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ROSEリポジトリいばらき（茨城大学学術情報リポジトリ）
A Certain Subclass of Meromorphically Convex Functions with Negative Coefficients

H.M. Srivastava,* H.M. Hosse and M.K. Aouf**

Abstract. In this paper we obtain coefficient inequalities, and distortion and closure theorems, for the class $A_k(\alpha,\beta,A,B)$ of meromorphically convex functions with negative coefficients, which we introduce here. We also obtain the class-preserving integral operator of the form:

$$F(z) = c \int_0^1 u^c f(uz) \, du \quad (c > 0)$$

for the class $A_k(\alpha,\beta,A,B)$. Conversely, when the image function $F(z) \in A_k(\alpha,\beta,A,B)$, we find the radius of convexity of the original function $f(z)$. Several interesting results involving the modified Hadamard product of functions belonging to the class $A_k(\alpha,\beta,A,B)$ are also derived.

§1. Introduction

Let $\Sigma$ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

which are analytic in the punctured unit disk

$$U^* = \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} = U \setminus \{0\}$$

with a simple pole at the origin with residue 1 there. Let $\Sigma_k$ denote the subclass of $\Sigma$ consisting of functions $f(z)$ which are convex with respect to the origin, that is, satisfying the condition:

$$\Re \left\{ - \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U^*).$$  \hspace{1cm} (1.2)
Let $\Sigma_k(\alpha)$ denote the subclass of $\Sigma$ consisting of functions $f(z)$ which are convex of order $\alpha$, that is, satisfying the condition:

$$\Re \left\{ -\left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} > \alpha \quad (z \in \mathcal{U}^*; \ 0 \leq \alpha < 1). \quad (1.3)$$

Let $\Sigma_k(\alpha, A, B)$ denote the class of functions $f(z)$ in $\Sigma$ which satisfy the condition that

$$\frac{(zf'(z))'}{f'(z)} \quad (z \in \mathcal{U}^*; \ 0 \leq \alpha < 1; -1 \leq A < B \leq 1; 0 < B \leq 1).$$

has a representation of the form given by

$$1 + \frac{zf''(z)}{f'(z)} = -\frac{1 + \{B + (A - B)(1 - \alpha)\} w(z)}{1 + Bw(z)} \quad (1.4)$$

Here $w(z)$ is analytic in $\mathcal{U}$ and satisfies the conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

The condition (1.4) is equivalent to

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \quad (z \in \mathcal{U}^*). \quad (1.5)$$

We note also that

$$\Sigma_k(\alpha, -1, 1) = \Sigma_k(\alpha).$$

Let $\Lambda$ denote the subclass of $\Sigma$ consisting of functions of the form:

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n. \quad (1.6)$$

**DEFINITION.** A function $f(z)$ in $\Sigma$ is in the class $\Sigma_k(\alpha, \beta, A, B)$ if it satisfies the condition

$$\frac{zf''(z)}{f'(z)} + 2 \left| \frac{zf''(z)}{f'(z)} + 2 \right| \quad (z \in \mathcal{U}^*; \ 0 \leq \alpha < 1; \ 0 < \beta \leq 1; -1 \leq A < B \leq 1; \ 0 < B \leq 1). \quad (1.7)$$

Let us write

$$\Lambda_k(\alpha, \beta, A, B) = \Sigma_k(\alpha, \beta, A, B) \cap \Lambda.$$
We note that
\[ \Lambda_k(\alpha, 1, -1, 1) = \Lambda_k(\alpha) \]
is the class of meromorphically convex functions of order \( \alpha \) with negative coefficients, which was studied by Uralegaddi and Ganigi [6]. Furthermore,
\[ \Lambda_k(0, 1, A, B) = \Lambda_k(A, B) \]
is the class of functions \( f(z) \in \Lambda \), satisfying the condition:
\[
\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1 \quad (1.8)
\]
\( (z \in \mathcal{U}^*; \quad 1 \leq A < B \leq 1; \quad 0 < B \leq 1) \).

In this paper we obtain coefficient inequalities and distortion theorems for the class \( \Lambda_k(\alpha, \beta, A, B) \). Further it is shown that \( \Lambda_k(\alpha, \beta, A, B) \) is closed under convex linear combinations. We also obtain the class-preserving integral operators of the form
\[
F(z) = c \int_0^1 u^c f(uz) \, du \quad (c > 0) \quad (1.9)
\]
for the class \( \Lambda_k(\alpha, \beta, A, B) \). Conversely, when \( F(z) \in \Lambda_k(\alpha, \beta, A, B) \), we obtain the radius of convexity of \( f(z) \) defined by (1.9). Thus we extend several known results of Goel and Sohi [2] and S.K. Bajpai [1]. We also derive many interesting results for the modified Hadamard products of functions belonging to the class \( \Lambda_k(\alpha, \beta, A, B) \). We employ techniques similar to those used earlier by Silverman [4] (see also Srivastava and Owa [5]).

