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An example of a positive definite function which is not of positive type on $\mathbb{Z}^2$

KOJI FURUTA* and NOBUHISA SAKAKIBARA**

Let $S = (S, +, \ast)$ be an abelian $\ast$-semigroup with the identity 0, $\mathcal{H}$ a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. A function $\varphi : S \to B(\mathcal{H})$ is called of positive type if

$$\sum_{i,j=1}^{n} \langle \varphi(s_i + s_j^*) \xi_i, \xi_j \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \ldots, s_n \in S$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$. Moreover, $\varphi$ is called positive definite if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \langle \varphi(s_i + s_j^*) \xi, \xi \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \ldots, s_n \in S$, $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and $\xi \in \mathcal{H}$. Every function of positive type is positive definite, and every scalar-valued, positive definite function is of positive type. But a positive definite function is not necessarily of positive type. In fact, T. M. Bisgaard demonstrated that there exists an explicit example of a positive definite function which is not of positive type on $(\mathbb{N}_0, +, x^* = x)$ where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ (see [1, Theorem 1]), and we did on $(\mathbb{Z}, +, x^* = x)$ (see [3, Theorem 3.7]). For abelian $\ast$-semigroups $(\mathbb{N}_0^2, +, x^* = x)$ and $(\mathbb{Z}^2, +, x^* = x)$, is there such an example? When $(\mathbb{N}_0^2, +, x^* = x)$, the answer is clear because we have the zero extension of Bisgaard's example (i.e. $\varphi(n, 0)$ is Bisgaard's and $\varphi(n, m) = 0$ for $m > 0$). In this paper, we shall show such an explicit example on $(\mathbb{Z}^2, +, x^* = x)$.

REMARK. A function $\varphi : S \to B(\mathcal{H})$ is called a operator moment function if there exists a $B(\mathcal{H})^*$-valued measure $F$ on $S^*$ such that

$$\langle \varphi(s) \xi, \eta \rangle = \int_{S^*} \rho(s)d(F(\rho)\xi, \eta)$$

for $s \in S$, $\xi, \eta \in \mathcal{H}$.

Every operator moment function is of positive type. But a function of positive type is not necessarily an operator moment function. In fact, even a scalar-valued, positive definite function is not necessarily a moment function. So an
Example of a positive definite function which is not of positive type on \( \mathbb{Z}^2 \)

Abelian \(*\)-semigroup \( S \) is called \textit{operator semiperfect} (resp. \textit{semiperfect}) if every function of positive type (resp. scalar-valued, positive definite function) on \( S \) is a operator moment function (resp. moment function). Operator semiperfect \(*\)-semigroups have been analyzed by Bisgaard ([2]), Stochel and Szafraniec ([4]), and the authors ([3]). Semiperfect \(*\)-semigroups have been more analyzed by Bisgaard and the authors.

Let us first define a linear order \( \ll \) on \( \mathbb{Z}^2 \) as follows:

\[
(n_1, m_1) \ll (n_2, m_2) \iff |n_1| + |m_1| < |n_2| + |m_2| \quad \text{or}
\]
\[
|n_1| + |m_1| = |n_2| + |m_2|, \quad m_1 > 0, m_2 > 0, n_1 < n_2 \quad \text{or}
\]
\[
|n_1| + |m_1| = |n_2| + |m_2|, m_1 > 0, m_2 \leq 0 \quad \text{or}
\]
\[
|n_1| + |m_1| = |n_2| + |m_2|, m_1 \leq 0, m_2 \leq 0, n_2 < n_1.
\]

Arrange points of \( \mathbb{Z}^2 \) by this order, i.e.

\[
\begin{align*}
x_0 &:= (0,0), & x_1 &:= (0,1), & x_2 &:= (1,0), & x_3 &:= (0,-1), \\
x_4 &:= (-1,0), & x_5 &:= (-1,1), & x_6 &:= (0,2), & \ldots
\end{align*}
\]

and let \( a_n := 2^{(n+2)}n, n \geq 1 \). The following is our theorem, in which the choices of \( 2 \times 2 \) matrices is similar to those in Bisgaard's example.

**Theorem.** Let \( \varphi : \mathbb{Z}^2 \to M_2(\mathbb{C}) \) be a function defined by

\[
\varphi(x_0) := \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(x_i) := \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad 1 \leq i \leq 5,
\]
\[
\varphi(x_6) := \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad \varphi(x_n) := \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix}, \quad n \geq 7.
\]

Then \( \varphi \) is positive definite and not of positive type on \( (\mathbb{Z}^2, +, x^* = x) \).

