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On Conditions for Denominator Ideals to Diffuse and Conditions for Elements to Be Exclusive in Anti-Integral Extensions

Susumu Oda* and Ken-ichi Yoshida**

Let \( R \) be a Noetherian integral domain and let \( \alpha \) be an element in an algebraic extension field of the quotient field of \( R \). Our objective is to show that a generalized denominator ideal \( I_{[\alpha]} \) satisfies \( I_{[\alpha]} R[\alpha] = R[\alpha] \) if and only if \( I_{[\alpha]} + I_{[\alpha^{-1}]} = R \) if and only if either \( \alpha \) or \( \alpha^{-1} \) is integral over \( R_p \) for each \( p \in \text{Spec}(R) \), provided that \( \alpha \) is an anti-integral element over \( R \). The other is to show that for the canonical map \( \phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R) \), \( Dp_1(R) \) is contained in the image of \( \phi \) if and only if \( \alpha \) is exclusive. The former result is closely concerned to [KY1] and [OY3], and the latter result is related to [OY2].

Notation and Conventions

Throughout this paper, we use the following notation unless otherwise specified:

Let \( R \) be a Noetherian domain (which is commutative and has a unit), let \( R[X] \) be a polynomial ring, let \( \alpha \) be an element of an algebraic extension field of the quotient field \( K \) of \( R \) and let \( \pi : R[X] \rightarrow R[\alpha] \) be the \( R \)-algebra homomorphism sending \( X \) to \( \alpha \). Let \( \varphi_\alpha(X) \) be the monic minimal polynomial of \( \alpha \) over \( K \) with \( \deg \varphi_\alpha(X) = d \) and write \( \varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \). Then \( \eta_i \in K \) \((1 \leq i \leq d)\) are uniquely determined by \( \alpha \). Put \( d = [K(\alpha) : K] \), \( I_{\eta_i} := R : R \eta_i \) and \( I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i} \). If \( \text{Ker}(\pi) = I_{[\alpha]} \varphi_\alpha(X) R[X] \), we say that \( \alpha \) is anti-integral over \( R \) (cf. [OSY]). Put \( J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \). For \( \beta \in K \), we put \( J_\beta := I_\beta(1, \beta) \). Then \( J_{[\alpha]} = c(I_{[\alpha]} \varphi_\alpha(X)) \), where \( c(\cdot) \) denotes the ideal generated by the coefficients of the polynomials in \( \varphi_\alpha(X) \), that is, the content ideal of \( \varphi_\alpha(X) \). If \( J_{[\alpha]} \not\subseteq p \) for all \( p \in Dp_1(R) := \{ p \in \text{Spec}(R) \mid \text{depth} R_p = 1 \} \), the element \( \alpha \) is called a super-primitive element over \( R \). A super-primitive element over \( R \) is anti-integral over \( R \) (cf.[OSY,Theorem 1.12]). It is also known that any algebraic element over a Krull domain \( R \) is super-primitive over \( R \) (cf.[OSY, Theorem 1.13]), and hence \( \alpha \) is anti-integral over \( R \). We also note here that \( I_{[\alpha]} = R \leftrightarrow R[\alpha] \) is integral over \( R \) and that \( J_{[\alpha]} = R \leftrightarrow R[\alpha] \) is flat over \( R \), provided that \( \alpha \) is anti-integral over \( R \).

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* Matsusaka Commercial High School Toyohara, Matsusaka, Mie 515-0205 Japan
  e-mail: 
** Department of Applied Math. Okayama University of Science Ridai-cho, Okayama 700-0005 Japan
  e-mail: 

§1. Conditions for generalized denominator ideals to diffuse in anti-integral extensions

We start with the following proposition. The implication (i) ⇒ (ii) is seen in [OY3, Proposition 7], but we give a proof for convenience.

**Proposition 1.1.** Assume that α is an anti-integral element over R. Then the following statements are equivalent:

(i) $I_\alpha R[\alpha] = R[\alpha]$;

(ii) $\eta_d \in R[\alpha]$, $I_\alpha = I_{\eta_d}$ and $J_{\eta_d} = R$.