§2. A Set of Coefficient Inequalities

**Theorem 1.** Let the function \( f(z) \) defined by (1.1) be analytic in \( \mathcal{U}^* \). If
\[
\sum_{n=1}^{\infty} \left\{ (n + 1) + \beta [Bn + (B - A) \alpha + A] \right\} n|a_n| \leq (B - A) \beta (1 - \alpha) \quad (2.1)
\]
\[ (0 \leq \alpha < 1; \quad 0 < \beta \leq 1; \quad -1 \leq A < B \leq 1; \quad 0 < B \leq 1), \]
then
\[ f(z) \in \Sigma_k(\alpha, \beta, A, B). \]

**Proof.** Suppose that (2.1) holds true for all admissible values of \( \alpha, \beta, A, \) and \( B \). Consider the expression
\[
H(f, f') = |zf''(z) + 2f'(z)| - \beta |B \{ f'(z) + zf''(z) \} + [B + (A - B)(1 - \alpha)] f'(z)|. \quad (2.2)
\]
Replacing $f$ and $f'$ by their series expansions, we have, for $0 < |z| = r < 1$,

$$H(f, f') = \left| \sum_{n=1}^{\infty} (n+1) n a_n z^{n-1} \right|$$

$$- \beta \left| \frac{(B - A)(1 - \alpha)}{z^2} + \sum_{n=1}^{\infty} [Bn + (B - A)\alpha + A] n a_n z^{n-1} \right|$$

or

$$r^2 H(f, f')$$

$$\leq \sum_{n=1}^{\infty} (n+1) n|a_n| r^{n+1}$$

$$- \beta \left\{ (B - a)(1 - \alpha) - \sum_{n=1}^{\infty} [Bn + (B - A)\alpha + A] n|a_n|r^{n+1} \right\}$$

$$= \sum_{n=1}^{\infty} \{ (n+1) + \beta [Bn + (B - A)\alpha + A] n|a_n|r^{n+1} \} - (B - A)\beta(1 - \alpha).$$

Since the above inequality holds true for all $r (0 < r < 1)$, letting $r \to 1-$, we have

$$H(f, f') \leq \sum_{n=1}^{\infty} \{ (n+1) + \beta [Bn + (B - A)\alpha + A] n|a_n|$$

$$- (B - A)\beta(1 - \alpha) \leq 0,$$

by (2.1). Hence it follows that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| < \beta \left| B \left( 1 + \frac{zf''(z)}{f'(z)} \right) + [B + (A - B)(1 - \alpha)] \right|,$$

so that $f(z) \in \Sigma_k(\alpha, \beta, A, B)$. This evidently completes the proof of Theorem 1.

**Theorem 2.** Let the function $f(z)$ defined by (1.6) be analytic in $U^*$. Then $f(z) \in \Lambda_k(\alpha, \beta, A, B)$ if and only if (2.1) is satisfied.

**Proof.** In view of Theorem 1, let us assume that the function $f(z)$ defined by (1.6) is in the class $\Lambda_k(\alpha, \beta, A, B)$. Then

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right|$$

$$\leq \sum_{n=1}^{\infty} (n+1) n|a_n| z^{n-1}$$

$$\leq \frac{(B - A)(1 - \alpha)}{z^2} - \sum_{n=1}^{\infty} [Bn + (B - A)\alpha + A] n|a_n| z^{n-1}$$

$$< \beta \quad (z \in U^*)$$
Using the fact that $\Re(z) \leq |z|$ for all $z$, we thus have

$$\Re \left\{ \sum_{n=1}^{\infty} (n+1) n|a_n| z^{n-1} \frac{(B-A)(1-\alpha)}{z^2} - \sum_{n=1}^{\infty} [Bn + (B-A)\alpha + A] n|a_n| z^{n-1} \right\} < \beta \quad (z \in U^*).$$

(2.3)

Now choose the values of $z$ on the real axis so that

$$1 + \frac{zf''(z)}{f'(z)}$$

is real. Upon clearing the denominator in (2.3) and letting $z \to 1-$ through real values, we obtain

$$\sum_{n=1}^{\infty} (n+1)n|a_n| \leq \beta \left\{ (B-A)(1-\alpha) - \sum_{n=1}^{\infty} [Bn + (B-A)\alpha + A] n|a_n| \right\}$$

or

$$\sum_{n=1}^{\infty} \{(n+1) + \beta [Bn + (B-A)\alpha + A]\} n|a_n| \leq (B-A)\beta(1-\alpha),$$

(2.4)

which proves Theorem 2.

**Corollary 1.** Let the function $f(z)$ defined by (1.6) be in the class $\Lambda_k(\alpha, \beta, A, B)$. Then

$$|a_n| \leq \frac{(B-A)\beta(1-\alpha)}{n \{(n+1) + \beta [Bn + (B-A)\alpha + A]\}} \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots \}),$$

(2.5)

where equality holds true for functions of the form:

$$f_n(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)}{n \{(n+1) + \beta [Bn + (B-A)\alpha + A]\}} z^n \quad (n \in \mathbb{N}).$$

(2.6)

§3. A Distortion Theorem

**Theorem 3.** Let the function $f(z)$ defined by (1.6) be in the class $\Lambda_k(\alpha, \beta, A, B)$. Then, for $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{(B-A)\beta(1-\alpha)}{2 + \beta [(B+A) + (B-A)\alpha]} r \leq |f(z)| \leq \frac{1}{r} + \frac{(B-A)\beta(1-\alpha)}{2 + \beta [(B+A) + (B-A)\alpha]} r$$

(3.1)
where equality holds true for the function

\[ f_1(z) = \frac{1}{z} - \frac{(B - A) \beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} z \quad (z = ir, r), \]  

(3.2)

and

\[ \frac{1}{r^2} - \frac{(B - A) \beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} \leq |f'(z)| \]

\[ \leq \frac{1}{r^2} + \frac{(B - A) \beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]}, \]  

(3.3)

where equality holds true for the function \( f_1(z) \) given by (3.2) at \( z = \pm r, \pm ir \).

