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Some Matrix Transformations Into the Cesàro Sequence Spaces of Non-absolute Type

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Abstract. The present paper is concerned with the neccessary and sufficient conditions in order for a matrix $A = (a_{nk})$ to belong to the classes $(\ell_\infty : X_p)$, $(bs : X_p)$ and $(bv : X_p)$ respectively, where $1 \le p \le \infty$. Furthermore, we prove that $A \in (bs : \mu)$ if and only if $B \in (\ell_\infty : \mu)$ and use this to characterise the class $(bs : X_p)$; where A and B are dual matrices and μ is any given sequence space.

1. Introduction, Definitions and Notation

In this paper ℓ_∞, c, c_0, bs, cs, bv and ℓ_p $(1 \le p < \infty)$ have their usual meanings which are the Banach spaces with the following finite norms:

$$\|x\|_{\ell_\infty} = \sup_k |x_k| \ ,$$

$$\|x\|_{bs} = \|x\|_{cs} = \sup_n \left|\sum_{k=1}^{n} x_k\right| \ ,$$

$$\|x\|_{bv} = \lim_k |x_k| \ + \ \sum_k |x_k - x_{k+1}|$$

and

$$\|x\|_{\ell_p} = \Big(\sum_k |x_k|^p\Big)^{1/p},$$

respectively. For simplicity in notation, here and after the summation without limits runs from 1 to ∞. By μ^β, we denote the β-dual (or generalized Köthe-Toeplitz dual) of a sequence space μ which is defined as

$$\mu^\beta = \big\{ \ x = (x_k) \in w \ : \ xy = (x_k y_k) \in cs \text{ for every } y \in \mu \ \big\}.$$

It is well-known that

$$(1) \qquad (bs)^\beta = bv_0 = bv \cap c_0 \text{ and } (\ell_\infty)^\beta = \ell_1$$

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(see Kamthan and Gupta [4], pp. 68-69). Some definitions and conventions are made in this section and two lemmas will be given in the Section 3. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}, (n, k \in \mathbb{N})$ and λ, μ two sequence spaces; where N denotes the set of all positive integers. Then, the matrix A defines a transformation from λ into μ, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = ((Ax)_n)$, the A-transform of x, exists and is in μ; where $(Ax)_n = \sum_k a_{nk}x_k$. By $(\lambda : \mu)$, we denote the class of all such matrices. A sequence x is said to be A-summable to b if Ax converges to b which is called as the A-limit of x. We shall also assume througout for brevitiy that $p, q \geq 1$ are conjugate exponents, i.e., $p^{-1} + q^{-1} = 1$, and denote the collection of all finite subsets of N by $\mathcal{F}$, and

$$a(n, k) = \sum_{i=1}^{n} a_{ik} \ , \ (n, k \in \mathbb{N}).$$

The Cesàro sequence spaces X_p of non-absolute type have introduced by Ng/Lee in [8] which may be defined as

$$X_p = \{ \ x = (x_k) \in w \ : \ C_1x \in \ell_p \ \}$$

and showed that these spaces are the Banach spaces with the following finite norm:

$$\|x\|_p = \|C_1x\|_{\ell_p} \ , \ (1 \leq p \leq \infty);$$

where C_1 denotes the method of arithmetic means. For the relevant terminology on the Cesàro sequence spaces X_p of non-absolute type and matrix transformations from X_p into the other spaces, the reader may refer to Ng/Lee [8] and Başar [1], respectively. To our knowledge, although the matrix transformations from X_p into certain spaces and the transformations between the X_p spaces and ℓ_p spaces have been characterised, the transformations from a sequence space different than the spaces X_p and ℓ_p into the X_p spaces have not been characterised, until quite recently. So, the main purpose of the present paper, which is a continuation of Başar [1, 2], is to achieve the characterisation of the matrix classes $(\ell_\infty : X_p)$, $(bs : X_p)$ and $(bv : X_p)$, respectively, where $1 \leq p \leq \infty$. Furthermore, we prove that $A \in (bs : \mu)$ if and only if $B \in (\ell_\infty : \mu)$, where A and B are dual matrices and μ is any given sequence space. We additionally establish the similar theorem obtained by changing the roles of the spaces bs and ℓ_∞ with μ and therefore to fill up some gaps in the existing literature.

2. The Dual Summability Methods

Lorentz introduced the concept of dual summability methods in [6] for the limitation methods dependent on a Stieltjes integral and passed to the discontinuous matrix methods by means of a suitable step function. Later, several authors, such as Kuttner [5], Öztürk [10], Orhan and Öztürk [9], Başar and Çolak [3], and the others, worked on the dual summability methods. Following Lorentz [6] and

Lorentz and Zeller [7], we firstly desire to define the dual summability methods in this section and nextly give a theorem on those methods.