**Proof.** If $I_\alpha = R$, then the implications (i) ⇔ (ii) are trivially valid because $\eta_d \in R$ by the definition of $I_\alpha$. So we may assume that $I_\alpha \neq R$.

(i) ⇒ (ii): Since $\eta_1, \ldots, \eta_d \in I_\alpha^{-1} R[\alpha] = I_\alpha^{-1} I_\alpha R[\alpha] \subseteq R[\alpha]$, we have $\eta_d \in R[\alpha]$. Put $C := R[\eta_1, \ldots, \eta_d]$. Then $R[\alpha]$ is a free $C$-module and $R[\alpha]$ is integral over $C$. So $I_\alpha R[\alpha] = R[\alpha]$ yields that $I_\alpha C = C$. Thus $C$ is flat over $R$. Since $R[\alpha]$ is flat over $C$, $R[\alpha]$ is flat over $R$, and hence $J_\alpha = R$.

Now we shall show that $I_\alpha = I_{\eta_d}$. It is easy to see that $I_\alpha \subseteq I_{\eta_d}$. Take $p \in \text{Spec}(R)$ with $p \supseteq I_\alpha$. Since $I_\alpha R[\alpha] = R[\alpha]$, we have $1 = a_0 + a_1 \alpha + \cdots + a_n \alpha^n$, where $a_i \in I_\alpha \subseteq p$. Since $1 - a_0$ is a unit in $R_p$, $\alpha^{-1}$ is integral over $R$. Since $\alpha$ is anti-integral over $R$, $\alpha^{-1}$ is also anti-integral over $R$. Thus $\varphi_{\alpha^{-1}}(X) = X^d + (\eta_d^{-1} \eta_{d-1} - 1) X^{d-1} + \cdots + (\eta_d^{-1} \eta_1) X + \eta_d^{-1} \in R_p[X]$. So there exists $x \in R_p$ such that $\eta_d = 1/x$, $\eta_i = y_i/x$ with some $y_i \in R_p$. Therefore we have $(I_\alpha)_p = x R_p = (I_{\eta_d})_p$. Since $p$ is arbitrary, we conclude that $I_\alpha = I_{\eta_d}$.

Next we shall show that $J_{\eta_d} = I_{\eta_d}(1, \eta_d) = R$. Suppose that $I_{\eta_d}(1, \eta_d) \subseteq p$ for some $p \in \text{Spec}(R)$. Then $I_\alpha \subseteq p$. Thus by using the above notation, we have $(I_{\eta_d}(1, \eta_d))_p = x(1, 1/x) R_p = R_p$, which contradicts the assumption $J_{\eta_d} = I_{\eta_d}(1, \eta_d) \subseteq p$. Therefore we conclude that $J_{\eta_d} = R$.

(ii) ⇒ (i) follows the implications: $R[\alpha] \supseteq I_\alpha R[\alpha] \supseteq I_\alpha(1, \eta_d) = I_{\eta_d}(1, \eta_d) = R \supsetneq 1$, and hence $R[\alpha] \supseteq I_\alpha R[\alpha] \supseteq R[\alpha]$. □

**Corollary 1.2.** Assume that α is an anti-integral element over R. If $I_\alpha R[\alpha] = R[\alpha]$, then $R[\alpha] \cap K = R[\eta_d]$.

**Proof.** Put $C := R[\eta_1, \ldots, \eta_d]$. Then by the proof of Proposition 1.1, we have $C \subseteq R[\alpha]$. Since $R[\alpha]$ is a free $C$-module $C + C\alpha + \cdots + C\alpha^{d-1}$, we conclude that $R[\alpha] \cap K = C$. Besides, as in the proof of Proposition 1, if $I_\alpha \subseteq p$, then there exists $x \in R_p$ such that $\eta_d = 1/x$ and $\eta_i = y_i/x \in R_p[\eta_d]$ for some $y_i \in R_p$. If $I_\alpha \not\subseteq p$, then $C_p = R_p = R_p[\eta_d]$. Therefore we have $C = R[\eta_d]$. □

