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Infinite Matrices and Cesàro Sequence Spaces of Non-absolute Type

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Abstract. In the present paper we essentially deal with to determine the neccessary and sufficient conditions in order for a matrix $A = (a_{nk})$ to belong to the classes $(X_p : bs), (X_p : fs), (X_1 : \ell_p)$, $(X_p : X_1)$ and $(\ell_p : X_1)$, respectively. Furthermore, we give the sufficient conditions on a matrix $A = (a_{nk})$ in the class $(X_p : \ell_p)$ for $1 < p < \infty$ and prove a Steinhaus type theorem concerning the disjointness of the classes $(X_p : fs)_r$ and $(bs : fs)$. Those sequence spaces are described, below.

1. Introduction, Definitions and Notation

Let us denote the space of all real or complex valued sequences by w whose any vector subspace is called as a sequence space. We shall employ the sequence spaces

$$\ell_\infty = \left\{ x = (x_k) \in w \ : \ \sup_k |x_k| < \infty \right\},$$

$$c = \left\{ x = (x_k) \in w \ : \ \lim_k \ x_k \text{ exists} \right\},$$

$$c_0 = \left\{ x = (x_k) \in w \ : \ \lim_k \ x_k = 0 \right\},$$

$$bs = \left\{ x = (x_k) \in w \ : \ \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

$$cs = \left\{ x = (x_k) \in w \ : \ \left(\sum_{k=1}^{n} x_k \right) \in c \right\}$$

and

$$\ell_p = \left\{ x = (x_k) \in w \ : \ \sum_k |x_k|^p < \infty \ , \ 1 \le p < \infty \right\}$$

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which are the Banach spaces with the following finite norms:

$$\|x\|_{\ell_\infty} = \sup_k |x_k|,$$

$$\|x\|_{bs} = \|x\|_{cs} = \sup_n \left| \sum_{k=1}^{n} x_k \right|$$

and

$$\|x\|_{\ell_p} = \Big(\sum_k |x_k|^p \Big)^{1/p}$$

as usual, respectively. For simplicity in notation, here and after we abbreviate the infinite sum notation $\sum_{k=1}^{\infty}$ as $\sum_k$. Further by f and fs, we denote the space of all almost convergent sequences, introduced by Lorentz in [5], and series, and V_σ also denotes the space of all bounded sequences all of whose invariant means are equal. A sequence space λ with a linear topology is called a K-space provided each of the maps $p_i : \lambda \to \mathbf{C}$ defined by $p_i(x) = x_i$ is continuous for $i = 1, 2, \ldots$, where $\mathbf{C}$ denotes the complex field. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space (see Choudhary and Nanda [2], pp.272-273). Some definitions and conventions are made in this section and three lemmas will be given as they become necessary. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}, (n, k \in \mathrm{N})$ and λ, μ two sequence spaces; where N denotes the set of all positive integers. Then, the matrix A defines a transformation from λ into μ, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = ((Ax)_n)$, the A-transform of x, exists and is in μ; where $(Ax)_n = \sum_k a_{nk}x_k$. By $(\lambda : \mu)$, we denote the class of all such matrices. A sequence x is said to be A-summable to b if Ax converges to b which is called as the A-limit of x. When there is some notion of limit or sum in λ and μ, we shall say that the method $A \in (\lambda : \mu)$ is multiplicative r if every $x \in \lambda$ is A-summable to r times of lim x, for some real number $r \neq 0$. We also denote the class of all r multiplicative matrices by $(\lambda : \mu)_r$. We shall assume througout for brevity that $p^{-1} + q^{-1} = 1$ for $p, q \geq 1$ and write

$$a(n,k) = \sum_{i=1}^{n} a_{ik} \text{ and } a(n,k,m) = \frac{1}{m} \sum_{i=1}^{m} a(n+i,k), \ (n,k,m \in \mathrm{N}).$$

We also denote the collection of all finite subsets of N by $\mathcal{F}$.

