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On Conditions for Denominator Ideals to Diffuse and Conditions for Elements to Be Exclusive in Anti-Integral Extensions

SUSUMU ODA* AND KEN-ICHI YOSHIDA**

Let $R$ be a Noetherian integral domain and let $\alpha$ be an element in an algebraic extension field of the quotient field of $R$. Our objective is to show that a generalized denominator ideal $I_{[\alpha]}$ satisfies

$$I_{[\alpha]}R[\alpha] = R[\alpha]$$

if and only if $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$ if and only if either $\alpha$ or $\alpha^{-1}$ is integral over $R_p$ for each $p \in \text{Spec}(R)$, provided that $\alpha$ is an anti-integral element over $R$. The other is to show that for the canonical map $\phi : \text{Spec}(R[\alpha]) \to \text{Spec}(R)$, $D_p(R)$ is contained in the image of $\phi$ if and only if $\alpha$ is exclusive. The former result is closely concerned to [KY1] and [OY3], and the latter result is related to [OY2].

Notation and Conventions

Throughout this paper, we use the following notation unless otherwise specified:

Let $R$ be a Noetherian domain (which is commutative and has a unit), let $R[X]$ be a polynomial ring, let $\alpha$ be an element of an algebraic extension field of the quotient field $K$ of $R$ and let $\varphi : R[X] \to R[\alpha]$ be the $R$-algebra homomorphism sending $X$ to $\alpha$. Let $\varphi_{\alpha}(X)$ be the monic minimal polynomial of $\alpha$ over $K$ with $\deg \varphi_{\alpha}(X) = d$ and write $\varphi_{\alpha}(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Then $\eta_i \in K$ $(1 \leq i \leq d)$ are uniquely determined by $\alpha$. Put $d = [K(\alpha) : K]$, $I_{\eta_i} := R : \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^{d} I_{\eta_i}$. If $\text{Ker}(\pi) = I_{[\alpha]}\varphi_{\alpha}(X)R[X]$, we say that $\alpha$ is anti-integral over $R$ (cf. [OSY]). Put $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. For $\beta \in K$, we put $J_{\beta} := I_{\beta}(1, \beta)$. Then $J_{[\alpha]} = c(I_{[\alpha]}\varphi_{\alpha}(X))$, where $c(\ )$ denotes the ideal generated by the coefficients of the polynomials in ( ), that is, the content ideal of ( ). If $J_{[\alpha]} \not\subseteq p$ for all $p \in D_p(R) := \{p \in \text{Spec}(R) \mid \text{depth}R_p = 1\}$, the element $\alpha$ is called a super-primitive element over $R$. A super-primitive element over $R$ is anti-integral over $R$ (cf. [OSY, Theorem 1.12]). It is also known that any algebraic element over a Krull domain $R$ is super-primitive over $R$ (cf. [OSY, Theorem 1.13]), and hence $\alpha$ is anti-integral over $R$. We also note here that $I_{[\alpha]} = R \iff R[\alpha]$ is integral over $R$ and that $J_{[\alpha]} = R \iff R[\alpha]$ is flat over $R$, provided that $\alpha$ is anti-integral over $R$.

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Conditions for generalized denominator ideals to diffuse in anti-integral extensions

We start with the following proposition. The implication (i) \(\Rightarrow\) (ii) is seen in [OY3, Proposition 7], but we give a proof for convenience.

**Proposition 1.1.** Assume that \(\alpha\) is an anti-integral element over \(R\). Then the following statements are equivalent:

(i) \(I_\alpha R[\alpha] = R[\alpha]\);
(ii) \(\eta_d \in R[\alpha], I_\alpha = I_{\eta_d}\) and \(J_{\eta_d} = R\).

**Proof.** If \(I_\alpha = R\), then the implications (i) \(\Leftrightarrow\) (ii) are trivially valid because \(\eta_d \in R\) by the definition of \(I_\alpha\). So we may assume that \(I_\alpha \neq R\).

