<table>
<thead>
<tr>
<th>タイトル</th>
<th>Some remarks on Krull's conjecture regarding almost integral elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>著者(s)</td>
<td>HABTE GEBRU</td>
</tr>
<tr>
<td>引用</td>
<td>Mathematical Journal of Ibaraki University, 30: 15-20</td>
</tr>
<tr>
<td>発行年月</td>
<td>1998</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10109/3065">http://hdl.handle.net/10109/3065</a></td>
</tr>
</tbody>
</table>

このリポジトリに収録されているコンテンツの著作権は、それぞれの著作権者に帰属します。引用、転載、複製等される場合は、著作権法を遵守してください。

お問合せ先

茨城大学学術企画部学術情報課（図書館） 情報支援係

http://www.lib.ibaraki.ac.jp/toiawase/toiawase.html
Some remarks on Krull’s conjecture regarding almost integral elements

HABTE GEBRU*

Introduction

A basic notion in algebraic number theory and algebraic geometry is the notion of integral dependence and integral closure. An element $x$ of a field $F$ is integral over a subdomain $D$ in $F$ if the domain $D[x]$ is a finitely generated $D$-module. If all $x \in F$ which are integral over $D$ are contained in $D$, then $D$ is called integrally closed in $F$. The elements of $F$ which are integral over $D$ form an integrally closed domain which is called the integral closure of $D$ in $F$. Examples of integrally closed domains in $F$ are the valuation rings $V$ of $F$. A weaker notion is that of almost integral dependence. An element $x$ of a field $F$ is almost integral over a subdomain $D$ in $F$ if $D[x]$ is contained in a finitely generated $D$-module. Obviously for a Noetherian domain $D$ both notions “integral” and “almost integral” coincide. A domain $D$ is called completely integrally closed in $F$, if every element of $F$ which is almost integral over $D$ belongs to $D$. The set of elements in $F$ almost integral over $D$ form a subring of $F$, but this subring is not necessarily completely integrally closed (cf.[4, page 99]). In contrast to the notion of integral dependence, the relation ”almost integral” is not transitive.

Unlike the integral closure, the construction of almost integral closure can be very complicated. The construction needs, in general, an uncountable iteration of almost integral extensions (cf.[6]), as soon as the rings under consideration are of dimension greater than one. But in the case of quasi-semi-local rings of dimension one, we show in this paper that only two steps are needed (Corollary 2.3).

It is a well known fact ([5]) that the integral closure of a domain $D$ in a field $F$ is the intersection of the family of all valuation rings on $F$ which contain $D$. On the other hand, if $V$ is a non trivial valuation ring of a field $F$, then $V$ is completely integrally closed if and only if $V$ is one dimensional (cf.[1]).

From the above facts the following conjecture was raised (cf. [1, page 232]):

KRULL’S CONJECTURE. If $F$ is a field and if $D$ is a subring of $F$, the complete integral closure of $D$ in $F$ is equal to the intersection of the family of valuation rings on $F$ which contain $D$ and have rank $\leq 1$.

Nakayama [7, 8] gave a strong counterexample to Krull’s conjecture by giving a completely integrally closed domain $D$ with quotient field $K \neq D$ such

Received October 9, 1997.

1991 Mathematics Subject Classification. Primary 20M14 Secondary 13A15.

* Department of Mathematics, University of Asmara, P.O.Box 1220, Eritrea.
that $D$ admits no rank one valuation. A simpler counter example was given by J. Ohm.

The conjecture does not hold in general even for Bezout domains of (Krull) dimension at least two. Positive answers seem to be possible in the case of one dimensional domains.

In this paper, we show that if $\bar{R} \neq R$, then $C(\bar{R}) \neq C(R)$, where $C(R)$ and $\bar{R}$ are the complete integral closure and the integral closure of $R$ respectively.

The general case of dimension one is still open.

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the set of natural, rational and real numbers, respectively.

1. Almost integral elements

**DEFINITION.** If $R$ is a ring with the total quotient ring $S$, and if $s \in S$, we say that $s$ is almost integral over $R$ if all powers of $s$ belong to a finite $R$-submodule of $S$. The set $C(R)$ of all elements of $S$ which are almost integral over $R$ is a subring called the complete integral closure of $R$. If $R = C(R)$, we say that $R$ is completely integrally closed.

**DEFINITION.** For an ordinal number $\sigma$, we define the $\sigma$-th complete integral closure $C^\sigma(D)$ of $D$ as follows: $C^0(D) = D$, $C^{\sigma+1}(D) = C(C^\sigma(D))$, if $\sigma = \lim \tau$, then $C^\sigma(D) = \cup C^\tau(D)$. In particular, if $n$ is a positive integer, the $n$-th complete integral closure $C^n(D)$ is given by induction as follows: $C^n(D) = C(C^{n-1}(D))$.