The class $(X_p : \mu)$ of infinite matrices has been characterised in the case $\mu = \ell_\infty, c$ by Ng [8], $\mu = cs, f$ by Savaş [11,12], $\mu = V_\sigma$ by Parashar [10] and $\mu = \Gamma$ by Mursaleen and Saifi [7]; where X_p denotes the Cesàro sequence spaces of non-absolute type, emphasised in the next section and Γ is the space of entire sequences introduced by Iyer in [3]. The object of this paper is to achieve the characterisation of the classes $(X_p : bs), (X_p : fs), (X_1 : \ell_p), (X_p : X_1)$ and $(\ell_p : X_1)$ respectively and give the sufficient conditions on a matrix belonging to the class $(X_p : \ell_p)$ for $1 < p < \infty$. Besides, we give a Steinhaus type theorem which asserts that there is no matrix belonging to the classes $(X_p : fs)_r$ and $(bs : fs)$.

2. The Cesàro Sequence Spaces of Non-absolute Type

Before explaining the notions related to the Cesàro sequence spaces of non-absolute type, we need to present some relevant terminology given by Ng and Lee [9]. Let $\lambda \subset w$ and ρ be a semi-norm on λ. The linear space λ_ρ defined by $\lambda_\rho = \{x \in \lambda \ : \ \rho(x) < \infty\}$ is known as a normed Köthe sequence space of non-absolute type with the semi-norm ρ. If λ_ρ is complete with respect to the norm ρ, then λ_ρ is called a Banach sequence space of non-absolute type provided the absolute property does not hold, i.e., $\rho(x) \neq \rho(|x|)$; where $|x| = (|x_k|)$.

Let λ_ρ be a Banach sequence space of non-absolute type and ρ be a given semi-norm. Then we define a new semi-norm ρ' as follows:

$$\rho'(x) = \sup\left\{ \left|\sum x_k y_k\right| \ : \ \rho(y) \leq 1 \right\}$$

and put $\rho'(x) = \infty$ if the series $\sum x_k y_k$ is divergent for some y satisfying $\rho(y) \leq 1$. The semi-norm ρ' is called the associate semi-norm of ρ and the space $\lambda_{\rho'} = \{x \in \lambda \ : \ \rho'(x) < \infty\}$ is called the associate space of λ_ρ. For any $x \in \lambda_\rho$ and $y \in \lambda_{\rho'}$ we always have $|\sum x_k y_k| \leq \rho(x)\rho'(y)$.

A semi-norm ρ is said to be saturated if for every non-empty subset E of N, there exists a non-empty subset F of E such that $\rho(x_F) < \infty$, where the sequence $x_F = (x_k)$ is defined as

$$x_k = \begin{cases} 1 & , \ (k \in F) \\ 0 & , \ (k \notin F). \end{cases}$$

It is not hard to observe that ρ is saturated if and only if λ_ρ contains all sequences having only finetely many non-zero terms.

We are now ready to define the Cesàro sequence spaces X_p $(1 \leq p \leq \infty)$ of non-absolute type. Ng and Lee [9] have introduced the spaces X_p which may be defined as

$$X_p = \left\{ \ x = (x_k) \in w \ : \ C_1 x \in \ell_p \ \right\}$$

and showed that these spaces are the Banach spaces with the following finite norm:

$$\|x\|_p = \|C_1 x\|_{\ell_p}$$

which are saturated except p=1; where C_1 denotes the method of arithmetic means and $1 \leq p \leq \infty$. Define the sequence $s = (s_k)$ as the C_1-transform of a sequence $x = (x_k)$, i.e.,

$$s_k = \frac{1}{k}\sum_{n=1}^{k} x_n \ , \ (k \in \mathrm{N}). \tag{1}$$

Now, we may begin with the following easy theorem given without proof by Ng and Lee in [9] which is essential in the text.

THEOREM 1. *X_p is linearly isomorphic to ℓ_p, i.e., $X_p \cong \ell_p, (1 \leq p \leq \infty)$.*

PROOF. For this, we must establish the existence of a linear bijection between the spaces X_p and ℓ_p. Consider the transformation g defined, with the notation of (1), from X_p into ℓ_p by

$$g \;:\; X_p \;\longrightarrow\; \ell_p \;;\; g(x) = s \;,\; (1 \leq p \leq \infty).$$

The linearity of g is trivial. Let $x, y \in X_p$ $(1 \leq p < \infty)$ and $g(y) = t$. In this situation, if $g(x) = g(y)$ then $s_k = t_k$ for every $k \in \mathrm{N}$ and we therefore get $x = y$. Hence, g is injective. Now, let $s \in \ell_p$ $(1 \leq p < \infty)$ and define the sequence $x = (x_n)$ by $x_n = ns_n - (n-1)s_{n-1}, (n = 1, 2, \ldots, s_0 = 0)$. Then it is immediate that

$$\|x\|_p = \Big(\sum_n \Big|\frac{1}{n}\sum_{k=1}^{n} x_k\Big|^p\Big)^{1/p} = \Big(\sum_n |s_n|^p\Big)^{1/p} = \|s\|_{\ell_p} < \infty$$

i.e., $x \in X_p$ and consequently g is surjective. This shows for the case $1 \leq p < \infty$ that g is bijective which completes the proof.