(i) \(\Leftrightarrow\) (ii): Since \(\eta_1, \ldots, \eta_d \in I_\alpha^{-1} R[\alpha] = I^{-1}_\alpha I_\alpha R[\alpha] \subseteq R[\alpha]\), we have \(\eta_d \in R[\alpha]\). Put \(C := R[\eta_1, \ldots, \eta_d]\). Then \(R[\alpha]\) is a free \(C\)-module and \(R[\alpha]\) is integral over \(C\). So \(I_\alpha^{-1} R[\alpha] = R[\alpha]\) yields that \(I_\alpha C = C\). Thus \(C\) is flat over \(R\). Since \(R[\alpha]\) is flat over \(C\), \(R[\alpha]\) is flat over \(R\), and hence \(J_\alpha = R\).

Now we shall show that \(I_\alpha = I_{\eta_d}\). It is easy to see that \(I_\alpha \subseteq I_{\eta_d}\). Take \(p \in \text{Spec}(R)\) with \(p \supseteq I_\alpha\). Since \(I_\alpha R[\alpha] = R[\alpha]\), we have \(1 = a_0 + a_1 \alpha + \cdots + a_n \alpha^n\), where \(a_i \in I_\alpha \subseteq p\). Since \(1 - a_0\) is a unit in \(R_p\), \(\alpha^{-1}\) is integral over \(R\). Since \(\alpha\) is anti-integral over \(R\), \(\alpha^{-1}\) is also anti-integral over \(R\). Thus \(\varphi_{\alpha^{-1}}(X) = X^d + (\eta_d^{-1} \eta_{d-1}) X^{d-1} + \cdots + (\eta_1^{-1} \eta_1) X + \eta_1^{-1} \in R_p[X]\). So there exists \(x \in R_p\) such that \(\eta_d = 1/x\), \(\eta_i \in y_i/x\) with some \(y_i \in R_p\). Therefore we have \((I_\alpha)_p = x R_p = (I_{\eta_d})_p\). Since \(p\) is arbitrary, we conclude that \(I_\alpha = I_{\eta_d}\).

Next we shall show that \(J_{\eta_d} = I_{\eta_d}\). Suppose that \(I_{\eta_d}(1, \eta_d) \subseteq p\) for some \(p \in \text{Spec}(R)\). Then \(I_\alpha \subseteq p\). Thus by the above notation, we have \((I_{\eta_d}(1, \eta_d))_p = x (1, 1/x) R_p = R_p\), which contradicts the assumption \(I_{\eta_d}(1, \eta_d) \subseteq p\). Therefore we conclude that \(J_{\eta_d} = R\).

(ii) \(\Rightarrow\) (i) follows the implications: \(R[\alpha] \supseteq I_\alpha R[\alpha] \supseteq I_\alpha(1, \eta_d) = I_{\eta_d}(1, \eta_d) = R \supseteq 1\), and hence \(R[\alpha] \supseteq I_\alpha R[\alpha] \supseteq R[\alpha]\). \(\square\)

**Corollary 1.2.** Assume that \(\alpha\) is an anti-integral element over \(R\). If \(I_\alpha R[\alpha] = R[\alpha]\), then \(R[\alpha] \cap K = R[\eta_d]\).

**Proof.** Put \(C := R[\eta_1, \ldots, \eta_d]\). Then by the proof of Proposition 1.1, we have \(C \subseteq R[\alpha]\). Since \(R[\alpha]\) is a free \(C\)-module \(C + C \alpha + \cdots + C \alpha^{d-1}\), we conclude that \(R[\alpha] \cap K = C\). Besides, as in the proof of Proposition 1, if \(I_\alpha \subseteq p\), then there exists \(x \in R_p\) such that \(\eta_d = 1/x\) and \(y_i \in y_i/x \in R_p[\eta_d]\) for some \(y_i \in R_p\). If \(I_\alpha \not\subseteq p\), then \(C_p = R_p = R_p[\eta_d]\). Therefore we have \(C = R[\eta_d]\). \(\square\)