**PROOF.** In view of Theorem 2, we have

\[ \sum_{n=1}^{\infty} |a_n| \leq \frac{(B - A) \beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} \].  

(3.4)

Thus, for \( 0 < |z| = r < 1 \),

\[ |f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| r^n \]

\[ \leq \frac{1}{r} + r \sum_{n=1}^{\infty} |a_n| \]

\[ \leq \frac{1}{r} + \frac{(B - A) \beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} r, \]  

(3.5)

and

\[ |f(z)| \geq \frac{1}{r} - \sum_{n=1}^{\infty} |a_n| r^n \]

\[ \geq \frac{1}{r} - r \sum_{n=1}^{\infty} |a_n| \]

\[ \geq \frac{1}{r} - \frac{(B - A) \beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} r, \]  

(3.6)

which, together, yield (3.1). Furthermore, it follows from Theorem 2 that

\[ \sum_{n=1}^{\infty} n|a_n| \leq \frac{(B - A) \beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]}. \]  

(3.7)
Hence

\[ |f'(z)| \leq \frac{1}{r^2} + \sum_{n=1}^{\infty} n|a_n|r^{n-1} \]
\[ \leq \frac{1}{r^2} + \sum_{n=1}^{\infty} n|a_n| \]
\[ \leq \frac{1}{r^2} + \frac{(B - A)\beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} \] (3.8)

and

\[ |f'(z)| \geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n|r^{n-1} \]
\[ \geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n| \]
\[ \geq \frac{1}{r^2} - \frac{(B - A)\beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]}, \] (3.9)

which, together, yield (3.3). It can easily be seen that the function \( f_1(z) \) defined by (3.2) is extremal for Theorem 3.

Putting \( (\alpha, \beta) = (0, 1) \) in Theorem 3, we have

**Corollary 2.** Let the function \( f(z) \) defined by (1.6) be in the class \( \Lambda_k(A,B) \). Then, for \( 0 < |z| = r < 1, \)

\[ \frac{1}{r} - \frac{B - A}{2 + B + A} r \leq |f(z)| \leq \frac{1}{r} + \frac{B - A}{2 + B + A} r \]

and

\[ \frac{1}{r^2} - \frac{B - A}{2 + B + A} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{B - A}{2 + B + A}. \]

The result is sharp.

\section*{§4. A Pair of Closure Theorems}

In this section we first prove

**Theorem 4.** The class \( \Lambda_k(\alpha,\beta,A,B) \) is closed under convex linear combinations.

**Proof.** Let each of the functions

\[ f_j(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_{n,j}|z^n \quad (j = 1, 2) \] (4.1)
be in the class $\Lambda_k(\alpha, \beta, A, B)$. It is sufficient to show that the function $h(z)$ defined by
\[
h(z) = (1 - \lambda) f_1(z) + \lambda f_2(z) \quad (0 \leq \lambda \leq 1) \tag{4.2}
\]
is also in the class $\Lambda_k(\alpha, \beta, A, B)$. Since
\[
h(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \left[ (1 - \lambda)|a_{n,1}| + \lambda|a_{n,2}| \right] z^n \quad (0 \leq \lambda \leq 1), \tag{4.3}
\]
with the aid of Theorem 2, we have
\[
\sum_{n=1}^{\infty} n \left\{ (n+1) + \beta \left[ Bn + (B - A) \alpha + A \right] \right\} \left[ (1 - \lambda)|a_{n,1}| + \lambda|a_{n,2}| \right] \\
= (1 - \lambda) \sum_{n=1}^{\infty} n \left\{ (n+1) + \beta \left[ Bn + (B - A) \alpha + A \right] \right\} |a_{n,1}| \\
+ \lambda \sum_{n=1}^{\infty} n \left\{ (n+1) + \beta \left[ Bn + (B - A) \alpha + A \right] \right\} |a_{n,2}| \\
\leq (1 - \lambda)(B - A)\beta(1 - \alpha) + \lambda(B - A)\beta(1 - \alpha) \\
= (B - A)\beta(1 - \alpha), \tag{4.4}
\]
which shows that $f \in \Lambda_k(\alpha, \beta, A, B)$. Hence we have Theorem 4.