In the case $p = \infty$, the proof is similar and so we omit it.

Let Y_q be the space of all $y \in w$ such that

$$|ky_k| \leq M, \;\; (k \in \mathrm{N}), \tag{2}$$

$$\big\|(k\Delta y_k)\big\|_{\ell_q} < \infty; \text{ where } 1 \leq q \leq \infty \text{ and } \Delta y_k = y_k - y_{k+1}, \;\; (k \in \mathrm{N}). \tag{3}$$

The following lemma , which gives the associate space of X_p, is due to Ng and Lee [9]:

LEMMA 2. *The associate space of X_p is the space Y_q, where $1 \leq p \leq \infty$.*

One can observe that $(ky_k) \in c_0$ whenever (y_k) is an element of the space Y_1 which is the associate space of X_∞. It is also known by Theorem 3.2 of Ng and Lee [9] with strict inclusion that $ces_p \subset X_p$ for $1 \leq p \leq \infty$; where ces_p denotes the Cesàro sequence spaces defined by Leibowitz in [4].

3. Matrix Transformations Related to the Cesàro Sequence Spaces of Non-absolute Type

In this section, we firstly give the characterisation of the mentioned classes of infinite matrices, above and later present a Steinhaus type theorem concerning the disjointness of the classes $(X_p : fs)_r$ and $(bs : fs)$.

THEOREM 3. *$A \in (X_p : bs)$ if and only if*

$$\sup_k |ka(n,k)| < \infty \ , \ (n \in \mathrm{N}), \tag{4}$$

$$\sup_n \sum_k |k\Delta a(n,k)|^q < \infty \ , \ (1 < p \leq \infty), \tag{5}$$

$$\sup_{k,n} |k\Delta a(n,k)| < \infty \ , \ (p = 1). \tag{6}$$

PROOF. Suppose the conditions (4), (5) and (6) hold, and take any $x \in X_p$ $(1 \leq p \leq \infty)$. Then the sequence $(a_{nk})_{k\in\mathrm{N}}$ is an element of the space Y_q for each $n \in \mathrm{N}$ and this implies the existence of the A-transform of x. Let us now consider the following equality obtained by applying the Abel's partial summation to the n, m^{th} partial sums of the double series $\sum_i \sum_k a_{ik}x_k$,

$$\sum_{i=1}^{n}\sum_{k=1}^{m} a_{ik}x_k = \sum_{k=1}^{m-1} k\Delta a(n,k)s_k + ma(n,m)s_m \ , \ (n, m \in \mathrm{N}); \tag{7}$$

where $\Delta a(n,k) = a(n,k) - a(n,k+1)$ Taking into account of the facts $s \in \ell_p \subset c_0$ and $s \in \ell_\infty$ with (5), in the cases $1 \leq p < \infty$ and $p = \infty$ respectively, if we pass to limit as $m \to \infty$ in (7) then the second term on the right hand tends to zero and we therefore have

$$\sum_{i=1}^{n}\sum_{k} a_{ik}x_k = \sum_k k\Delta a(n,k)s_k \tag{8}$$

for every $n \in \mathrm{N}$. Now, by passing to supremum over n in (8) we derive by Hölder's inequality for $1 < p < \infty$ that

$$\begin{aligned} \|Ax\|_{bs} &= \sup_n \left|\sum_k k\Delta a(n,k)s_k\right| \\ &\leq \sup_n \left[\sum_k |k\Delta a(n,k)|^q\right]^{1/q} \cdot \left(\sum_k |s_k|^p\right)^{1/p} < \infty \end{aligned}$$

which proves the sufficiency of the case $1 < p < \infty$.

Besides this, we easily observe by (8) that

$$\|Ax\|_{bs} = \sup_n \left|\sum_k k\Delta a(n,k)s_k\right| \leq \|s\|_{\ell_\infty} \cdot \sup_n \sum_k |k\Delta a(n,k)| < \infty$$

which proves the sufficiency of the case $p = \infty$.