**THEOREM 1.1.** For any domain $D$, the $\omega_1$-th complete integral closure $C^{\omega_1}(D)$ is completely integrally closed, where $\omega_1$ is the first uncountable ordinal.

**PROOF.** Let $K$ be the quotient field of $D$ and assume that $x \in K$ is almost integral over $C^{\omega_1}(D)$. Then there is $0 \neq a \in C^{\omega_1}(D)$ for which $ax^n \in C^{\omega_1}(D)$ for all $n$. Then $a \in C^{\sigma(0)}(D)$ and $ax^n \in C^{\sigma(n)}(D)$, where each $\sigma(n)$ for $n = 0, 1, \ldots$ is a countable ordinal. Then $\sup_n \sigma(n) = \sigma$ is also countable and $x \in C^{\sigma+1}(D) \subseteq C^{\omega_1}(D)$. This shows that $C^{\omega_1}(D)$ is completely integrally closed.

2. Some positive results on Krull's conjecture

**DEFINITION.** The pseudoradical of a domain $D$ is defined as the intersection of all nonzero prime ideals of $D$.

**REMARK.** If $V$ is a valuation ring with a quotient field $K$ and a prime ideal $P$, then

(1) If for all $x, y \in K, xy \in P$, then $x \in P$ or $y \in P$.

(2) $P = PV_P$.

**PROOF.** (1) If $x, y \in V$, then we are done, since $P$ is a prime ideal. If $x \notin V$, then $x^{-1} \in V$. Consequently, $y = x^{-1}xy \in P$. Similarly, if $y \notin V$, then $x \in P$.

(2) Clearly, $P = PV \subseteq PV_P$. On the other hand, $x \in PV_P$ implies $x = ps^{-1}, p \in P, s \notin P$. But then $p = s(s^{-1}p) \in P$. Using (1), since $s \notin P$, we have $x = s^{-1}p \in P$. Hence $P = PV_P$. 

PROPOSITION 2.1 (cf. [3]). Let $D$ be an integrally closed domain with quotient field $K$ and a nonzero pseudoradical $Q$. Then $C(D)$ is the intersection of rank one valuation rings which contain $D$, and hence is completely integrally closed.

PROOF. Let $\{V_\alpha\}$ be the collection of all non-trivial valuation rings between $D$ and $K$. For any $\alpha$, $V_\alpha$ has nonzero pseudoradical since $D$ does. Hence the valuation ring $V_\alpha$ contains a minimal prime ideal $P_\alpha \neq \{0\}$ and so $V_\alpha$ is contained in the rank one valuation ring $W_\alpha = (V_\alpha)_{P_\alpha}$ with maximal ideal $P_\alpha$. Clearly, $P_\alpha \cap D$ is a proper prime ideal of $D$ so that $Q \subseteq P_\alpha \cap D$. Since $W_\alpha$ is a real valuation ring it follows that $W_\alpha$ and $\bigcap W_\alpha$ are completely integrally closed. Consequently, $C(D) \subseteq \bigcap W_\alpha$.

To see the converse, let $x \in \bigcap W_\alpha$. Let $0 \neq q \in Q$ and $n$ a positive integer. Then $q^nx \in QW_\alpha \subseteq P_\alpha W_\alpha = P_\alpha$, by the above remark. Therefore $q^nx \in \bigcap V_\alpha = D$, i.e., $x \in C(D)$. Hence $C(D) = \bigcap W_\alpha$.

DEFINITION. A domain $R$ is said to be quasi-local (resp., semi-quasi-local) if it has exactly one maximal ideal (resp., finitely many maximal ideals).

COROLLARY 2.2. Let $R$ be a quasi-semi-local one dimensional domain, then $C(R) = \bigcap V_i$, where $V_i$ are the one dimensional valuation overrings of $R$.

PROOF. Notice that the pseudoradical of $R$ is nonzero. Using the "lying above theorem", each nonzero prime ideal of $\hat{R}$ meets $R$ in a nonzero prime ideal of $\hat{R}$. Hence the pseudoradical of $\hat{R}$ is nonzero. By Proposition 2.1, we have the required result.

COROLLARY 2.3. If $R$ is a one dimensional quasi-semi-local domain, then $C^3(R) = C^2(R) = C^2(\hat{R}) = C(\hat{R})$.

PROOF. $R \subseteq \hat{R} \subseteq C(R)$ implies $C(R) \subseteq C(\hat{R}) \subseteq C^2(R)$. Consequently, since $C(\hat{R})$ is completely integrally closed, we have $C^2(R) \subseteq C^2(\hat{R}) = C(\hat{R}) \subseteq C^2(R)$.

PROPOSITION 2.4. Let $R$ be a one dimensional quasi-local domain with maximal ideal $M$. Let $K$ be the quotient field of $R$. Then the following hold true:

(a) Let $0 \neq x \in M$. Then $K = R[x^{-1}]$ and every maximal overring $S$ of $R$ which does not contain $x^{-1}$ is a real valuation ring.