**THEOREM 5.** Let $f_0(z) = \frac{1}{z}$ and
\[
f_n(z) = \frac{1}{z} - \frac{(B - A)\beta(1 - \alpha)}{n \left\{ (n+1) + \beta \left[ Bn + (B - A)\alpha + A \right] \right\}} z^n \quad (n \in \mathbb{N}). \tag{4.5}
\]
Then $f(z) \in \Lambda_k(\alpha, \beta, A, B)$ if and only if it can be expressed in the form:
\[
f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \tag{4.6}
\]
where
\[
\lambda_n \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \lambda_n = 1.
\]
**PROOF.** Suppose that
\[
f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \\
= \frac{1}{z} - \sum_{n=1}^{\infty} \lambda_n \frac{(B - A)\beta(1 - \alpha)}{n \left\{ (n+1) + \beta \left[ Bn + (B - A)\alpha + A \right] \right\}} z^n. \tag{4.7}
\]
Since
\[ \sum_{n=1}^{\infty} \frac{n \{ (n+1) + \beta [ Bn + (B-A)\alpha + A] \} \lambda_n}{(B-A)\beta(1-\alpha)} = \sum_{n=1}^{\infty} \frac{(B-A)\beta(1-\alpha)}{n \{ (n+1) + \beta [ Bn + (B-A)\alpha + A] \}} \lambda_n = 1 - \lambda_0 \leq 1, \] (4.8)
we have \( f(z) \in \Lambda_k(\alpha, \beta, A, B) \), by Theorem 2.
Conversely, suppose that the function \( f(z) \) defined by (1.6) belongs to the class \( \Lambda_k(\alpha, \beta, A, B) \).
Since
\[ |a_n| \leq \frac{(B-A)\beta(1-\alpha)}{n \{ (n+1) + \beta [ Bn + (B-A)\alpha + A] \}} \] (n \( \in \mathbb{N} \)), (4.9)
by Corollary 1, setting \( \lambda_n = \frac{n \{ (n+1) + \beta [ Bn + (B-A)\alpha + A] \}}{(B-A)\beta(1-\alpha)} |a_n| \) \( (n \in \mathbb{N}) \) (4.10)
and
\[ \lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n, \] (4.11)
it follows that
\[ f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z). \]
This completes the proof of Theorem 5.

§5. A Family of Integral Operators

THEOREM 6. Let the function \( f(z) \) defined by (1.6) be in the class \( \Lambda_k(\alpha, \beta, A, B) \). Then
\[ F(z) = c \int_0^1 u^c f(uz) \, du = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{c}{c+n+1} |a_n| z^n \quad (c > 0) \]
belongs to the class \( \Lambda_k(\gamma, \beta, A, B) \), where
\[ \gamma = \gamma(\alpha, \beta, A, B, c) \]
\[ = \frac{(1 + \beta B) + \{ 3 + \beta [2(B-A)\alpha + B + 2A] + 2\alpha c(1 + \beta B) \}}{(1 + \beta B) + \{ 3 + \beta [2(B-A)\alpha + B + 2A] + 2c(1 + \beta B) \}} \] (5.1)
for $F(z) \neq 0$ ($z \in \mathcal{U}^*$). The result is sharp for the function

$$f(z) = \frac{1}{z} - \frac{(B - A)\beta(1 - \alpha)}{2 + \beta[Bn + (B - A)\alpha + A]} z.$$  \hspace{1cm} (5.2)

**Proof.** Suppose that $f(z) \in \Lambda_k(\alpha, \beta, A, B)$. Then

$$\sum_{n=2}^{\infty} n \left\{ (n+1) + \beta[Bn + (B - A)\alpha + A] \right\} \frac{|a_n|}{(B - A)\beta(1 - \alpha)} \leq 1. \hspace{1cm} (5.3)$$

In view of Theorem 2, we shall find the largest $\gamma$ for which

$$\sum_{n=2}^{\infty} n \left\{ (n+1) + \beta[Bn + (B - A)\gamma + A] \right\} \left( \frac{c}{c + n + 1} \right) \frac{|a_n|}{(B - A)\beta(1 - \gamma)} \leq 1. \hspace{1cm} (5.4)$$

It suffices to find the range of values of $\gamma$ for which

$$\frac{c \left\{ (n+1) + \beta[Bn + (B - A)\gamma + A] \right\}}{(1 - \gamma)(c + n + 1)} \leq \frac{(n+1) + \beta[Bn + (B - A)\alpha + A]}{1 - \alpha}$$

for each $n \in \mathbb{N}$. From the above inequality, we obtain

$$\gamma \leq \frac{n^2(1 + \beta B) + n(k_1 + \alpha k_2) + \{\alpha(k_2 + k_3) + k_4\}}{n^2(1 + \beta B) + n(k_1 + k_2) + (k_2 + \alpha k_3 + k_4)} \hspace{1cm} (5.5)$$

where

$$\begin{align*}
k_1 &= 2 + \beta [(B - A)\alpha + (B + A)]; \\
k_2 &= c(1 + \beta B); \\
k_3 &= (B - A)\beta; \\
k_4 &= 1 + \beta A.
\end{align*} \hspace{1cm} (5.6)$$