**Lemma 1.3.** $I_{\alpha^{-1}} = \eta_d I_\alpha$.

**Proof.** Since $\varphi_{\alpha^{-1}}(X) = X^d + (\eta_d^{-1} \eta_{d-1}) X^{d-1} + \cdots + (\eta_d^{-1} \eta_1) X + \eta_d^{-1}$, we have $I_{\alpha^{-1}} = \bigcap_{i=1}^{d-1} I_{\eta_d^{-1}} \cap I_{\eta_d^{-1}}$. Take $x = \eta_d y \in \eta_d I_\alpha$ with $y \in I_\alpha$. Then $x \eta_d^{-1} = y \eta_i \in R$ and hence $x \eta_d^{-1} = y \in R$. Thus $I_{\alpha^{-1}} \supseteq \eta_d I_\alpha$. Conversely,
take $x \in I_{[\alpha^{-1}]}$. Then $x\eta_d^{-1} = y \in R$, $y\eta_d = x \in R$ and $x\eta_d^{-1} \eta_i = y\eta_i \in R$. Hence $y \in I_{[\alpha]}$. Therefore we have $I_{[\alpha^{-1}]} = \eta_d I_{[\alpha]}$. □

**Remark 1.4.** (i) An element $\alpha$ is anti-integral over $R$ if and only if so is $\alpha^{-1}$ (cf. [KY]).

(ii) An element $\alpha$ is super-primitive over $R$ if and only if so is $\alpha^{-1}$. Indeed, the minimal monic polynomial $\varphi_{\alpha^{-1}}$ of $\alpha^{-1}$ is $X^d + (\eta_{d-1}/\eta_d)X^{d-1} + \cdots + (\eta_1/\eta_d)X + (1/\eta_d)$. Hence we have

\[
J_{[\alpha^{-1}]} = I_{[\alpha^{-1}]}(1, \eta_{d-1}/\eta_d, \cdots, \eta_1/\eta_d) = \eta_d I_{[\alpha]}(1, \eta_{d-1}/\eta_d, \cdots, \eta_1/\eta_d) = I_{[\alpha]}(1, \eta_1, \cdots, \eta_d) = J_{[\alpha]},
\]

here we use Lemma 1.3. So our assertion follows from the definition of super-primitiveness.

**Theorem 1.5.** Assume that $\alpha$ is an anti-integral element over $R$ and that $\eta_d \in R[\alpha]$. The following statements are equivalent:

(i) $I_{[\alpha]} R[\alpha] = R[\alpha]$;

(ii) $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$;

(iii) either $\alpha$ or $\alpha^{-1}$ is integral over $R_p$ for each $p \in \text{Spec}(R)$.

**Proof.** (i) $\Rightarrow$ (ii): We have $\eta_d \in R[\alpha], I_{[\alpha]} = I_{[\eta_d]}$ and $J_{[\eta_d]} = R$ by Proposition 1.1. So we obtain $R = J_{[\eta_d]} = I_{[\eta_d]}(1, \eta_d) = I_{[\alpha]}(1, \eta_d) = I_{[\alpha]} + \eta_d I_{[\alpha]} = I_{[\alpha]} + I_{[\alpha^{-1}]}$ (cf. Lemma 1.3).

(ii) $\Rightarrow$ (i): We have $R = I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]} + \eta_d I_{[\alpha]} = I_{[\alpha]}(1, \eta_d)$. Thus $R[\alpha] = I_{[\alpha]}(1, \eta_d)R[\alpha] = I_{[\alpha]}R[\alpha] = I_{[\alpha]}\eta_1 R[\alpha] \subseteq I_{[\alpha]} R[\alpha]$ because $\eta_d \in R[\alpha]$. The converse inclusion is obvious. Thus $I_{[\alpha]} R[\alpha] = R[\alpha]$.

(ii) $\Rightarrow$ (iii): Take $p \in \text{Spec}(R)$. Then $I_{[\alpha]} \not\subseteq p$ or $I_{[\alpha^{-1}]} \not\subseteq p$ because $I_{[\alpha]} + I_{[\alpha^{-1}]} = R$. If $I_{[\alpha]} \not\subseteq p$, we have $\eta_1, \ldots, \eta_d \in R_p$. Hence $\alpha^d + \eta_1 \alpha^{d-1} + \cdots + \eta_d = 0$, which means that $\alpha$ is integral over $R_p$. Next, we assume that $I_{[\alpha^{-1}]} \not\subseteq p$. The $\eta_d^{-1} \eta_1, \ldots, \eta_d^{-1} \eta_{d-1}, \eta_d^{-1} \in R_p$. Thus $(\alpha^{-1})^d + (\eta_d^{-1} \eta_{d-1}) \alpha^{d-1} + \cdots + \eta_d^{-1} \eta_1 \alpha + \eta_d^{-1} = 0$, which means that $\alpha^{-1}$ is integral over $R_p$.