**Lemma 1.3.** \(I_{\alpha^{-1}} = \eta_d I_\alpha\).

**Proof.** Since \(\varphi_{\alpha^{-1}}(X) = X^d + (\eta_d^{-1} \eta_{d-1}) X^{d-1} + \cdots + (\eta_1^{-1} \eta_1) X + \eta_1^{-1}\), we have \(I_{\alpha^{-1}} = \bigcap_{i=1}^{d-1} I_{\eta_d^{-1}, \eta_i} \cap I_{\eta_d^{-1}}\). Take \(x = \eta_d y \in \eta_d I_\alpha\) with \(y \in I_\alpha\). Then \(x \eta_d^{-1} \eta_i = y \eta_i \in R\) and hence \(x \eta_d^{-1} = y \in R\). Thus \(I_{\alpha^{-1}} \supseteq \eta_d I_\alpha\). Conversely,
take $x \in I_{[\alpha^{-1}]}$. Then $x\eta_{d}^{-1} = y \in R$, $y\eta_{d} = x \in R$ and $x\eta_{d}^{-1}\eta_{i} = y\eta_{i} \in R$. Hence $y \in I_{[\alpha]}$. Therefore we have $I_{[\alpha^{-1}]} = \eta_{d} I_{[\alpha]}$.

Remark 1.4. (i) An element $\alpha$ is anti-integral over $R$ if and only if so is $\alpha^{-1}$ (cf. [KY]).

(ii) An element $\alpha$ is super-primitive over $R$ if and only if so is $\alpha^{-1}$. Indeed, the minimal monic polynomial $\varphi_{\alpha^{-1}}$ of $\alpha^{-1}$ is $X^{d} + (\eta_{d-1}/\eta_{d})X^{d-1} + \cdots + (\eta_{1}/\eta_{d})X + (1/\eta_{d})$. Hence we have

$$J_{[\alpha^{-1}]} = I_{[\alpha^{-1}]}(1, \eta_{d-1}/\eta_{d}, \ldots, \eta_{1}/\eta_{d}) = \eta_{d} I_{[\alpha]}(1, \eta_{d-1}/\eta_{d}, \ldots, \eta_{1}/\eta_{d}) = I_{[\alpha]}(1, \eta_{1}, \ldots, \eta_{d}) = J_{[\alpha]}$$

here we use Lemma 1.3. So our assertion follows from the definition of super-primitiveness.

Theorem 1.5. Assume that $\alpha$ is an anti-integral element over $R$ and that $\eta_{d} \in R[\alpha]$. The following statements are equivalent:

(i) $I_{[\alpha]} R[\alpha] = R[\alpha]$;

(ii) $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$;

(iii) either $\alpha$ or $\alpha^{-1}$ is integral over $R_{p}$ for each $p \in \text{Spec}(R)$.

Proof. (i) $\Rightarrow$ (ii): We have $\eta_{d} \in R[\alpha], I_{[\alpha]} = I_{\eta_{d}}$ and $J_{\eta_{d}} = R$ by Proposition 1.1. So we obtain $R = J_{\eta_{d}} = I_{\eta_{d}}(1, \eta_{d}) = I_{[\alpha]}(1, \eta_{d}) = I_{[\alpha]} + \eta_{d} I_{[\alpha]} = I_{[\alpha]} + I_{[\alpha^{-1}]}$ (cf. Lemma 1.3).

(II) $\Rightarrow$ (i): We have $R = I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]} + \eta_{d} I_{[\alpha]} = I_{[\alpha]}(1, \eta_{d})$. Thus $R[\alpha] = I_{[\alpha]}(1, \eta_{d}) R[\alpha] = I_{[\alpha]} R[\alpha] + I_{[\alpha]} \eta_{d} R[\alpha] \subseteq I_{[\alpha]} R[\alpha]$ because $\eta_{d} \in R[\alpha]$. The converse inclusion is obvious. Thus $I_{[\alpha]} R[\alpha] = R[\alpha]$.

