 \right]
= \frac{1}{\ln b} \sum_{1 \leq i < j \leq n} p_i p_j \frac{(\xi_i - \xi_j)^2}{\xi_i \xi_j}.
\end{equation}

The equality holds in (1.2) iff $\xi_1 = \ldots = \xi_n$.

They applied inequality (1.2) in Information Theory for the entropy function, conditional entropy, mutual information, etc.

An integral version of (1.2) was employed by Dragomir and Goh in [22] to obtain different bounds for the entropy, conditional entropy and mutual information of continuous random variables. In addition, some applications of an integral counterpart of Jensen’s inequality were done by Dragomir and Keady in 1996 [20].

For recent generalizations, for the both discrete and continuous case, as well as extensions for functions defined on normed linear spaces, see M. Matic’s Ph.D. Dissertation [17], where further applications in Information Theory are given.

2. Some Inequalities for Convex Functions.

The following result holds.

Theorem 3. Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on $(a, b)$ and $m \leq f''(x) \leq M$ for all $x \in [a, b]$. If $x_i \in [a, b] \ (i = 1, \ldots, n)$ and $p = (p_i)_{i=1,\ldots,n}$ is a probability distribution, then

\begin{equation}
\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) \left( f'(x_i) - f'(x_j) \right) - \frac{1}{4} M \sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2
\leq \sum_{i=1}^{n} p_i f(x_i) - f \left( \sum_{i=1}^{n} p_i x_i \right)
\leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) \left( f'(x_i) - f'(x_j) \right) - \frac{1}{4} m \sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2.
\end{equation}

Proof. Consider the function $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - \frac{1}{2} m x^2$. Then $g$ is twice differentiable on $(a, b)$ and

\begin{align*}
g'(x) &= f'(x) - mx, \ x \in (a, b); \\
g''(x) &= f''(x) - m, \ x \in (a, b),
\end{align*}

which shows that the function \( g \) is convex on \([a, b]\).

We apply inequality (1.1) for the convex function \( g \), i.e.,

\[
0 \leq \sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i\right) \\
\leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) \left(g'(x_i) - g'(x_j)\right)
\]

to obtain

\[
0 \leq \sum_{i=1}^{n} p_i \left[f(x_i) - \frac{1}{2} m x_i^2\right] - f\left(\sum_{i=1}^{n} p_i x_i\right) + \frac{1}{2} m \left(\sum_{i=1}^{n} p_i x_i\right)^2 \\
\leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) \left[(f'(x_i) - f'(x_j)) - m (x_i - x_j)\right] \\
= \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) \left(f'(x_i) - f'(x_j)\right) - \frac{m}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2,
\]

which is equivalent to

\[
\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\
\leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) \left(f'(x_i) - f'(x_j)\right) + \frac{1}{2} m \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \\
- \frac{1}{2} m \sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2 \\
= \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) \left(f'(x_i) - f'(x_j)\right) - \frac{1}{4} m \sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2,
\]

and the second inequality is proved.

The proof of the first inequality goes likewise for the function \( h : [a, b] \to \mathbb{R}, h(x) = \frac{1}{2} M x^2 - f(x) \) and we omit the details. \( \square \)

The following corollary holds.

**Corollary 1.** Let \( x_i \in [m, M] \subset (0, \infty) \) and \( p_i > 0 \) \((i = 1, \ldots, n)\) with \( \sum_{i=1}^{n} p_i = 1 \). Then we have the inequality

\[
\frac{1}{4m^2} \sum_{i,j=1}^{n} p_i p_j \cdot \frac{(x_i - x_j)^2 (2m^2 - x_i x_j)}{x_i x_j} \\
\leq \ln \left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i \ln x_i \\
\leq \frac{1}{4M^2} \sum_{i,j=1}^{n} p_i p_j \cdot \frac{(x_i - x_j)^2 (2M^2 - x_i x_j)}{x_i x_j}.
\]
The equality holds in either inequality of (2.3) if and only if \( x_1 = \ldots = x_n \).