Again, we get from (8) with (6) that

$$\|Ax\|_{bs} = \sup_n \left|\sum_k k\Delta a(n,k)s_k\right| \leq \sup_{k,n} |k\Delta a(n,k)| \cdot \sum |s_k| < \infty$$

which shows the sufficiency of the case $p = 1$.

Conversely, suppose that $A \in (X_p : bs)$. Then, since Ax exists for every $x \in X_p$, we have $(a_{nk})_{k\in\mathbb{N}} \in Y_q$ $(1 \leq q \leq \infty)$ for each $n \in \mathbb{N}$. Thus, we obtain by (2) that $(ka_{nk})_{k\in\mathbb{N}} \in \ell_\infty$ for all $n \in \mathbb{N}$ which gives the necessity of (4) for $1 \leq p \leq \infty$. On the other hand, since X_p and bs are the BK-spaces and it is known that a matrix transformation between the BK-spaces is continuous (see [14], p.29), there exists some real constant M such that

$$\|Ax\|_{bs} \leq M \cdot \|x\|_p$$

for all $x \in X_p$. Therefore, we again have from (8) for $1 < p < \infty$ by Hölder's inequality that

$$\begin{aligned}
\frac{\|Ax\|_{bs}}{\|s\|_{\ell_p}} &= \sup_n \frac{|\sum_k k\Delta a(n,k)s_k|}{\|s\|_{\ell_p}} \\
&\leq \sup_n \frac{\left[\sum_k |k\Delta a(n,k)|^q\right]^{1/q} \cdot \left(\sum_k |s_k|^p\right)^{1/p}}{\|s\|_{\ell_p}} \\
&= \sup_n \left[\sum_k |k\Delta a(n,k)|^q\right]^{1/q} < \infty
\end{aligned}$$

which proves the necessity of (5). Since the necessity of the cases $p = \infty$ and $p = 1$ may be proved in the similar manner, we leave them to the reader. Thus, the proof is completed.

THEOREM 4. *$A \in (X_p : fs)$ if and only if (4), (5) and (6) hold, and*

$$f - \lim\ k\Delta a(n,k) = \alpha_k\ ,\ (k \in \mathbb{N}). \tag{9}$$

PROOF. We consider only the case $1 < p < \infty$ and omit the proof of the sufficiency of the cases $p = \infty$ and $p = 1$, since it may be proved in a similar way. Let the conditions (4), (5) and (9) hold, $x \in X_p$ and $n \in \mathbb{N}$ be fixed. Then we have from (9) that

$$|k\Delta a(n,k,m)|^q \to |\alpha_k|^q \text{ as } m \to \infty$$

uniformly in n for each $k \in \mathbb{N}$ which leads us with (5) that the inequality

$$\sum_{j=1}^{k} |\alpha_j|^q \leq \sup_{n,m} \sum_j |j\Delta a(n,j,m)|^q = M < \infty$$

holds for every $k \in \mathbb{N}$ which gives that $(\alpha_k) \in \ell_q$. Since $x \in X_p$ by the hypothesis, Theorem 1 gives that $s \in \ell_p$. Therefore, we derive by Hölder's inequality that $\sum |\alpha_k s_k| < \infty$ for each $s \in \ell_p$. For any given $\varepsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$ such that

$$\left(\sum_{k=k_0+1}^{\infty} |s_k|^p \right)^{1/p} < \frac{\varepsilon}{4M^{1/q}}.$$

Now, define a matrix $B = (b_{nk})$ with $b_{nk} = k\Delta a(n,k), (n,k \in \mathbb{N})$. Then, there is some $m_0 \in \mathbb{N}$ by (9) such that

$$\left| \sum_{k=1}^{k_0} \Big[b(n,k,m) - \alpha_k \Big] s_k \right| < \frac{\varepsilon}{2}$$