**Proof.** To see that \( \varphi \) is positive definite on \( (\mathbb{Z}^2, +, x^* = x) \), we shall prove

\[
D_n(\xi) := \begin{vmatrix}
\langle \varphi(x_0 + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_0 + x_n)\xi, \xi \rangle \\
\vdots & \ddots & \vdots \\
\langle \varphi(x_n + x_0)\xi, \xi \rangle & \cdots & \langle \varphi(x_n + x_n)\xi, \xi \rangle
\end{vmatrix} > 0
\]

for \( n \geq 0 \) and \( \xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2 \) with \( |\alpha|^2 + |\beta|^2 = 1 \). We have

\[
D_0(\xi) = 4|\alpha|^2 + |\beta|^2 \geq 1,
\]
\[
D_1(\xi) = (4|\alpha|^2 + |\beta|^2)(|\alpha|^2 + 4|\beta|^2) - 16(\text{Re}(\alpha\overline{\beta}))^2
\]
\[
= 4|\alpha|^4 + 4|\beta|^4 + 17|\alpha\beta|^2 - 16(\text{Re}(\alpha\overline{\beta}))^2
\]
\[
\geq 4|\alpha|^4 + 4|\beta|^4 + |\alpha|^2|\beta|^2
\]
\[
= 4 - 7|\alpha|^2|\beta|^2 \geq 4 - \frac{7}{4} > 1.
\]
Let $n \geq 2$ and suppose that $D_{n-1}(\xi) \geq 1$. We show that $D_n(\xi) \geq 1$. Suppose the following:

(*) For every $n \geq 2$, there exists a natural number $m \geq n, m \geq 8$ such that

$$\begin{align*}
&||\varphi(x_n + x_n)|| = a_m, \\
&||\varphi(x_j + x_k)|| \leq a_{m-1}, \quad 0 \leq j \leq n, \quad 0 \leq k \leq n - 1.
\end{align*}$$

Then

$$D_n(\xi) = \langle \varphi(x_n + x_n) \xi, \xi \rangle D_{n-1}(\xi)$$

$$\geq m - n a_{m-1} n!$$

$$\geq 2^{(m+2)!} - (n+1)2^{(m+1)!}(n+1)!$$

$$\geq 2^{(m+1)!}(m+1)(2^{(m+1)!} - (n+1)!$$

$$\geq 2^{(m+1)!} - (m+1)!$$

$$\geq 1,$$

where

$$D_k(\xi) := \begin{vmatrix}
\langle \varphi(x_0 + x_0) \xi, \xi \rangle & \cdots & \langle \varphi(x_0 + x_{n-1}) \xi, \xi \rangle \\
\vdots & \ddots & \vdots \\
\langle \varphi(x_{k-1} + x_0) \xi, \xi \rangle & \cdots & \langle \varphi(x_{k-1} + x_{n-1}) \xi, \xi \rangle \\
\langle \varphi(x_{k+1} + x_0) \xi, \xi \rangle & \cdots & \langle \varphi(x_{k+1} + x_{n-1}) \xi, \xi \rangle \\
\vdots & \ddots & \vdots \\
\langle \varphi(x_n + x_0) \xi, \xi \rangle & \cdots & \langle \varphi(x_n + x_{n-1}) \xi, \xi \rangle
\end{vmatrix},$$

$k = 0, 1, 2, \ldots, n - 1$. Thus, by induction, we get $D_n(\xi) \geq 1$ for $n \geq 0$. Suppose that

(**) for every $n \geq 2$, $x_n + x_k \ll x_n + x_n$, $k = 0, 1, 2, \ldots, n - 1$.

Then we can easily prove (*). Therefore let us prove (**).

Put $x_n : = (p, q)$ and $x_k : = (r, s)$. Then $x_n + x_n = (2p, 2q)$ and $x_n + x_k = (p + r, q + s)$. In case that $|r| + |s| < |p| + |q|$ or $pr < 0$ or $qs < 0$, it is easily seen that $|p + r| + |q + s| < |2p| + |2q|$. Hence $x_n + x_k \ll x_n + x_n$. Suppose that $|r| + |s| = |p| + |q|, pr \geq 0$ and $qs \geq 0$. When $p > 0$ and $q \geq 0$, the condition $x_k \ll x_n$ implies $p > r \geq 0$ and $s \geq 0$. Hence $(p + r) + (q + s) = 2p + 2q$, $0 \leq 2q < q + s$ and $p + r < 2p$. Therefore $x_n + x_k \ll x_n + x_n$. When $p \geq 0$ and $q < 0$, we have $r > p \geq 0$ and $0 \geq s > q$. Hence $(p + r) + (q + s) = 2p + |2q|$, $2q < q + s < 0$ and $p + r > 2p$. Therefore $x_n + x_k \ll x_n + x_n$. When $p \leq 0$
and $q > 0$, since $0 \leq p > r$ and $q > s > 0$, we have $|p + r| + (q + s) = |2p| + 2q$, $0 < q + s < 2q$ and $p + r < 2p$. Therefore $x_n + x_k \ll x_n + x_n$. When $p < 0$ and $q \leq 0$, since $0 \geq r > p$ and $s < q \leq 0$, we have $|p + r| + |q + s| = |2p| + |2q|$, $q + s < 2q \leq 0$ and $2p < p + r$. Therefore $x_n + x_k \ll x_n + x_n$. We finished the proof of (**).

Put $\xi_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\xi_1 := \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Then

$$\sum_{i,j=0}^1 \langle \varphi(x_i + x_j)\xi_i, \xi_j \rangle = -2 < 0,$$

which implies that $\varphi$ is not of positive type. This completes the proof. $\blacksquare$

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