**Proof.** Consider the function \( f : [m, M] \subseteq (0, \infty) \to \mathbb{R} \) given by \( f(x) = -\ln x \).

Then \( f'(x) = -\frac{1}{x} \), \( f''(x) = \frac{1}{x^2} \) and then

\[
\inf_{x \in (m, M)} f(x) = \frac{1}{M^2}, \quad \sup_{x \in (m, M)} f(x) = \frac{1}{m^2}.
\]

Applying (2.1) for this function we can write

\[
\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} - \frac{1}{4m^2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2
\]

\[
\leq \ln \left( \sum_{i=1}^{n} p_i x_i \right) - \sum_{i=1}^{n} p_i \ln x_i
\]

\[
\leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} - \frac{1}{4M^2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2,
\]

which is equivalent to (2.3).

The case of equality follows by the strict convexity of the functions \( g(x) = -\ln x - \frac{1}{2M} x^2 \), \( h(x) = \ln x + \frac{1}{2m} x^2 \) on the interval \([m, M]\). We shall omit the details. \( \square \)

The following inequality is well-known in the literature as the **arithmetic mean - geometric mean - harmonic mean inequality**:

\[ A_n(p, x) \leq G_n(p, x) \leq H_n(p, x), \]

where

\[
A_n(p, x) := \sum_{i=1}^{n} p_i x_i \quad \text{the arithmetic mean,}
\]

\[
G_n(p, x) := \prod_{i=1}^{n} x_i^{p_i} \quad \text{the geometric mean,}
\]

\[
H_n(p, x) := \frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}} \quad \text{the harmonic mean}
\]

and \( \sum_{i=1}^{n} p_i = 1 \) and \( p_i \geq 0, \ x_i > 0 \ (i = 1, \ldots , n) \).

The following corollaries are then valid.

**Corollary 2.** Let \( x_i \in [m, M] \subseteq (0, \infty) \) and \( p_i > 0 \ (i = 1, \ldots , n) \) with \( \sum_{i=1}^{n} p_i = 1 \). Then we have the inequality

\[
\exp \left[ \frac{1}{4m^2} \sum_{i,j=1}^{n} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} \right] \leq \frac{A_n(p, x)}{G_n(p, x)} \leq \exp \left[ \frac{1}{4M^2} \sum_{i,j=1}^{n} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} \right].
\]
The equality holds in either inequality of (2.5) if and only if $x_1 = \ldots = x_n$.

The proof is obvious by (2.3).

If in (2.5), we put instead of $x$, $\frac{1}{x}$, we obtain

**Corollary 3.** Let $x_i, p_i$ ($i = 1, \ldots, n$) be as in Corollary 2. Then we have the inequality

\[
\exp\left[ \frac{1}{4m^2} \sum_{i,j=1}^{n} p_ip_j \frac{(x_i - x_j)^2 (2m^2 x_ix_j - 1)}{x_i^2 x_j^2} \right] \leq \frac{G_n(p,x)}{H_n(p,x)} \leq \exp\left[ \frac{1}{4M^2} \sum_{i,j=1}^{n} p_ip_j \frac{(x_i - x_j)^2 (2M^2 x_ix_j - 1)}{x_i^2 x_j^2} \right].
\]

The equality holds in either inequality of (2.6) if and only if $x_1 = \ldots = x_n$.

**Remark 1.** If in the inequality (2.5) we choose $x_i$ close enough to $m$, i.e., $x_i \in [m, \sqrt{2m}]$, then $2m^2 \geq x_ix_j$ for all $i, j \in \{1, \ldots, n\}$ and then

\[
\exp\left[ \frac{1}{4m^2} \sum_{i,j=1}^{n} p_ip_j \frac{(x_i - x_j)^2 (2m^2 - x_ix_j)}{x_i x_j} \right] \geq 1,
\]

showing the fact that (2.5) is a refinement of the classical result

\[
\frac{A_n(p,x)}{G_n(p,x)} \geq 1.
\]

Similar comments apply for the inequality (2.6), but we omit the details.