(III) $\Rightarrow$ (iii): Take $p \in \text{Spec}(R)$. Then $I_{[\alpha]} \not\subseteq p$ or $I_{[\alpha^{-1}]} \not\subseteq p$ because $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$. If $I_{[\alpha]} \not\subseteq p$, we have $\eta_{1}, \ldots, \eta_{d} \in R_{p}$. Hence $\alpha^{d} + \eta_{1} \alpha^{d-1} + \cdots + \eta_{d} = 0$, which means that $\alpha$ is integral over $R_{p}$. Next, we assume that $I_{[\alpha^{-1}]} \not\subseteq p$. The $\eta_{d}^{-1}\eta_{1}, \ldots, \eta_{d}^{-1}\eta_{d-1}, \eta_{d}^{-1} \in R_{p}$. Thus $(\alpha^{-1})^{d} + (\eta_{d}^{-1})\alpha^{d-1} + \cdots + \eta_{d}^{-1}\eta_{1}\alpha + \eta_{d}^{-1} = 0$, which means that $\alpha^{-1}$ is integral over $R_{p}$.

(iii) $\Rightarrow$ (ii): Suppose that there exists a prime ideal $p$ of $R$ such that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$. Note that $\alpha$ or $\alpha^{-1}$ is integral over $R_{p}$. Assume that $\alpha$ is integral over $R_{p}$. Then $\alpha$ satisfies a monic relation of degree $d$ over $R_{p}$ because $\alpha$ is anti-integral over $R_{p}$. But since $\varphi_{\alpha}(X) = X^{d} + \eta_{1} X^{d-1} + \cdots + \eta_{d} \in R_{p}[X]$, we have $I_{[\alpha]} \not\subseteq p$, a contradiction. Similarly we come to a contradiction when we assume $\alpha^{-1}$ is integral over $R_{p}$.

Proposition 1.6. Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. If $\eta_{d} \in R[\alpha]$, then $\sqrt{I_{[\alpha]} R[\alpha]} \cap R = \sqrt{I_{[\alpha]} + I_{[\alpha^{-1}]}}$.

Proof. Take $p \in \text{Spec}(R)$. Then $I_{[\alpha]} R[\alpha] \cap R \not\subseteq p \Leftrightarrow I_{[\alpha]} R_{p}[\alpha] = R_{p}[\alpha] \Leftrightarrow (I_{[\alpha]} + I_{[\alpha^{-1}]})_{p} = R_{p} \Leftrightarrow I_{[\alpha]} + I_{[\alpha^{-1}]} \not\subseteq p$. Thus we come to our conclusion by Theorem 1.5. 

§2. Conditions for anti-integral elements to be exclusive.

Recall first that an algebraic element $\alpha$ over $R$ is called to be exclusive if $R[\alpha] \cap K = R$. Put $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$, an ideal of $R$.

**Remark 2.1.** Assume that $R$ contains a field $k$. Let $x$ be an indeterminate and put $S := R \otimes_k k(x)$. Then $S$ contains the infinite field $k(x)$. Put $I_{[\alpha]}^S := \bigcap_{i=1}^d (S : s \eta_i)$, $J_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \ldots, \eta_d)S$ and $\tilde{J}_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \ldots, \eta_{d-1})S$.

Since $S$ is faithfully flat over $R$, $I_{[\alpha]}^S = I_{[\alpha]}^S$, $J_{[\alpha]}^S = J_{[\alpha]}^S$ and $\tilde{J}_{[\alpha]}^S = \tilde{J}_{[\alpha]}^S$. So if $\alpha$ is super-primitive (resp. anti-integral) over $R$, then so is $\alpha$ over $S$. Moreover $\alpha$ is exclusive over $S$ if and only if $\alpha$ is exclusive over $R$; that is, $\alpha$ is exclusive over $S$, i.e., $S[\alpha] \cap K(x) = S[\alpha]$ if and only if $\alpha$ is exclusive over $R$, i.e., $R[\alpha] \cap K = R$.

**Lemma 2.2.** Assume that $\alpha$ is super-primitive over $R$. Then the following statements (i) and (ii) are equivalent:

(i) $\bigcap_{i=1}^{d-1} I_{\eta_i} \subset I_{\eta_d}$;
(ii) grade($\tilde{J}_{[\alpha]}$) $> 1$ or $\tilde{J}_{[\alpha]} = R$.

Furthermore if $R$ contains a field, then the following (iii) is equivalent to (i):

(iii) $\alpha$ is exclusive over $R$.