(iii) $\Rightarrow$ (ii): Suppose that there exists a prime ideal $p$ of $R$ such that $I_{[\alpha]} + I_{[\alpha^{-1}]} \subseteq p$. Note that $\alpha$ or $\alpha^{-1}$ is integral over $R_p$. Assume that $\alpha$ is integral over $R_p$. Then $\alpha$ satisfies a monic relation of degree $d$ over $R_p$ because $\alpha$ is anti-integral over $R_p$. But since $\varphi_{\alpha}(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \in R_p[X]$, we have $I_{[\alpha]} \not\subseteq p$, a contradiction. Similarly we come to a contradiction when we assume $\alpha^{-1}$ is integral over $R_p$. □

**Proposition 1.6.** Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. If $\eta_d \in R[\alpha]$, then $\sqrt{I_{[\alpha]} R[\alpha]} \cap R = \sqrt{I_{[\alpha]} + I_{[\alpha^{-1}]}^{1/2}}$.

**Proof.** Take $p \in \text{Spec}(R)$. Then $I_{[\alpha]} R[\alpha] \cap R \not\subseteq p \iff I_{[\alpha]} R_p[\alpha] = R_p[\alpha] \iff (I_{[\alpha]} + I_{[\alpha^{-1}]})_p = R_p \iff I_{[\alpha]} + I_{[\alpha^{-1}]} \not\subseteq p$. Thus we come to our conclusion by Theorem 1.5. □
§2. Conditions for anti-integral elements to be exclusive.

Recall first that an algebraic element $\alpha$ over $R$ is called to be exclusive if $R[\alpha] \cap K = R$. Put $\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$, an ideal of $R$.

REMARK 2.1. Assume that $R$ contains a field $k$. Let $x$ be an indeterminate and put $S := R \otimes_k k(x)$. Then $S$ contains the infinite field $k(x)$. Put $I_{[\alpha]}^S := \bigcap_{i=1}^d (S : s \eta_i)$, $J_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \ldots, \eta_d)S$ and $\tilde{J}_{[\alpha]}^S := I_{[\alpha]}^S(1, \eta_1, \ldots, \eta_{d-1})S$. Since $S$ is faithfully flat over $R$, $I_{[\alpha]}^S = I_{[\alpha]}S$, $J_{[\alpha]}^S = J_{[\alpha]}S$ and $\tilde{J}_{[\alpha]}^S = \tilde{J}_{[\alpha]}S$. So if $\alpha$ is super-primitive (resp. anti-integral) over $R$, then so is $\alpha$ over $S$. Moreover $\alpha$ is exclusive over $S$ if and only if so is $\alpha$ over $R$; $\bigcap_{i=1}^d I_{[\alpha]} \subseteq I_{[\alpha]}$ if and only if $\bigcap_{i=1}^d I_{[\alpha]}^S \subseteq I_{[\alpha]}^S$; grade($\tilde{J}_{[\alpha]}$) > 1 if and only if grade($\tilde{J}_{[\alpha]}^S$) > 1; and $\alpha$ is exclusive over $S$, i.e., $S[\alpha] \cap K(x) = S[\alpha]$ if and only if $S[\alpha]$ is exclusive over $R$, i.e., $R[\alpha] \cap K = R$.

LEMMA 2.2(cf. [OY2, Theorem 5]) Assume that $\alpha$ is super-primitive over $R$. Then the following statements (i) and (ii) are equivalent:

(i) $\bigcap_{i=1}^d I_{[\alpha]} \subseteq I_{[\alpha]}$;
(ii) grade($\tilde{J}_{[\alpha]}$) > 1 or $\tilde{J}_{[\alpha]} = R$.