3. **Applications for Shannon's Entropy.**

Let $X$ be a random variable with the range $R = \{x_1, \ldots, x_n\}$ and the probability distribution $p_1, \ldots, p_n$ ($p_i > 0$, $i = 1, \ldots, n$). Define the Shannon entropy as

\[
H(X) := -\sum_{i=1}^{n} p_i \ln p_i.
\]

The following theorem is well known in the literature and concerns the maximum possible value of $H(X)$ in terms of the size of $R$ [15, p. 27]:

**Theorem 4.** Let $X$ be as above. Then

\[
0 \leq H(X) \leq \ln n.
\]

Furthermore, $H(X) = 0$ iff $p_i = 1$ for some $i$ and $H(X) = \ln n$ iff $p_i = \frac{1}{n}$ for all $i \in \{1, \ldots, n\}$.

In the recent paper [14], Dragomir and Goh proved the following counterpart result:
Theorem 5. Let $X$ be defined as above. Then

\begin{align*}
0 \leq \ln n - H(X) &\leq \sum_{1 \leq i < j \leq n} (p_i - p_j)^2 .
\end{align*}

Equality holds in both inequalities iff $p_i = \frac{1}{n}$ for all $i \in \{1, \ldots, n\}$.

The following theorem holds.

Theorem 6. Let $X$ be a random variable with the probability distribution $p_i$ ($i = 1, \ldots, n$). Assume that $p = \min \{p_i | i = 1, \ldots, n\} > 0$ and $P = \max \{p_i | i = 1, \ldots, n\}$. Then we have the inequality:

\begin{align*}
\frac{1}{4} \sum_{i,j=1}^{n} \frac{(p_i - p_j)^2 (2p_i p_j - P^2)}{p_i p_j} &\leq \ln n - H(X) \\
&\leq \frac{1}{4} \sum_{i,j=1}^{n} \frac{(p_i - p_j)^2 (2p_i p_j - p^2)}{p_i p_j}.
\end{align*}

The equality holds in either inequality of (3.3) if and only if $p_i = \frac{1}{n}$, $i = 1, \ldots, n$.

Proof. We know, by Corollary 1, that

\begin{align*}
\frac{1}{4m^2} \sum_{i,j=1}^{n} p_i p_j \cdot \frac{(x_i - x_j)^2 (2m^2 - x_i x_j)}{x_i x_j} &\leq \ln \left( \sum_{i=1}^{n} p_i x_i \right) - \sum_{i=1}^{n} p_i \ln x_i \\
&\leq \frac{1}{4M^2} \sum_{i,j=1}^{n} p_i p_j \cdot \frac{(x_i - x_j)^2 (2M^2 - x_i x_j)}{x_i x_j},
\end{align*}

provided $x_i \in [m, M] \subset (0, \infty)$.

If we choose $x_i = \frac{1}{p_i} \in \left[ \frac{1}{P}, \frac{1}{p} \right]$ in (3.4), we can deduce (with $m = \frac{1}{p}$, $M = \frac{1}{p}$) that

\begin{align*}
\frac{P^2}{4} \sum_{i,j=1}^{n} p_i p_j \cdot \frac{\left( \frac{1}{p_i} - \frac{1}{p_j} \right)^2 \left( 2 \frac{1}{p_j^2} - \frac{1}{p_i} \cdot \frac{1}{p_j} \right)}{\frac{1}{p_i} \cdot \frac{1}{p_j}} &\leq \ln n - H(X) \\
&\leq \frac{P^2}{4} \sum_{i,j=1}^{n} p_i p_j \cdot \frac{\left( \frac{1}{p_i} - \frac{1}{p_j} \right)^2 \left( 2 \frac{1}{p_i^2} - \frac{1}{p_i p_j} \right)}{\frac{1}{p_i p_j}},
\end{align*}

which, with some elementary calculation, is equivalent to the desired inequality. □
**Remark 2.** If we assume that the probability distribution is taken so that \( \sqrt{2p} \geq P (\geq p) \), then \( 2p_i p_j - P^2 \geq 2p^2 - P^2 \geq 0 \) and then

\[
\frac{1}{4} \sum_{i,j=1}^{n} \frac{(p_i - p_j)^2 (2p_i p_j - P^2)}{p_i p_j} \geq 0
\]

showing that, with the above assumption on \( p \) and \( P \), the inequality (3.3) improves the first inequality in (3.2). That is, \( \ln n - H (X) \geq 0 \).