Let us suppose throughout that the sequence $v = (v_k)$ and the infinite series $\sum u_n$ are connected with the following relation:

$$v_k = \sum_{n=1}^{k} u_n \ , \quad (k \in \mathrm{N}), \tag{2}$$

and let the A-transform of the sequence $u = (u_n)$ be $z = (z_n)$ and the B-transform of the sequence $v = (v_k)$ be $t = (t_n)$, that is to say that,

$$z_n = \left(Au\right)_n = \sum_k a_{nk} u_k \ , \quad (n \in \mathrm{N}) \tag{3}$$

and

$$t_n = \left(Bv\right)_n = \sum_k b_{nk} v_k \ , \quad (n \in \mathrm{N}). \tag{4}$$

Further we assume that the series

$$\sum_k b_{nk} \tag{5}$$

converges for each $n \in \mathrm{N}$ which is a much weaker assumption than the conditions on the matrix B which are necessary and sufficient for it to be in any matrix class, in general.

We shall say in this situation that the methods A and B in (3), (4) are dual if z_n reduces to t_n (or t_n reduces to z_n) under the application of the formal summation by parts. This statement is formally equivalent to the relation

$$a_{nk} = \sum_{i=k}^{\infty} b_{ni} \ (\text{or } b_{nk} = \Delta a_{nk} \text{ where } \Delta a_{nk} = a_{nk} - a_{n,k+1}) \ , \quad (n, k \in \mathrm{N}). \tag{6}$$

Now one can immediately deduce that the condition (5) is required in order for the infinite matrix $A = (a_{nk})$, which is dual to the matrix $B = (b_{nk})$, to be defined and t_n reduces to z_n as follows:

$$t_n = \sum_k b_{nk} v_k = \sum_k b_{nk} \left(\sum_{i=1}^{k} u_i \right) = \sum_i \sum_{k=i}^{\infty} b_{nk} u_i = z_n. \tag{7}$$

But the order of summation may not be reversed and thus the methods B, A are not necessarily equivalent.

Now we may give the following theorem related to the dual matrices which is a consequence of the one-to-one correspondence between the spaces bs and ℓ_∞, defined by (2).

THEOREM 1. *Let $A = (a_{nk})$ and $B = (b_{nk})$ be dual matrices and μ be any given sequence space. Then $A \in (bs : \mu)$ if and only if $B \in (\ell_\infty : \mu)$.*

PROOF. Suppose that $A = (a_{nk})$ and $B = (b_{nk})$ be dual matrices, that is to say that (6) holds, and μ be any given sequence space and take into account the relations in (1).

Let $A \in (bs : \mu)$ and take any $v \in \ell_\infty$. Then the series $\sum_k a_{nk}u_k$ converges for each $n \in \mathbb{N}$ and each $u \in bs$ and this implies that $(a_{nk})_{k\in\mathbb{N}} \in bv_0$ which gives to us that $(b_{nk})_{k\in\mathbb{N}} \in \ell_1, (n \in \mathbb{N})$. Hence, Bv exists for every $v \in \ell_\infty$. Therefore, we derive from the equality

$$\sum_{k=1}^{m} b_{nk}v_k = \sum_{k=1}^{m} a_{nk}u_k - a_{n,m+1}v_m \ ; \ (n, m \in \mathbb{N}),$$

by passing to limit as $m \to \infty$ with (8) that $Bv = Au$ and this gives that $B \in (\ell_\infty : \mu)$.

Conversely, let $B \in (\ell_\infty : \mu)$ and take any $u \in bs$. Then the series $\sum_k b_{nk}v_k$ converges for each $n \in \mathbb{N}$ and each $v \in \ell_\infty$ and this implies that $(b_{nk})_{k\in\mathbb{N}} \in \ell_1$, $(n \in \mathbb{N})$, and (8) holds. This yields the fact that $(a_{nk})_{k\in\mathbb{N}} \in bv_0$, $(n \in \mathbb{N})$. Hence, Au exists for every $u \in bs$ and thus letting $m \to \infty$ in the equality

$$\sum_{k=1}^{m} a_{nk}u_k = \sum_{k=1}^{m-1} b_{nk}v_k + a_{nm}v_m \ ; \ (n, m \in \mathbb{N}),$$

we get that $Au = Bv$ which leads us to the consequence $A \in (bs : \mu)$. Thus, the proof is completed.