for every $m \geq m_0$, uniformly in n. Therefore, we have

$$\begin{aligned}
&\left| \frac{1}{m} \sum_{i=1}^{m} (Bs)_{n+i} - \sum_k \alpha_k s_k \right| \\
&\leq \left| \sum_{k=1}^{k_0} \Big[b(n,k,m) - \alpha_k \Big] s_k \right| + \left| \sum_{k=k_0+1}^{\infty} \Big[b(n,k,m) - \alpha_k \Big] s_k \right| \\
&< \frac{\varepsilon}{2} + \left\{ \sum_{k=k_0+1}^{\infty} \Big[|b(n,k,m)| + |\alpha_k| \Big]^q \right\}^{1/q} \cdot \left(\sum_{k=k_0+1}^{\infty} |s_k|^p \right)^{1/p} \\
&< \frac{\varepsilon}{2} + 2M^{1/q} \frac{\varepsilon}{4M^{1/q}} = \varepsilon
\end{aligned}$$

for all sufficiently large m uniformly in n, hence $Bs \in f$ and this yields by (8) that $Ax \in fs$ which is what we wished to prove.

Conversely, suppose that $A \in (X_p : fs)$. Then, since $fs \subset bs$, the necessities of (4), (5) and (6) evidently follow from Theorem 3. To prove the necessity of (9), consider the sequence $x = x^{(k)} = (x_n^{(k)})_{n \in \mathbb{N}} \in X_p$ for every fixed $k \in \mathbb{N}$ defined by

$$x_n^{(k)} = \begin{cases} (-1)^{n-k} \cdot k & , (k \leq n \leq k+1) \\ 0 & , (1 \leq n < k \text{ or } n > k+1). \end{cases}$$

Since Ax exists and is in fs for every $x \in X_p$, one can easily see that $Ax^{(k)} = (k\Delta a_{nk})_{n \in \mathbb{N}} \in fs$ for each $k \in \mathbb{N}$ which shows the necessity of (9). This concludes the proof.

We should note our gratitude to the referee who kindly pointed out to us the failure of the proof of sufficiency part of Theorem 4 and the adding the condition (6) to Theorems 3 and 4.

It is trivial that Theorem 4 reduced to the characterisation of the class $(X_p : cs)$ given by Savaş in [11], when the f-limit is replaced by the ordinary limit.

By Theorem 4, we have

COROLLARY 5. *$A \in (X_p : fs)_r$ if and only if (4), (5) and (6) hold, and (9) also holds with $\alpha_k = r$ for each $k \in \mathbb{N}, (1 \leq p < \infty)$.*

The Steinhaus type theorems were formulated by Maddox in [6] as follows: Consider the class $(\lambda : \mu)_1$ of 1-multiplicative matrices and ν be a sequence space such that $\nu \supset \lambda$. Then the result of the form $(\lambda : \mu)_1 \cap (\nu : \mu) = \emptyset$ is called a theorem of Steinhaus type, where $\emptyset$ denotes the empty set. Now, we may give a Steinhaus type theorem whose proof requires the following lemma due to Başar and Solak [1]:

LEMMA 6. *$A \in (bs : fs)$ if and only if*

$$\sup_n \sum_k |\Delta a(n,k)| < \infty, \tag{10}$$

$$\lim_k a_{nk} = 0 \ , \ (n \in \mathbb{N}), \tag{11}$$

$$f - \lim \ a(n,k) = \alpha_k \ , \ (k \in \mathbb{N}), \tag{12}$$

$$\lim_m \sum_k \frac{1}{m} \left| \sum_{i=1}^{m} \sum_{j=1}^{n+i} \Delta \left(a_{jk} - \alpha_k\right) \right| = 0 \text{ uniformly in } n. \tag{13}$$

THEOREM 7. *The classes $(X_p : fs)_r$ and $(bs : fs)$ are disjoint, $(1 \leq p < \infty)$.*

PROOF. Suppose that the classes $(X_p : fs)_r$ and $(bs : fs)$ are not disjoint. Then there is at least one matrix A satisfying the conditions of both Lemma 6 and Corollary 5. Therefore, we obtain by the fact $A \in (X_p : fs)_r$ for $x = e^k \in X_p$ that $Ae^k = (a_{nk})_{n \in \mathbb{N}} \in fs$ with

$$f - \lim \ a(n,k) = r \ , \ (k \in \mathbb{N}); \tag{14}$$

where e^k denotes the sequence defined by the Kronecker delta, i.e., $e^k = (\delta_{nk})_{n \in \mathbb{N}}, (k \in \mathbb{N})$. We also derive by combining (13) and (14) that

$$\lim_m \sum_k |\Delta a(n,k,m)| = 0 \text{ uniformly in } n. \tag{15}$$

Additionally the series $\sum_k \Delta a(n,k,m)$ is uniformly convergent in n by (15) and thus we have by (11) and (14) that

$$\lim_m \sum_k \Delta a(n,k,m) = f - \lim \ a(n,1) = r$$

which contradicts (15) and this completes the proof.