**Remark 3.** It is important to remark that as \( p_i p_j \geq 1 \), then \( \frac{1}{2} (2p_i p_j - p^2) < 1 \) and thus

\[
\frac{1}{4} \frac{(2p_i p_j - p^2) (p_i - p_j)^2}{p_i p_j} \leq \frac{1}{2} \frac{(p_i - p_j)^2}{p_i p_j}
\]

for all \( i, j \in \{1, \ldots, n\} \), which implies

\[
\frac{1}{4} \sum_{i,j=1}^{n} \frac{(2p_i p_j - p^2) (p_i - p_j)^2}{p_i p_j} \leq \frac{1}{2} \sum_{i,j=1}^{n} \frac{(p_i - p_j)^2}{p_i p_j}.
\]

This shows that the upper bound provided by Theorem 6 for \( \ln n - H (X) \) is better than the bound in Theorem 5.

### 4. Applications for Rényi’s Entropy

Let \( X \) be a random variable taking values in \( R = \{x_1, x_2, \ldots, x_n\} \) and having the probability distribution \( p_1, \ldots, p_n \). Consider the Rényi’s entropy of order \( \alpha \) \((\alpha \in (0,1) \cup (1,\infty))\) given by [19]

\[
H_\alpha (X) := \frac{1}{1 - \alpha} \ln \left( \sum_{i=1}^{n} p_i^\alpha \right).
\]

Using the classical Jensen's discrete inequality for convex functions, that is,

\[
f \left( \sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i f (x_i),
\]

where \( f : I \subseteq R \rightarrow R \) is a convex function on \( I \), \( x_i \in I \ (i = 1, \ldots, n) \) and \((p_i)_{i=1,\ldots,n}\) is a probability distribution, for the convex function \( f (x) = -\ln x \), we have

\[
\ln \left( \sum_{i=1}^{n} p_i x_i \right) \geq \sum_{i=1}^{n} p_i \ln x_i.
\]

Choose \( x_i := p_i^{\alpha-1} \ (i = 1, \ldots, n) \) in (4.3) to obtain

\[
\ln \left( \sum_{i=1}^{n} p_i^\alpha \right) \geq (\alpha - 1) \sum_{i=1}^{n} p_i \ln p_i,
\]

which is equivalent to

\[
(1 - \alpha) [H_\alpha (X) - H (X)] \geq 0.
\]

Now, if \( \alpha \in (0,1) \), then \( H_\alpha (X) \geq H (X) \), and if \( \alpha \in (1,\infty) \), then \( H_\alpha (X) \leq H (X) \). Equality holds iff \((p_i)_{i=1,\ldots,n}\) is a uniform distribution and this fact follows by the strict convexity of \(-\ln (\cdot)\).

The following result, which is related to (4.5) holds.
**Theorem 7.** Let $X$ be a random variable with the probability distribution $p_i$ $(i = 1, \ldots, n)$. Assume that $p = \min \{p_i | i = 1, \ldots, n\} > 0$ and $P = \max \{p_i | i = 1, \ldots, n\}$.

(i) If $\alpha \in (0, 1)$, then we have the inequality

\[
\frac{1}{4} \sum_{i,j=1}^{n} p_i^{2\alpha-1} p_j^{2\alpha-1} (p_i^{1-\alpha} - p_j^{1-\alpha})^2 \left(2p_i^{1-\alpha} p_j^{1-\alpha} - P^{2(1-\alpha)}\right)
\leq (1 - \alpha) [H_\alpha (X) - H (X)]
\leq \frac{1}{4} \sum_{i,j=1}^{n} p_i^{2\alpha-1} p_j^{2\alpha-1} (p_i^{1-\alpha} - p_j^{1-\alpha})^2 \left(2p_i^{1-\alpha} p_j^{1-\alpha} - p^{2(1-\alpha)}\right).
\]