By changing the roles of the spaces bs and ℓ_∞ with μ, we have

THEOREM 2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} = a(n, k) \ ; \ (n, k \in \mathbb{N}) \tag{8}$$

and μ be a sequence space. Then, $A \in (\mu : bs)$ if and only if $B \in (\mu : \ell_\infty)$.

PROOF. Let $x \in \mu$ and consider the equality with (8) that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_j = \sum_{j=1}^{m} b_{nj}x_j \ , \ (m, n \in \mathbb{N}) \tag{9}$$

which yields as $m \to \infty$ that

$$\sum_{i=1}^{n} (Ax)_i = (Bx)_n \ ; \ (n \in \mathbb{N}). \tag{10}$$

Now, we immediately have by taking the ℓ_∞-norm from (10) that $\|Ax\|_{bs} = \|Bx\|_{\ell_\infty}$ which means that $Ax \in bs$ whenever $x \in \mu$ if and only if $Bx \in \ell_\infty$ whenever $x \in \mu$. This completes the proof.

3. The Characterisation of the Classes $(\ell_\infty : X_p)$, $(bs : X_p)$ and $(bv : X_p)$ of Infinite Matrices

In this section, we wish to characterise the classes $(\ell_\infty : X_p)$, $(bs : X_p)$ and $(bv : X_p)$, $(1 \leq p \leq \infty)$, of infinite matrices. We begin with two lemmas due to Zeller and Beekmann [12] and Stieglitz and Tietz [11], respectively, which are needed in the proof of our theorems.

LEMMA 3. *The matrix mappings between the BK-spaces are continuous.*

LEMMA 4. $A \in (\ell_\infty : \ell_p)$ *if and only if*

$$\sup_{F\in\mathcal{F}} \sum_n \left| \sum_{k\in F} a_{nk} \right|^p < \infty \ , \ (1 \leq p < \infty). \tag{11}$$

THEOREM 5. $A \in (\ell_\infty : X_p)$ *if and only if*

$$\sup_{F\in\mathcal{F}} \sum_n \left| \sum_{k\in F} \frac{1}{n} a(n,k) \right|^p < \infty \ , \ (1 \leq p < \infty), \tag{12}$$

$$\sup_n \sum_k \left| \frac{1}{n} a(n,k) \right| < \infty \ , \ (p = \infty). \tag{13}$$

PROOF. Let $A \in (\ell_\infty : X_p)$ and $1 \leq p < \infty$. Then, since ℓ_∞ and X_p are the BK-spaces there exists some real constant $K > 0$ by Lemma 3 such that

$$\left\|Av\right\|_p \leq K \cdot \left\|v\right\|_{\ell_\infty} \tag{14}$$

for every $v \in \ell_\infty$. Since this inequality will be fulfilled for the sequence $v_F = (v_k)$ defined by

$$v_k = \begin{cases} 1 & , (k \in F) \\ 0 & , (k \notin F) \end{cases}$$

belonging to ℓ_∞ we thus have for $1 \leq p < \infty$ that

$$\begin{aligned} \left\|Av_F\right\|_p &= \left[\sum_n \left| \frac{1}{n} \sum_{i=1}^n (Av_F)_i \right|^p \right]^{1/p} = \left[\sum_n \left| \sum_{k\in F} \frac{1}{n} a(n,k) \right|^p \right]^{1/p} \\ &\leq K \cdot \left\|v_F\right\|_{\ell_\infty} = K \end{aligned}$$

which shows the necessity of (12), where $F \in \mathcal{F}$. On the other hand, the necessity of (13) easily follows from (14) with $p = \infty$ and $v = (v_k)$ such that $|v_k| = 1$, $(k \in \mathbb{N})$.

Conversely suppose that (12) holds and take any $v \in \ell_\infty$. Define the matrix $B = (b_{nk})$ with $b_{nk} = n^{-1} \cdot a(n,k)$ for all $n,k \in \mathbb{N}$. Then the condition (11) holds for the matrix B and hence $B \in (\ell_\infty : \ell_p)$, by Lemma 4. Therefore we have at this stage that

$$\begin{aligned}\|Av\|_p &= \Big[\sum_n \Big|\frac{1}{n}\sum_{i=1}^{n}(Av)_i\Big|^p\Big]^{1/p} = \Big[\sum_n |(Bv)_n|^p\Big]^{1/p} \\ &= \|Bv\|_{\ell_p} < \infty\end{aligned}$$

and this step concludes the proof, for $1 \le p < \infty$. Since the sufficiency of the case $p = \infty$ may be proved in the similar manner, we leave it to the reader.