For each $\alpha, \beta, A,$ and $B$, and for $c$ fixed, let

$$F(n) = \frac{n^2(1 + \beta B) + n(k_1 + \alpha k_2) + \{\alpha(k_2 + k_3) + k_4\}}{n^2(1 + \beta B) + n(k_1 + k_2) + (k_2 + \alpha k_3 + k_4)}.$$

Then

$$F(n + 1) - F(n) = \frac{c(1 - \alpha)(1 + \beta B)^2(n + 1)(n + 2)}{\{n^2(1 + \beta B) + n(k_1 + k_2) + (k_2 + \alpha k_3 + k_4)\}} \cdot \frac{1}{(n + 1)^2(1 + \beta B) + (n + 1)(k_1 + k_2) + (k_2 + \alpha k_3 + k_4)} > 0$$

for each $n \in \mathbb{N}$. Hence $F(n)$ is an increasing function of $n$. Since

$$F(1) = \frac{(1 + \beta B) + \{3 + \beta [2(B - A)\alpha + B + 2A] + 2\alpha c(1 + \beta B)\}}{(1 + \beta B) + \{3 + \beta [2(B - A)\alpha + B + 2A] + 2c(1 + \beta B)\}},$$

the assertion of Theorem 6 follows.

From the proof of Theorem 6 with $(\alpha, \beta) = (0, 1)$, the following result follows easily:
COROLLARY 3. If the function \( f(z) \) defined by (1.6) satisfies the inequality:

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{2 + (B + A)}{2 + (B + A) + c(1 + B)} \quad (z \in U^*),
\]

then \( F(z) \) defined by (1.9) belongs to the class \( \Lambda_k(A, B) \) for \( F(z) \neq 0 \) \((z \in U^*)\).

REMARK 1. Corollary 3 extends a result of Goel and Sohi [2] for functions with negative coefficients. In fact, Goel and Sohi [2] proved that, if the function \( f(z) \) defined by (1.1) satisfies the inequality:

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{1}{2(c + 1)} \quad (z \in U^*),
\]

then \( F(z) \) defined by (1.9) is convex in \( U^* \) for \( F(z) \neq 0 \) \((z \in U^*)\).

Putting \( c = 1 \) in Corollary 3, we obtain

COROLLARY 4. If the function \( f(z) \) defined by (1.6) satisfies the inequality:

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{2 + (B + A)}{3 + (2B + A)} \quad (z \in U^*),
\]

then

\[
F(z) = \int_0^1 u f(uz) \, du
\]

belongs to the class \( \Lambda_k(A, B) \) for \( F(z) \neq 0 \) \((z \in U^*)\).

REMARK 2. Corollary 4 extends a result of Bajpai [1] for functions with negative coefficients. In fact, Bajpai [1] has proved that, if the function \( f(z) \) defined by (1.1) is convex in \( U^* \), then so is

\[
F(z) = z^{-2} \int_0^z t f(t) \, dt.
\]

THEOREM 7. Let

\[
F(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n
\]

be in the class \( \Lambda_k(\alpha, \beta, A, B) \) and put

\[
f(z) = \frac{1}{c} \left[ (c + 1) F(z) + z F'(z) \right]
\]

\[
= \frac{1}{z} - \sum_{n=1}^{\infty} \left( \frac{c + n + 1}{c} \right) |a_n| z^n \quad (c > 0).
\]

Then

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < -\phi \quad (0 < |z| < r; \ 0 \leq \phi < 1)
\]
where
\[ r = r(\alpha, \beta, A, B, \phi) \]
\[ = \inf_{n \in \mathbb{N}} \left\{ \frac{c(1 - \phi) \left\{ (n + 1) + \beta \left[ Bn + (B - A)\alpha + A \right] \right\}}{(B - A)\beta(1 - \alpha)(n + \phi)(c + n + 1)} \right\}^{1/(n+1)} \]
(5.9)

The result is sharp for
\[ F(z) = \frac{1}{z} - \frac{(B - A)\beta(1 - \alpha)}{n \left\{ (n + 1) + \beta \left[ Bn + (B - A)\alpha + A \right] \right\}} z^n \]
for some \( n \in \mathbb{N} \).

PROOF. It suffices to show that
\[ \left| \frac{zf''(z)}{f'(z)} + 2 \right| < 1 \quad (0 < |z| < r), \]
where \( r \) is given by (5.9). Indeed we have
\[ \left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq \sum_{n=1}^{\infty} \frac{n(n+1) \left( \frac{c+n+1}{c} \right) |a_n| |z|^{n+1}}{2(1 - \phi) - \sum_{n=1}^{\infty} n(n-1+2\phi) \left( \frac{c+n+1}{c} \right) |a_n| |z|^{n+1}}. \]
The last expression is bounded above by 1 if
\[ \sum_{n=1}^{\infty} \frac{n(n+\phi)(c+n+1)}{(1 - \phi) c} |a_n| |z|^{n+1} \leq 1. \]
(5.11)

From Theorem 2, we have
\[ \sum_{n=1}^{\infty} \frac{n \left\{ (n + 1) + \beta \left[ Bn + (B - A)\alpha + A \right] \right\} |a_n|}{(B - A)\beta(1 - \alpha)} \leq 1. \]
Hence (5.11) is satisfied if
\[ \frac{(n + \phi)(c+n+1)}{(1 - \phi) c} |z|^{n+1} \leq \frac{(n + 1) + \beta \left[ Bn + (B - A)\alpha + A \right]}{(B - A)\beta(1 - \alpha)} \quad (n \in \mathbb{N}). \]
(5.12)