THEOREM 8. *$A \in (X_1 : \ell_p)$ if and only if*

$$\sup_k |ka_{nk}| < \infty \ , \ (n \in \mathrm{N}), \tag{16}$$

$$\sup_k \sum_n |k\Delta a_{nk}|^p < \infty \ , \ (1 \le p < \infty), \tag{17}$$

$$C = \sup_{k,n} |k\Delta a_{nk}| < \infty \ , \ (p = \infty). \tag{18}$$

PROOF. Let $A \in (X_1 : \ell_p)$. It is obvious that the condition (16) must be satisfied in order for A to be applicable to the every element of X_1. Hence, (16) is necessary in the both cases of $1 \le p < \infty$ and $p = \infty$. By the similar arguments used in the proof of the necessity of (5), since X_1 and ℓ_p are the BK-spaces and a matrix transformation between the BK-spaces is continuous, there exists some real constant K such that

$$\|Ax\|_{\ell_p} \le K \cdot \|x\|_1 \tag{19}$$

for all $x \in X_1$ and $1 \le p < \infty$. Therefore we get from (19) with $x = x^{(k)} \in X_1$ that

$$\|Ax^{(k)}\|_{\ell_p} = \|(k\Delta a_{nk})_{n\in\mathrm{N}}\|_{\ell_p} = \Big(\sum_n |k\Delta a_{nk}|^p\Big)^{1/p} \le K$$

for all $k \in \mathrm{N}$ which proves the necessity of (17), where $x^{(k)}$ is defined as in the proof of Theorem 4. Now, since the proof of the necessity of (18) is trivial by putting ℓ_∞ instead of ℓ_p in (19), we omit it.

Now consider the following equality derived in a similar way of (7):

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m-1} k\Delta a_{nk}s_k + ma_{nm}s_m \ ; \ (n, m \in \mathrm{N}) \tag{20}$$

which gives us by regarding (16) and letting $m \to \infty$ that

$$\sum_k a_{nk}x_k = \sum_k k\Delta a_{nk}s_k \ ; \ (n \in \mathrm{N}). \tag{21}$$

Let us argue the case $1 \le p < \infty$, firstly. Let $A = (a_{nk})$ satisfy (16), (17) and $x \in X_1$. Then, $s \in \ell_1$ and $|k\Delta a_{nk}| \le M^{1/p}$ for all $n, k \in \mathrm{N}$ and so the series $\sum_k |k\Delta a_{nk}s_k| < \infty$ for each $n \in \mathrm{N}$ and each $s \in \ell_1$. Therefore considering (21), we get by Minkowski's inequality

$$\Big[\sum_n |(Ax)_n|^p\Big]^{1/p} = \Big(\sum_n \Big|\sum_k k\Delta a_{nk}s_k\Big|^p\Big)^{1/p}$$
$$\le \sum_k |s_k| \cdot \Big(\sum_n |k\Delta a_{nk}|^p\Big)^{1/p} < \infty$$

which means for $1 \leq p < \infty$ that $A \in (X_1 : \ell_p)$.

Let us turn to the case $p = \infty$, and $A = (a_{nk})$ satisfy (16), (18) and $x \in X_1$. Then, Ax exists and we again have by (21) that

$$\sup_n \left|(Ax)_n\right| = \sup_n \left|\sum_k k\Delta a_{nk} s_k\right| \leq \sup_n \sum_k |k\Delta a_{nk}| \; |s_k| \leq C \cdot \sum_k |s_k| < \infty$$

and hence $Ax \in \ell_\infty$. This step completes the proof.

We wish to present a theorem which gives the sufficient conditions for a matrix $A = (a_{nk})$ to be an element of the class $(X_p : \ell_p)$ for $1 < p < \infty$.

THEOREM 9. *Suppose that a matrix $A = (a_{nk})$ satisfies*

$$K = \sup_n \sum_k |k\Delta a_{nk}| < \infty \; , \tag{22}$$

$$L = \sup_k \sum_n |k\Delta a_{nk}| < \infty \; . \tag{23}$$

Then, $A \in (X_p : \ell_p)$ for $1 < p < \infty$.