**Proof.** (i) $\Leftrightarrow$ (ii) follows from [OY2, Lemma 3].

Next assume that $R$ contains a field. We may assume that $R$ contains an infinite field by Remark 2.1. Hence our conclusion (ii) $\Leftrightarrow$ (iii) follows from [OY2, Theorem 5].

**Remark 2.3.** Assume that $R$ contains a field.

(i) When $\alpha$ is a super-primitive element over $R$, $\alpha$ is exclusive over $R$ if and only if grade($\tilde{J}_{[\alpha]}$) $> 1$ by Lemma 2.2.

(ii) $\tilde{J}_{[\alpha]} = I_{[\alpha]}(\eta_1, \ldots, \eta_d)$. This follows from the similar argument of Remark 1.4(ii).

(iii) By Remark 1.4, $\alpha$ is super-primitive over $R$ if and only if $\alpha$ is exclusive over $R$. Hence our conclusion (i) $\Leftrightarrow$ (ii) above.

(iii) If grade($I_{[\alpha]}(\eta_1, \ldots, \eta_{d-1})$) $> 1$, then both $\alpha$ and $\alpha^{-1}$ are exclusive over $R$ by (i), (ii) and (iii).

**Proposition 2.4.** Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. If $I_{[\alpha]} R[\alpha] P = R[\alpha] P$ for every $P \in \text{DP}_1(R[\alpha])$, then $R[\alpha] \cap K = R[\eta_1, \ldots, \eta_d]$.

**Proof.** It follows that $R[\alpha] P \supset I_{[\alpha]}^{-1} I_{[\alpha]} R[\alpha] P = I_{[\alpha]}^{-1} R[\alpha] P \supset I_{[\alpha]}^{-1} \cap \eta_1, \ldots, \eta_d$ because $\bigcap_{P \in \text{DP}_1(R)} R[\alpha] P = R[\alpha]$. Thus $R[\alpha] \cap K \supset R[\eta_1, \ldots, \eta_d] = C$, $\alpha$ is integral over $C$ and $\alpha$ is anti-integral over $C$. Since
$K(C) = K$ and $[K(\alpha) : K] = d$, $R[\alpha]$ is a free $C$-module $C + C\alpha + \cdots + C\alpha^{d-1}$. Hence $R[\alpha] \cap K = C$, as was to be shown.

**Theorem 2.5.** Assume that $\alpha$ is a super-primitive element of degree $d$ over $R$ and that $R$ contains a field. Let $\phi : \text{Spec}(R[\alpha]) \to \text{Spec}(R)$ be the canonical map induced from the inclusion $R \subseteq R[\alpha]$. Then the following statements are equivalent:

(i) $D_{p_1}(R)$ is contained in the image of $\phi$, i.e., $\text{Im}(\phi) \supseteq D_{p_1}(R)$;

(ii) $\alpha$ is exclusive over $R$.

**Proof.** $\text{Im}(\phi) \supseteq D_{p_1}(R)$ if and only if $J_{[\alpha]}/p = J_{[\alpha]}/p = R_p$ for every $p \in D_{p_1}(R)$ by [OY2, Lemma 2]. Since $\alpha$ is super-primitive over $R$, we see that grade$(J_{[\alpha]}) > 1$. Hence we have grade$(J_{[\alpha]}) > 1$, that is, $J_{[\alpha]}/p = R_p$ for every $p \in D_{p_1}(R)$. So we conclude that $\alpha$ is exclusive over $R$ by Lemma 2.2.

Conversely, the set of diffusing points (i.e., $p \in \text{Spec}(R)$ such that $pR[\alpha]/p = R[\alpha]/p$) is given by $\bigcap_{i=1}^{d-1} V(I_{[\alpha]}/\eta_i) \setminus V(I_{[\alpha]}/\eta_d) = V(J_{[\alpha]}) \setminus V(I_{[\alpha]}/\eta_d)$ by [OY2, Lemma 2]. So take $p \in D_{p_1}(R)$ such that $pR[\alpha]/p = R[\alpha]/p$. Then $p \supseteq J_{[\alpha]}$ and $p \supseteq I_{[\alpha]}/\eta_d$. Thus we have grade$(J_{[\alpha]}) = 1$, which yields that $\alpha$ is not exclusive over $R$ by Lemma 2.2. Hence $qR[\alpha]/q \neq R[\alpha]/q$ for all $q \in \text{Im}(\phi)$.