Furthermore if $R$ contains a field, then the following (iii) is equivalent to (i):

(iii) $\alpha$ is exclusive over $R$.

PROOF. (i) $\Leftrightarrow$ (ii) follows from [OY2, Lemma 3].

Next assume that $R$ contains a field. We may assume that $R$ contains an infinite field by Remark 2.1. Hence our conclusion (ii) $\Leftrightarrow$ (iii) follows from [OY2, Theorem 5].

REMARK 2.3. Assume that $R$ contains a field.

(i) When $\alpha$ is a super-primitive element over $R$, $\alpha$ is exclusive over $R$ if and only if grade($\tilde{J}_{[\alpha]}$) > 1 by Lemma 2.2.

(ii) $\tilde{J}_{[\alpha]} = I_{[\alpha]}(\eta_1, \ldots, \eta_d)$. This follows from the similar argument of Remark 1.4(ii).

(iii) By Remark 1.4, $\alpha$ is super-primitive over $R$ if and only if $\alpha$ is $\alpha^{-1}$. So by (i) and (ii) above, we have $\alpha^{-1}$ is exclusive over $R$ if and only if grade($I_{[\alpha]}(\eta_1, \ldots, \eta_d)$) > 1.

(iv) If grade($I_{[\alpha]}(\eta_1, \ldots, \eta_{d-1})$) > 1, then both $\alpha$ and $\alpha^{-1}$ are exclusive over $R$ by (i), (ii) and (iii).

PROPOSITION 2.4. Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. If $I_{[\alpha]} R[\alpha]_P = R[\alpha]_P$ for every $P \in D_{P_1}(R[\alpha])$, then $R[\alpha] \cap K = R[\eta_1, \ldots, \eta_d]$.

PROOF. It follows that $R[\alpha]_P \supseteq I_{[\alpha]}^{-1} I_{[\alpha]} R[\alpha]_P = I_{[\alpha]}^{-1} R[\alpha]_P \supseteq I_{[\alpha]}^{-1} \ni \eta_1, \ldots, \eta_d$ because $\bigcap_{P \in D_{P_1}(R)} R[\alpha]_P = R[\alpha]$. Thus $R[\alpha] \cap K \supseteq R[\eta_1, \ldots, \eta_d] =: C$, $\alpha$ is integral over $C$ and $\alpha$ is anti-integral over $C$. Since
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\[ K(C) = K \text{ and } [K(\alpha) : K] = d, \ R[\alpha] \text{ is a free } C\text{-module } C + C\alpha + \cdots + C\alpha^{d-1}. \]

Hence \( R[\alpha] \cap K = C \), as was to be shown. \[
\]

**Theorem 2.5.** Assume that \( \alpha \) is a super-primitive element of degree \( d \) over \( R \) and that \( R \) contains a field. Let \( \phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R) \) be the canonical map induced from the inclusion \( R \subseteq R[\alpha] \). Then the following statements are equivalent:

(i) \( Dp_1(R) \) is contained in the image of \( \phi \), i.e., \( \text{Im}(\phi) \supseteq Dp_1(R) \);
(ii) \( \alpha \) is exclusive over \( R \).

**Proof.** \( \text{Im}(\phi) \supseteq Dp_1(R) \) if and only if \( J_{[\alpha]}p = J_{[\alpha]}p = R_p \) for every \( p \in Dp_1(R) \) by [OY2, Lemma 2]. Since \( \alpha \) is super-primitive over \( R \), we see that \( \text{grade}(J_{[\alpha]}) > 1 \). Hence we have \( \text{grade}(J_{[\alpha]}) > 1 \), that is, \( J_{[\alpha]}p = R_p \) for every \( p \in Dp_1(R) \). So we conclude that \( \alpha \) is exclusive over \( R \) by Lemma 2.2.