THEOREM 6. *$A \in (bs : X_p)$ if and only if*

$$\lim_k a_{nk} = 0 \ , \ (n \in \mathbb{N}), \tag{15}$$

$$\sup_{F \in \mathcal{F}} \sum_n \Big|\sum_{k \in F} \frac{1}{n}\Delta a(n,k)\Big|^p < \infty \ , \ (1 \le p < \infty), \tag{16}$$

$$\sup_n \sum_k \Big|\frac{1}{n}\Delta a(n,k)\Big| < \infty \ , \ (p = \infty). \tag{17}$$

PROOF. One can easily observe via (1) that the condition (15) is necessary in order for the A-transform of every element in the space bs to be defined. As an application of Theorem 1 with X_p instead of μ, we see that $Au \in X_p$ whenever $u \in bs$ if and only if $Bv \in X_p$ whenever $v \in \ell_\infty$; where A and B are the dual matrices. Thus the proof immediately follows from Theorem 5.

THEOREM 7. *$A \in (bv : X_p)$ if and only if*

$$\sup_m \sum_n \Big|\sum_{k=m}^{\infty} \frac{1}{n} a(n,k)\Big|^p < \infty \ , \ (1 \le p < \infty) \ , \tag{18}$$

$$\sup_{m,n} \Big|\sum_{k=m}^{\infty} \frac{1}{n} a(n,k)\Big| < \infty \ , \ (p = \infty). \tag{19}$$

PROOF. Let $A \in (bv : X_p), 1 \le p < \infty$. Then, since bv and X_p are the BK-spaces there is some real constant $K > 0$ by Lemma 3 such that

$$\|Ax\|_p \le K \cdot \|x\|_{bv}$$

for every $x \in bv$. Thus we have

$$\sum_n \left| \frac{1}{n} \sum_{i=1}^{n} (Ax)_i \right|^p = \sum_n \left| \frac{1}{n} \sum_{i=1}^{n} \sum_k a_{ik} x_k \right|^p \leq \left(K \cdot \|x\|_{bv} \right)^p.$$

This yields for $x = x^{(m)} = (x_k^{(m)})_{k \in \mathbb{N}}$, $(m \in \mathbb{N})$, defined by

$$x_k^{(m)} = \begin{cases} 0 & , (1 \leq k \leq m-1) \\ 1 & , (k \geq m) \end{cases}$$

for which $\|x\|_{bv} = 2$ that

$$\sum_n \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{k=m}^{\infty} a_{ik} \right|^p \leq (2K)^p < \infty.$$

Since this holds for every $m \in \mathbb{N}$, the necessity of (18) is now trivial.

Conversely, assume that the condition (18) holds and take any $x \in bv$ with $\eta = \sum_{k=0}^{\infty} \Gamma \Delta x_k \Gamma$; where $\Delta x_0 = -x_1$ and $\Delta x_k = x_k - x_{k+1}$, $(k \in \mathbb{N})$, and define the matrix $B = (b_{nk})$ as in the second paragraph of the proof of Theorem 5, above. Then, since $x_k = -\sum_{m=0}^{k-1} \Delta x_m$ we derive by Minkowski's inequality that

$$\begin{aligned}
\|Ax\|_p^p &= \sum_n \left| \sum_k b_{nk} x_k \right|^p = \sum_n \left| \sum_k \sum_{m=0}^{k-1} b_{nk} \Delta x_m \right|^p \\
&= \sum_n \left| \sum_{m=0}^{\infty} \Big(\sum_{k=m+1}^{\infty} b_{nk} \Big) \Delta x_m \right|^p \leq \eta^p \cdot \sum_n \Big(\sum_{m=0}^{\infty} \left| \sum_{k=m+1}^{\infty} b_{nk} \right| \frac{|\Delta x_m|}{\eta} \Big)^p \\
&\leq \eta^p \cdot \sum_{m=0}^{\infty} \frac{|\Delta x_m|}{\eta} \Big(\sum_n \left| \sum_{k=m+1}^{\infty} b_{nk} \right|^p \Big) \\
&\leq \eta^p \cdot \sup_m \sum_n \left| \sum_{k=m}^{\infty} b_{nk} \right|^p .
\end{aligned}$$

It is easily checked that $\eta \leq 2\|x\|_{bv}$ and we thus have by (18) that

$$\|Ax\|_p \leq 2\|x\|_{bv} \cdot \Big(\sup_m \sum_n \left| \sum_{k=m}^{\infty} b_{nk} \right|^p \Big)^{1/p} < \infty ,$$

which means that $Ax \in X_p$. This completes the proof of the case $1 \leq p < \infty$.

It is trivial in the case $p = \infty$ that the condition (19) is necessary and sufficient for $A \in (bv : X_\infty)$, which is proved in the way similar to that of (18) and so we omit it.

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