**Proposition 2.6.** Assume that $\alpha_1, \ldots, \alpha_n$ are super-primitive elements over $R$ and that $R$ contains a field. Put $A := R[\alpha_1, \ldots, \alpha_n]$ and let $\phi : \text{Spec}(A) \to \text{Spec}(R)$ be the canonical map induced from the inclusion $R \subseteq A$. If $D_{p_1}(R)$ is contained in the image of $\phi$, then each $\alpha_i$ $(1 \leq i \leq n)$ is exclusive over $R$.

**Proof.** Let $\psi_i : \text{Spec}(A) \to \text{Spec}(R[\alpha_i])$ and $\phi_i : \text{Spec}(R[\alpha_i]) \to \text{Spec}(R)$ be the canonical maps induced from the inclusion $R[\alpha_i] \subseteq A$ and $R \subseteq R[\alpha_i]$, respectively. Then $\phi = \phi_i \cdot \psi_i$ induces the inclusions $\text{Im}(\phi_i) \supseteq \text{Im}(\phi) \supseteq D_{p_1}(R)$. Thus our conclusion follows Theorem 2.5.

**Problem.** Is the converse statement of Proposition 2.6 valid?

**Proposition 2.7.** Assume that $\alpha$ is a super-primitive element of degree $d$ over $R$. If $\eta_d \in R$, then the canonical map $\phi : \text{Spec}(R[\alpha]) \to \text{Spec}(R)$ is surjective.

**Proof.** Note that $\phi : \text{Spec}(R[\alpha]) \to \text{Spec}(R)$ is surjective if and only if $V(J_{[\alpha]}) = V(J_{[\alpha]}/\eta_d) \subseteq V(I_{[\alpha]}/\eta_d)$ (cf. [KY1, Theorem 7], [OY2, Lemma 2]). Since $\eta_d \in R$, we have $\eta_d I_{[\alpha]} \subseteq I_{[\alpha]}$, which implies that $V(J_{[\alpha]}) = V(J_{[\alpha]})$ by the construction of $J_{[\alpha]}$. So we have our conclusion.

Note that $\phi : \text{Spec}(R[\alpha]) \to \text{Spec}(R)$ is surjective and $\phi$ is flat if and only if $\phi$ is faithfully flat. So we obtain the following corollary.

**Corollary 2.8.** Assume that $\eta_d \in R$. If $R[\alpha]$ is flat over $R$, then $R[\alpha]$ is faithfully flat over $R$.

The following proposition gives rise to a condition for $\eta_d \in R$. 
PROPOSITION 2.9. Assume that $\alpha$ is a super-primitive element of degree $d$ over $R$ and that $R$ contains a field. Then the following statement are equivalent:

(i) $\eta_d \in R$;
(ii) $\alpha$ is exclusive over $R$ and $I_{\eta_d}R[\alpha] = R[\alpha]$.

**Proof.** (i) $\Rightarrow$ (ii) follows from Lemma 2.2.
(ii) $\Rightarrow$ (i): Since $\eta_d \in I_{\eta_d}^{-1}R[\alpha] = I_{\eta_d}^{-1}I_{\eta_d}R[\alpha] \subseteq R[\alpha]$, we conclude that $\eta_d \in R[\alpha] \cap K = R$. \(\square\)

REMARK 2.10. Let $I$ denote an ideal of $R$. Then

$$\text{grade}(I) > 1 \iff I^{-1} = R,$$

where $I^{-1} := R :_K I$.