Solving (5.12) for \( |z| \) we obtain
\[ |z| \leq \left\{ \frac{c(1 - \phi) \left\{ (n + 1) + \beta \left[ Bn + (B - A)\alpha + A \right] \right\}}{(B - A)\beta(1 - \alpha)(n + \phi)(c + n + 1)} \right\}^{1/n+1} \quad (n \in \mathbb{N}). \]
(5.13)

Writing \( |z| = r(\alpha, \beta, A, B, \phi) \) in (5.13), we complete the proof of Theorem 7.

Putting \( (\alpha, \beta) = (0, 1) \) and \( \phi = 0 \) in Theorem 7, we get
COROLLARY 5. Let the function $F(z)$ defined by (5.7) be in the class $\Lambda_k(A, B)$ and let $f(z)$ be given by (5.8). Then

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 0 \quad (0 < |z| < r_1),$$

where

$$r_1 = r_1(A, B) = \inf_{n \in \mathbb{N}} \left\{ \frac{c[n(1 + B) + (1 + A)]}{(B - A)n(c + n + 1)} \right\}^{1/(n+1)}.$$

The result is sharp for some $n \in \mathbb{N}$.

REMARK 3. Corollary 5 extends a result of Goel and Sohi [2] for functions with negative coefficients. In fact, Goel and Sohi [2] proved the following result:

If

$$F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

is convex in $U^*$ and if

$$f(z) = \frac{1}{c} \left( (c + 1) F(z) + z F'(z) \right) \quad (c > 0),$$

then $f(z)$ is convex in

$$0 < |z| < \sqrt{\frac{c}{c + 2}}.$$

The result is sharp.

REMARK 4. Putting $A = -1$ and $B = \beta = 1$ in the above results, we get the corresponding results of Uralegaddi and Ganigi [6].

§6. Properties Involving the Modified Hadamard Products

Let the functions $f_j(z)$ ($j = 1, 2$) be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 \ast f_2)(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_{n,1}| |a_{n,2}| z^n. \quad (6.1)$$

THEOREM 8. Let the functions $f_j(z)$ ($j = 1, 2$) defined by (4.1) be in the class $\Lambda_k(\alpha, \beta, A, B)$. 
Then
\[(f_1 * f_2)(z) \in \Lambda_k (\zeta(\alpha, \beta, A, B), \beta, A, B),\]
where
\[
\zeta(\alpha, \beta, A, B) = 1 - \frac{2(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{[2 + \beta [(B + A) + (B - A)\alpha]]^2 + (B - A)^2\beta^2(1 - \alpha)^2}.
\]
\[(6.2)\]

The result is sharp.

**Proof.** Employing the technique used earlier by Schild and Silverman [3], we need to find the largest \( \zeta = \zeta(\alpha, \beta, A, B) \) such that
\[
\sum_{n=1}^{\infty} \frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\xi] \}}{(B - A)\beta(1 - \zeta)} |a_{n,1}| |a_{n,2}| \leq 1.
\]
\[(6.3)\]

Since
\[
\sum_{n=1}^{\infty} \frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}{(B - A)\beta(1 - \alpha)} |a_{n,1}| \leq 1
\]
and
\[
\sum_{n=1}^{\infty} \frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}{(B - A)\beta(1 - \alpha)} |a_{n,2}| \leq 1,
\]
by the Cauchy-Schwarz inequality, we have
\[
\sum_{n=1}^{\infty} \frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}{(B - A)\beta(1 - \alpha)} \sqrt{|a_{n,1}| |a_{n,2}|} \leq 1.
\]
\[(6.6)\]

Thus it is sufficient to show that
\[
\frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\xi] \}}{(B - A)\beta(1 - \zeta)} |a_{n,1}| |a_{n,2}|
\leq \frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}{(B - A)\beta(1 - \alpha)} \sqrt{|a_{n,1}| |a_{n,2}|}, \quad (n \in \mathbb{N}),
\]
\[(6.7)\]
that is, that
\[
\sqrt{|a_{n,1}| |a_{n,2}|} \leq \frac{(1 - \zeta) \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}{(1 - \alpha) \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}.
\]
\[(6.8)\]

Note that
\[
\sqrt{|a_{n,1}| |a_{n,2}|} \leq \frac{(B - A)\beta(1 - \alpha)}{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}} \quad (n \in \mathbb{N}).
\]
\[(6.9)\]

Consequently, we need only to prove that
\[
\frac{(B - A)\beta(1 - \alpha)}{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}} \leq \frac{(1 - \zeta) \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}{(1 - \alpha) \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}}, \quad (n \in \mathbb{N}),
\]
\[(6.10)\]
or, equivalently, that
\[ \zeta \leq 1 - \frac{(n + 1)(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{n \left\{ (n + 1) + \beta \left[ (Bn + A) + (B - A)\alpha \right] \right\}^2 (B - A)^2 \beta^2 (1 - \alpha)^2} \quad (n \in \mathbb{N}). \quad (6.11) \]