Conversely, the set of diffusing points (i.e., \( p \in \text{Spec}(R) \) such that \( pR[\alpha]p = R[\alpha]p \)) is given by \( \bigcap_{i=1}^{d-1} V(I_{[\alpha]}/\eta_i) \setminus V(I_{[\alpha]}\eta_d) = V(J_{[\alpha]}) \setminus V(I_{[\alpha]}\eta_d) \) by [OY2, Lemma 2]. So take \( p \in Dp_1(R) \) such that \( pR[\alpha]p = R[\alpha]p \). Then \( p \supseteq J_{[\alpha]}p \) for all \( p \in Dp_1(R) \). Thus we have \( \text{grade}(J_{[\alpha]}) = 1 \), which yields that \( \alpha \) is not exclusive over \( R \) by Lemma 2.2. Hence \( qR[\alpha]q \neq R[\alpha]q \) for all \( q \in \text{Im}(\phi) \).

**Proposition 2.6.** Assume that \( \alpha_1, \ldots, \alpha_n \) are super-primitive elements over \( R \) and that \( R \) contains a field. Put \( A := R[\alpha_1, \ldots, \alpha_n] \) and let \( \phi : \text{Spec}(A) \rightarrow \text{Spec}(R[\alpha]) \) be the canonical map induced from the inclusion \( R \subseteq A \). If \( Dp_1(R) \) is contained in the image of \( \phi \), then each \( \alpha_i \) (\( 1 \leq i \leq n \)) is exclusive over \( R \).

**Proof.** Let \( \psi_i : \text{Spec}(A) \rightarrow \text{Spec}(R[\alpha_i]) \) and \( \phi_i : \text{Spec}(R[\alpha_i]) \rightarrow \text{Spec}(R) \) be the canonical maps induced from the inclusion \( R[\alpha_i] \subseteq A \) and \( R \subseteq R[\alpha_i] \), respectively. Then \( \phi = \phi_i \cdot \psi_i \) induces the inclusions \( \text{Im}(\phi_i) \supseteq \text{Im}(\phi) \supseteq Dp_1(R) \). Thus our conclusion follows Theorem 2.5.

**Problem.** Is the converse statement of Proposition 2.6 valid?

**Proposition 2.7.** Assume that \( \alpha \) is a super-primitive element of degree \( d \) over \( R \). If \( \eta_d \in R \), then the canonical map \( \phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R) \) is surjective.

**Proof.** Note that \( \phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R) \) is surjective if and only if \( V(J_{[\alpha]}) = V(J_{[\alpha]}) \subseteq V(I_{[\alpha]}\eta_d) \) (cf. [KY1, Theorem 7], [OY2, Lemma 2]). Since \( \eta_d \in R \), we have \( \eta_d I_{[\alpha]} \subseteq I_{[\alpha]} \), which implies that \( V(J_{[\alpha]}) = V(J_{[\alpha]}) \) by the construction of \( J_{[\alpha]} \). So we have our conclusion.

Note that \( \phi : \text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R) \) is surjective and \( \phi \) is flat if and only if \( \phi \) is faithfully flat. So we obtain the following corollary.

**Corollary 2.8.** Assume that \( \eta_d \in R \). If \( R[\alpha] \) is flat over \( R \), then \( R[\alpha] \) is faithfully flat over \( R \).

The following proposition gives rise to a condition for \( \eta_d \in R \).
Proposition 2.9. Assume that $\alpha$ is a super-primitive element of degree $d$ over $R$ and that $R$ contains a field. Then the following statements are equivalent:

(i) $\eta_d \in R$;
(ii) $\alpha$ is exclusive over $R$ and $I_{\eta_d} R[\alpha] = R[\alpha]$.

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 2.2.
(ii) $\Rightarrow$ (i): Since $\eta_d \in I_{\eta_d}^{-1} R[\alpha] = I_{\eta_d}^{-1} I_{\eta_d} R[\alpha] \subseteq R[\alpha]$, we conclude that $\eta_d \in R[\alpha] \cap K = R$. □

Remark 2.10. Let $I$ denote an ideal of $R$. Then

$$\text{grade}(I) > 1 \iff I^{-1} = R,$$

where $I^{-1} := R : K I$.