(ii) If $\alpha \in (1, \infty)$, then we have the inequality:

\[
\frac{1}{4p^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i^{2-\alpha} p_j^{2-\alpha} (p_i^{\alpha-1} - p_j^{\alpha-1})^2 \left(2p^{2(\alpha-1)} - p_i^{\alpha-1} p_j^{\alpha-1}\right)
\leq (\alpha - 1) [H (X) - H_\alpha (X)]
\leq \frac{1}{4P^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i^{2-\alpha} p_j^{2-\alpha} (p_i^{\alpha-1} - p_j^{\alpha-1})^2 \left(2P^{2(\alpha-1)} - p_i^{\alpha-1} p_j^{\alpha-1}\right).
\]

**Proof.**

(i) If $x_i := p_i^{\alpha-1}$ $(i = 1, \ldots, n)$, $\alpha \in (0, 1)$, then $P^{\alpha-1} \leq x_i \leq p^{\alpha-1}$ and then, by (2.3), we deduce that

\[
\frac{1}{4p^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i p_j \frac{(p_i^{\alpha-1} - p_j^{\alpha-1})^2 \left(2P^{2(\alpha-1)} - p_i^{\alpha-1} p_j^{\alpha-1}\right)}{p_i^{\alpha-1} p_j^{\alpha-1}}
\leq \ln \left(\sum_{i=1}^{n} p_i^{\alpha}\right) - (\alpha - 1) \sum_{i=1}^{n} p_i \ln p_i
\leq \frac{1}{4p^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i p_j \frac{(p_i^{\alpha-1} - p_j^{\alpha-1})^2 \left(2P^{2(\alpha-1)} - p_i^{\alpha-1} p_j^{\alpha-1}\right)}{p_i^{\alpha-1} p_j^{\alpha-1}}.
\]

However, a simple calculation shows that

\[
\frac{1}{4P^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i p_j \frac{(p_i^{\alpha-1} - p_j^{\alpha-1})^2 \left(2P^{2(\alpha-1)} - p_i^{\alpha-1} p_j^{\alpha-1}\right)}{p_i^{\alpha-1} p_j^{\alpha-1}}
= \frac{1}{4} \sum_{i,j=1}^{n} p_i^{2\alpha-1} p_j^{2\alpha-1} (p_j^{1-\alpha} - p_i^{1-\alpha})^2 \left(2p_i^{1-\alpha} p_j^{1-\alpha} - p^{2(1-\alpha)}\right),
\]
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\[
\ln \left( \sum_{i=1}^{n} p_i^\alpha \right) - (\alpha - 1) \sum_{i=1}^{n} p_i \ln p_i = (1 - \alpha) [H_\alpha (X) - H (X)],
\]

and

\[
\frac{1}{4p^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i p_j \left( p_i^\alpha - p_j^\alpha \right)^2 \frac{2p^{2(\alpha-1)} - p_i^\alpha p_j^\alpha}{p_i^\alpha p_j^\alpha - p_i^\alpha - p_j^\alpha}
\]

\[
= \frac{1}{4} \sum_{i,j=1}^{n} p_i^{2\alpha-1} p_j^{2\alpha-1} \left( p_j^{1-\alpha} - p_i^{1-\alpha} \right)^2 \left( 2p_i^{1-\alpha} p_j^{1-\alpha} - p^{2(1-\alpha)} \right)
\]

and the inequality (4.6) is proved.

(ii) If \( x_i = p_i^{\alpha-1} \) (\( i = 1, \ldots, n \)), \( \alpha \in (1, \infty) \), then \( p^{\alpha-1} \leq x_i \leq P^{\alpha-1} \) and then, by (2.3), we deduce that

\[
\frac{1}{4p^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i p_j \left( p_i^\alpha - p_j^\alpha \right)^2 \frac{2p^{2(\alpha-1)} - p_i^\alpha p_j^\alpha}{p_i^\alpha p_j^\alpha - p_i^\alpha - p_j^\alpha}
\]

\[
\leq (\alpha - 1) [H (X) - H_\alpha (X)]
\]

\[
\leq \frac{1}{4p^{2(\alpha-1)}} \sum_{i,j=1}^{n} p_i p_j \left( p_i^\alpha - p_j^\alpha \right)^2 \frac{2p^{2(\alpha-1)} - p_i^\alpha p_j^\alpha}{p_i^\alpha p_j^\alpha - p_i^\alpha - p_j^\alpha}
\]

and the inequality (4.7) is proved. \( \Box \)

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References