PROOF. Let the conditions (22) and (23) hold and take any $x \in X_p$ for $1 < p < \infty$. Then the series $\sum_k |k\Delta a_{nk} s_k| < \infty$ for each $n \in \mathrm{N}$ and each $s \in \ell_p$. This allows us to use (21) and therefore we obtain by Hölder's inequality

$$\begin{aligned}
\sum_n \left|(Ax)_n\right|^p &= \sum_n \left|\sum_k k\Delta a_{nk} s_k\right|^p \\
&\leq \sum_n \left[\left(\sum_k |k\Delta a_{nk}|\right)^{p/q} \cdot \left(\sum_k |k\Delta a_{nk}| \, |s_k|^p\right)\right] \\
&\leq K^{p/q} \cdot \sum_k \left(\sum_n |k\Delta a_{nk}| \, |s_k|^p\right) \leq K^{p/q} \cdot L \cdot \sum_k |s_k|^p < \infty
\end{aligned}$$

which means that $Ax \in \ell_p$ for $1 < p < \infty$, as was desired.

We should note here that the case $p = 1$ of Theorem 9 is included in the first part of Theorem 8 and additionally (16) and (22) are necessary and sufficient in order for $A \in (X_\infty : \ell_\infty)$ but we omit the proof, since it is not hard to prove this fact in a similar fashion. Finally, we state that the necessity part of Theorem 9 remains open since we are not able to prove that the conditions (22) and (23), or which conditions, are necessary in order for $A \in (X_p : \ell_p)$ for $1 < p < \infty$.

To our knowledge, although the matrix transformations from X_p into certain spaces have been characterised neither the transformations between the X_p spaces nor the transformations from the ℓ_p spaces or any sequence space into the X_p spaces have been characterised, until quite recently. So we wish to characterise the classes $(X_p : X_1)$ and $(\ell_p : X_1)$ of infinite matrices, as a beginning. We additionally state that the characterisation of the classes $(X_1 : X_p), (\ell_1 : X_p)$, $(X_p : X_\infty)$ and $(\ell_p : X_\infty)$ is being worked out and will be subjected with the relation, called the duality of the new type, between a pair of infinite matrices one of them applied to the sequences belonging to the space X_p and the other one to the sequences belonging to the space ℓ_p, in the seperate paper. To prove Theorem 11 below, we shall need the following lemma due to Stieglitz and Tietz [13]:

LEMMA 10. *$A \in (\ell_p : \ell_1)$ if and only if*

$$\sup_{S\in\mathcal{F}} \sum_k \left| \sum_{n\in S} a_{nk} \right|^q < \infty \ , \ (1 < p \leq \infty). \tag{24}$$

THEOREM 11. *$A \in (X_p : X_1)$ if and only if (4) holds, and*

$$\sup_{S\in\mathcal{F}} \sum_k \left| \sum_{n\in S} \frac{k}{n} \Delta a(n,k) \right|^q < \infty \ , \ (1 < p \leq \infty). \tag{25}$$

PROOF. The necessity of (4) follows by an argument similar to that used in proving Theorem 3. Now, since (8) holds for every $x \in X_p$, we see that $Ax \in X_1$ whenever $x \in X_p$ if and only if $Bs \in \ell_1$ whenever $s \in \ell_p$, and thus the proof is immediately obtained from Lemma 10 with B instead of A; where $B = (b_{nk})$ such that $b_{nk} = n^{-1}k \cdot \Delta a(n,k)$ for all $n, k \in \mathbb{N}$.

THEOREM 12. *$A \in (\ell_p : X_1)$ if and only if*

$$\sup_{S\in\mathcal{F}} \sum_k \left| \sum_{n\in S} \frac{1}{n} a(n,k) \right|^q < \infty \ , \ (1 < p \leq \infty). \tag{26}$$

PROOF. This is immediate from the fact that $A \in (\ell_p : X_1)$ if and only if $B \in (\ell_p : \ell_1)$ which is a consequence of the isomorphism $X_1 \cong \ell_1$; where $B = (b_{nk})$ with $b_{nk} = n^{-1} \cdot a(n,k), (n,k \in \mathbb{N})$. Now, the proof follows by Lemma 10. We should remark the reader from now on that Theorem 12 does not need the corresponding condition to the condition (4) of Theorems 3, 4 and 11, since the nature of ℓ_q, the β-dual of ℓ_p, is different than that of Y_q.

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