**Theorem 2.11.** Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. Let $f : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)$ and $g : \text{Spec}(R[\eta_d]) \rightarrow \text{Spec}(R)$ be the canonical maps obtained from the inclusions $R \subseteq R[\alpha]$ and $R \subseteq R[\eta_d]$, respectively. If $\text{grade}_{R[\alpha]}(J_1R[\alpha]) > 1$, $\text{grade}_{R[\alpha]}(J_{\eta_d}R[\alpha]) > 1$ and $\text{grade}_{R[\eta_d]}(J_1R[\eta_d]) > 1$, then $R[\alpha] \cap K = R[\eta_d]$ and $\text{Im}(f) = \text{Im}(g)$.

**Proof.** First note that $J_1R[\eta_d] \subseteq J_{R[\eta_d]}^0$ (here we use the notation as in Remark 2.1; put $S = R[\eta_d]$). So $\text{grade}_{R[\eta_d]}(J_1R[\eta_d]) > 1$ yields that $\text{grade}_{R[\alpha]}(J_{1R[\alpha]}) > 1$, so that $\alpha$ is super-primitive over $R[\eta_d]$. Take $P \in \text{Dp}_1(R[\alpha])$ and put $p := P \cap R$. Since $J_1R[\alpha] \not\subseteq P$, we have either $J_1[\alpha] \not\subseteq p$ or $I_{\eta_d} \not\subseteq p$. Thus $R_p[\alpha]$ is flat over $R_p$ (cf. [OY]). So we have $I_{\eta_d}R_p[\alpha] = (R :_{R_p} \eta_d)R_p[\alpha] = R_p[\alpha] :_{R_p} \eta_d = R_p[\alpha]$. Thus $\eta_d \in R_p[\alpha]$ by Proposition 1.1. Therefore $\eta_d \in \bigcap_{P \in \text{Dp}_1(R[p])} R[p] = R[\alpha]$. As mentioned above, $\alpha$ is super-primitive over $R[\eta_d]$. Since $\eta_d \in R[\eta_d]$, $\alpha$ is exclusive over $R[\eta_d]$. Hence the canonical map $\psi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R[\eta_d])$ is surjective by Proposition 2.7. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\text{Spec}(R[\eta_d][\alpha]) & \longrightarrow & \text{Spec}(R[\alpha]) \\
\phi \downarrow & & \downarrow g \\
\text{Spec}(R[\eta_d]) & \longrightarrow & \text{Spec}(R),
\end{array}
$$

here we use that $\eta_d \in R[\alpha]$. Since $\psi$ is surjective and $f \cdot \psi = g$, we conclude that $\text{Im}(f) = \text{Im}(g)$. \(\square\)

We say that $\alpha$ is an ultra-primitive element of degree $d$ over $R$ if $\text{grade}(I_{\alpha} + C(R/R)) > 1$, where $R$ denotes the integral closure of $R$ in $K$ and $C(R/R)$ denotes the conductor between $R$ and $\overline{R}$ (cf. [OY3]).

**Proposition 2.12.** Assume that an ultra-primitive element of degree $d$. If $\text{grade}(I_{\alpha} : R I_{\alpha-1}) > 1$, then $\eta_d \in R$.

**Proof.** Take $P \in \text{Dp}_1(R)$. Then either $I_{\alpha} \not\subseteq p$ or $C(R/R) \not\subseteq p$. If $I_{\alpha} \not\subseteq p$, then $J_1 \subseteq I_{\eta_d} \not\subseteq p$, that is, $\eta_d \in R_p$. If $I_{\alpha} \subseteq p$, then $C(R/R) \not\subseteq p$ and hence...
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$R_p$ is a normal domain. Note that $I_{[\alpha^{-1}]p} = \eta_d I_{[\alpha]p} \subseteq I_{[\alpha]p}$ (cf. Lemma 1.3). The latter inclusion $\eta_d I_{[\alpha]p} \subseteq I_{[\alpha]p}$, that is, $\eta_d \subseteq (I_{[\alpha]} : R I_{[\alpha]})_p$ implies that $\eta_d$ is integral over $R_p$, noting that $I_{[\alpha]p}$ is finitely generated over $R_p$. So we have $\eta_d \in R_p$. Therefore $\eta_d \in \bigcap_{p \in \mathcal{DP}(R)} R_p = R$. □

References