Since
\[ \Psi(n) = 1 - \frac{(n + 1)(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{n \left\{ (n + 1) + \beta \left[ (Bn + A) + (B - A)\alpha \right] \right\}^2 + (B - A)^2 \beta^2 (1 - \alpha)^2} \]

is an increasing function of \( n \ (n \in \mathbb{N}) \), letting \( n = 1 \) in (6.12), we obtain
\[ \zeta \leq \Psi(1) = 1 - \frac{2(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{2 + \beta \left[ (B + A) + (B - A)\alpha \right]^2 + (B - A)^2 \beta^2 (1 - \alpha)^2}, \quad (6.13) \]

which completes the proof of Theorem 8.

Finally, by taking the functions
\[ f_j(z) = \frac{1}{z} - \frac{(B - A)\beta(1 - \alpha)}{2 + \beta \left[ (B + A) + (B - A)\alpha \right]} z \quad (j = 1, 2), \quad (6.14) \]

we can see that the result is sharp.

**Corollary 6.** For \( f_1(z) \) and \( f_2(z) \) as in Theorem 8, the function
\[ h(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \sqrt{|a_{n,1}| |a_{n,2}|} z^n \quad (6.15) \]

belongs to the class \( \Lambda_k(\alpha, \beta, A, B) \).

The result follows from the inequality (6.6). It is sharp for the same functions \( f_j(z) \) \((j = 1, 2)\) as in Theorem 8.

**Theorem 9.** Let the function \( f_1(z) \) defined by (4.1) be in the class \( \Lambda_k(\alpha, \beta, A, B) \) and the function \( f_2(z) \) defined by (4.1) be in the class \( \Lambda_k(\gamma, \beta, A, B) \). Then
\[ (f_1 * f_2)(z) \in \Lambda_k(\delta, \beta, A, B), \]

where
\[ \delta = \delta(\alpha, \gamma, \beta, A, B) = 1 - \frac{2(B - A)\beta(1 + \beta B)(1 - \alpha)(1 - \gamma)}{\Theta(\alpha)\Theta(\gamma) + (B - A)^2 \beta^2 (1 - \alpha)(1 - \gamma)}. \quad (6.16) \]

\((\Theta(\lambda) := 2 + \beta([B + A] + (B - A)\lambda])). \)
The result is best possible for the functions

\[ f_1(z) = \frac{1}{z} - \frac{(B - A)\beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} z \]  

(6.17)

and

\[ f_2(z) = \frac{1}{z} - \frac{(B - A)\beta(1 - \gamma)}{2 + \beta [(B + A) + (B - A)\gamma]} z. \]  

(6.18)

**PROOF.** Proceeding as in the proof of Theorem 8, we get

\[ \delta = \delta(\alpha, \gamma, \beta, A, B) \]

\[ \leq 1 - \frac{(n + 1)(B - A)\beta(1 + \beta B)(1 - \alpha)(1 - \gamma)}{n \{(n + 1) + \beta \Phi(\alpha)\} \{(n + 1) + \beta \Phi(\gamma)\} + N_{\gamma}} \]  

(6.19)

where

\[ \Phi(\lambda) := (Bn + A) + (B - A)\lambda, \]

(6.20)

As in Theorem 8, \( \delta \) is easily seen to be an increasing function of \( n \in \mathbb{N} \). Therefore, setting \( n = 1 \) in (6.19), we get

\[ \delta \leq 1 - \frac{2(B - A)\beta(1 + \beta B)(1 - \alpha)(1 - \gamma)}{\{2 + \beta [(B + A) + (B - A)\alpha]\} \{2 + \beta [(B + A) + (B - A)\gamma]\} + N_{\gamma}}, \]

where \( N_{\gamma} \) is given by (6.20). This completes the proof of Theorem 9.

**COROLLARY 7.** Let the functions \( f_j(z) \) (\( j = 1, 2, 3 \)) defined by (4.1) be in the class \( \Lambda_k(\alpha, \beta, A, B) \). Then

\[ (f_1 * f_2 * f_3)(z) \in \Lambda_k(\xi, \beta, A, B), \]

where

\[ \xi = \xi(\alpha, \beta, A, B) = 1 - \frac{2(B - A)^2\beta^2(1 + \beta B)(1 - \alpha)^3}{\{2 + \beta [(B + A)\alpha]\}^3 + (B - A)^3\beta^3(1 - \alpha)^3}. \]  

(6.21)

The result is best possible for the functions

\[ f_j(z) = \frac{1}{z} - \frac{(B - A)\beta(1 - \alpha)}{2 + \beta [(B + A) + (B - A)\alpha]} z \]  

(6.22)

**PROOF.** From Theorem 8 we have

\[ (f_1 * f_2)(z) \in \Lambda_k(\xi, \beta, A, B), \]

where \( \zeta = \zeta(\alpha, \beta, A, B) \) is given by (6.2). Using Theorem 9, we get

\[ (f_1 * f_2 * f_3)(z) \in \Lambda_k(\xi, \beta, A, B), \]
where
\[ \xi = \xi(\alpha, \beta, A, B) \]
\[ = 1 - \frac{2(B - A)\beta(1 + \beta B)(1 - \alpha)(1 - \xi)}{\{2 + \beta \[(B + A) + (B - A)\alpha]\}^2 + (B - A)^3} \]
where \( \xi \) is given by (6.20). This completes the proof of Corollary 7.