Theorem 2.11. Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. Let $f : \text{Spec}(R[\alpha]) \to \text{Spec}(R)$ and $g : \text{Spec}(R[\eta_d]) \to \text{Spec}(R)$ be the canonical maps obtained from the inclusions $R \subseteq R[\alpha]$ and $R \subseteq R[\eta_d]$, respectively. If $\text{grade}_{R[\alpha]}(J[\alpha] R[\alpha]) > 1$, $\text{grade}_{R[\alpha]}(J_{\eta_d} R[\alpha]) > 1$ and $\text{grade}_{R[\eta_d]}(J[\alpha] R[\eta_d]) > 1$, then $R[\alpha] \subseteq K = R[\eta_d]$ and $\text{Im}(f) = \text{Im}(g)$.

Proof. First note that $J[\alpha] R[\eta_d] \subseteq J_{\eta_d}[\alpha]$ (here we use the notation as in Remark 2.1; put $S = R[\eta_d]$). So $\text{grade}_{R[\eta_d]}(J[\alpha] R[\eta_d]) > 1$ yields that $\text{grade}_{R[\alpha]}(J[\alpha] R[\alpha]) > 1$, so that $\alpha$ is super-primitive over $R[\eta_d]$. Take $P \in \text{Dp}_1(R[\alpha])$ and put $p := P \cap R$. Since $J[\alpha] R[\alpha] \not\subseteq P$, we have either $J[\alpha] \not\subseteq p$ or $I_{\eta_d} \not\subseteq p$. Thus $R_p[\alpha]$ is flat over $R_p$ (cf. [OSY]). So we have $I_{\eta_d} R_p[\alpha] = (R : R_{\eta_d}) R_p[\alpha] = R_p[\alpha] : R[\alpha] \eta_d = R_p[\alpha]$. Thus $\eta_d \in R_p[\alpha]$ by Proposition 1.1. Therefore $\eta_d \in \bigcap_{P \in \text{Dp}_1(R[\alpha])} R[\alpha] P = R[\alpha]$. As mentioned above, $\alpha$ is super-primitive over $R[\eta_d]$. Since $\eta_d \in R[\eta_d]$, $\alpha$ is exclusive over $R[\eta_d]$. Hence the canonical map $\psi : \text{Spec}(R[\alpha]) \to \text{Spec}(R[\eta_d])$ is surjective by Proposition 2.7. Consider the following commutative diagram:

$$\begin{array}{ccc}
\text{Spec}(R[\eta_d][\alpha]) & \longrightarrow & \text{Spec}(R[\alpha]) \\
\psi \downarrow & & \downarrow g \\
\text{Spec}(R[\eta_d]) & \longrightarrow & \text{Spec}(R),
\end{array}$$

here we use that $\eta_d \in R[\alpha]$. Since $\psi$ is surjective and $f \cdot \psi = g$, we conclude that $\text{Im}(f) = \text{Im}(g)$. □

We say that $\alpha$ is an ultra-primitive element of degree $d$ over $R$ if $\text{grade}(I[\alpha] + C(R/R)) > 1$, where $R$ denotes the integral closure of $R$ in $K$ and $C(R/R)$ denotes the conductor between $R$ and $\overline{R}$ (cf. [OY3]).

Proposition 2.12. Assume that an ultra-primitive element of degree $d$. If $\text{grade}(I[\alpha] : R I[\alpha-1]) > 1$, then $\eta_d \in R$.

Proof. Take $p \in \text{Dp}_1(R)$. Then either $I[\alpha] \not\subseteq p$ or $C(R/R) \not\subseteq p$. If $I[\alpha] \not\subseteq p$, then $I[\alpha] \subseteq I_{\eta_d} \not\subseteq p$, that is, $\eta_d \in R_p$. If $I[\alpha] \subseteq p$, then $C(R/R) \not\subseteq p$ and hence
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$R_p$ is a normal domain. Note that $I_{[\alpha^{-1}]p} = \eta_d I_{[\alpha]p} \subseteq I_{[\alpha]p}$ (cf. Lemma 1.3). The latter inclusion $\eta_d I_{[\alpha]p} \subseteq I_{[\alpha]p}$, that is, $\eta_d \in (I_{[\alpha]} :_{R_p} I_{[\alpha]})_p$ implies that $\eta_d$ is integral over $R_p$, noting that $I_{[\alpha]p}$ is finitely generated over $R_p$. So we have $\eta_d \in R_p$. Therefore $\eta_d \in \bigcap_{p \in \mathbb{P}(R)} R_p = R$. □

References