**THEOREM 10.** Let the functions \( f_j(z) \) (\( j = 1,2 \)) defined by (4.1) be in the class \( \Lambda_k(\alpha, \beta, A, B) \). Then the function
\[ h(z) = \frac{1}{z} - \sum_{n=1}^{\infty} (|a_{n,1}|^2 + |a_{n,2}|^2) z^n \] (6.23)
belongs to the class
\[ \Lambda_k(\eta, \beta, A, B), \]
where
\[ \eta = \eta(\alpha, \beta, A, B) \]
\[ = 1 - \frac{4(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{\{2 + \beta \[(B + A) + (B - A)\alpha]\}^2 + 2(B - A)^2\beta^2(1 - \alpha)^2}. \] (6.24)
The result is sharp for the functions \( f_j(z) \) (\( j = 1,2 \)) defined by (6.14).

**PROOF.** By virtue of Theorem 2, we obtain
\[ \sum_{n=1}^{\infty} \left[ n \left\{ (n + 1) + \beta \[(Bn + A) + (B - A)\alpha]\right\} \frac{1}{(B - A)\beta(1 - \alpha)} \right]^2 |a_{n,1}|^2 \leq 1 \] (6.25)
and
\[ \sum_{n=1}^{\infty} \left[ n \left\{ (n + 1) + \beta \[(Bn + A) + (B - A)\alpha]\right\} \frac{1}{(B - A)\beta(1 - \alpha)} \right]^2 |a_{n,2}|^2 \leq 1. \] (6.26)
It follows from (6.25) and (6.26) that
\[ \sum_{n=1}^{\infty} \frac{1}{2} \left[ n \left\{ (n + 1) + \beta \[(Bn + A) + (B - A)\alpha]\right\} \frac{1}{(B - A)\beta(1 - \alpha)} \right]^2 (|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1. \] (6.27)
Therefore, we need to find the largest $\eta = \eta(\alpha, \beta, A, B)$ such that

\[
\frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \} (B - A)\beta(1 - \alpha)}{(B - A)\beta(1 - \eta)} \leq \frac{1}{2} \left[ \frac{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}^2}{(B - A)\beta(1 - \alpha)} \right]^2 \quad (n \in \mathbb{N}), \tag{6.28}
\]

that is, that

\[
\eta \leq 1 - \frac{2(n + 1)(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}^2 + 2(B - A)^2\beta^2(1 - \alpha)^2} \quad (n \in \mathbb{N}). \tag{6.29}
\]

Since

\[
\Psi(n) = 1 - \frac{2(n + 1)(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \}^2 + 2(B - A)^2\beta^2(1 - \alpha)^2}
\]

is an increasing function of $n$, we readily have

\[
\eta \leq \Psi(1) = 1 - \frac{4(B - A)\beta(1 + \beta B)(1 - \alpha)^2}{2 + \beta [(B + A) + (B - A)\alpha] \}^2 + 2(B - A)^2\beta^2(1 - \alpha)^2}, \tag{6.30}
\]

and Theorem 10 follows at once.

**Theorem 11.** Let the function

\[
f_1(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_{n,1}| z^n
\]

be in the class $\Lambda_k(\alpha, \beta, A, B)$ and let

\[
f_2(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_{n,2}| z^n \quad (|a_{n,2}| \leq 1; \ n \in \mathbb{N}).
\]

Then

\[
(f_1 \ast f_2)(z) \in \Lambda_k(\alpha, \beta, A, B).
\]

**Proof.** Since

\[
\sum_{n=1}^{\infty} n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \} |a_{n,1}a_{n,2}|
\]

\[
= \sum_{n=1}^{\infty} n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \} |a_{n,1}| |a_{n,2}|
\]

\[
\leq \sum_{n=1}^{\infty} n \{ (n + 1) + \beta [(Bn + A) + (B - A)\alpha] \} |a_{n,1}|
\]

\[
\leq (B - A)\beta(1 - \alpha),
\]
it follows from Theorem 2 that

$$(f_1 * f_2)(z) \in \Lambda_k(\alpha, \beta, A, B).$$

**COROLLARY 8.** If $f_1(z) \in \Lambda_k(\alpha, \beta, A, B)$ and

$$f_2(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_{n,2} z^n (0 \leq a_{n,2} \leq 1; \ n \in \mathbb{N})$$

then

$$(f_1 * f_2)(z) \in \Lambda_k(\alpha, \beta, A, B).$$

**REMARK 5.** Putting $A = -1$ and $B = \beta = 1$ in Theorems 8, 9, 10, and 11, and also in Corollaries 6, 7, and 8, we get the corresponding results for the class $\Lambda_k(\alpha)$ introduced by Uralegaddi and Ganigi [6].

**References**