(b) If $S$ is a ring such that $R \subseteq S \subseteq K$, then $S$ is contained in a real valuation ring of $K$.

(c) If $y \in K$ and $yM \subseteq M$, then $y$ is almost integral over $R$.

(d) Let $R$ be integrally closed. If $y \in K$ and $y$ is not almost integral over $R$, then there exists a real valuation ring $S \supseteq R$ such that $y \notin S$.

(e) Let $R$ be completely integrally closed, then $R$ can be written as an intersection of real valuation rings, i.e., Krull's conjecture is true for $R$.

PROOF. (a) We show first $K = R[x^{-1}]$, in other words, $0 \neq r \in R$ implies $r^{-1} \in R[x^{-1}]$. Hence it is sufficient to show $x^n \in (r)$ for some $n$. 

...
Notice that \( R'' := R/(r) \) is a zero dimensional quasi-local ring. Therefore every non-unit \( x'' \in R'' \), where \( x'' = x + (r) \) for some \( x \in R \) is nilpotent. But then \( x''^n = 0 \) implies \( (x + (r))^n = (r) \), i.e., \( x^n \in (r) \).

On the other hand, let \( S \) be a maximal overring of \( R \) in \( K \) which does not contain \( x^{-1} \). Then \( S \) is a valuation ring ([5]). If \( S \) is not a real valuation ring, then there exists a non-trivial prime \( P \) in \( S \) which is not a maximal ideal. Then \( P \cap R = M \). Notice that \( x \in M \) implies \( x \in P \). But then \( S_P \) does not contain \( x^{-1} \). This is a contradiction since \( S \) is maximal without \( x^{-1} \). So \( S \) is a real valuation ring.

(b) If \( R \subseteq S \subseteq K \), then there exists \( 0 \neq x \in M \) such that \( x \) is not invertible in \( S \). But then, using (a), a maximal overring of \( S \) which does not contain \( x^{-1} \) is a real valuation ring.

(c) Let \( 0 \neq x \in M \). \( y^n M \subseteq M \) for all positive integers \( n \). Hence \( xy^n \in R \).

(d) We consider two cases:

Case 1: \( yM \subseteq R \). Notice that \( yM \not\subseteq M \) by (c). Hence \( yM = R \), i.e., \( y^{-1} \in M \), then according to (a), there exists a real valuation ring \( S \) which does not contain \( y \).

Case 2: \( yM \not\subseteq R \). In this case, there exists \( x \in M \) such that \( yx = z \not\in R \). Observe that \( R \) can be written as an intersection of valuation rings containing \( R \) ([5]). Hence there exists a valuation ring \( S_0 \supseteq R \) in \( K \) such that \( z \not\in S_0 \). But then \( z^{-1} \in S_0 \). Let \( S \) be a maximal overring of \( S_0 \). Let \( v \) be the corresponding valuation of \( S \). Clearly, \( v(z) \leq 0 \) and \( v(x) > 0 \). But then \( xy = z \) implies \( v(y) < 0 \). Hence \( y \not\in S \).

(e) follows from (d).

REMARK. One can also use Proposition 2.1 to prove (e) as follows: \( R = C(R) \) implies \( R = \bar{R} \). The pseudoradical of \( R \) is nonzero, and hence the result. Proposition 2.4 (d) gives an alternate proof for Corollary 2.2.

**Lemma 2.5** (cf. [1, page 280]). Let \( V_1, V_2, \ldots, V_n \) be a finite collection of valuation rings on a field \( K \) such that \( V_i \not\subseteq V_j \) for \( i \neq j \) and let \( R = \bigcap_{i=1}^n V_i \). Then \( R \) is a Prüfer domain with quotient field \( K \); \( R \) has exactly \( n \) maximal ideals, which are the centers of the valuation rings.

**Proposition 2.6.** Let \( R \) be a quasi-semi-local one dimensional domain with the maximal ideals \( M_1, M_2, \ldots, M_n \). Then \( R = \bigcap R_{M_i} \) if and only if \( R \) is a Prüfer ring.

**Proof.** The necessity is given by Lemma 2.5. On the other hand, if \( R \) is a Prüfer ring, then \( R_{M_i} \) is a real valuation ring and \( R = \bigcap_{i=1}^n R_{M_i} \).

**Definition.** Let \( T \) be a commutative monoid and let \( S \) be a submonoid of \( T \). An element \( t \in T \) is said to be integral over \( S \) if \( nt \in S \) for some positive integer \( n \). The set \( \bar{S} \) of elements \( t \in T \) that are integral over \( S \) is a submonoid of \( T \) containing \( S \); \( \bar{S} \) is called the integral closure of \( S \) in \( T \). If \( S = \bar{S} \), we say \( S \) is integrally closed in \( T \). If \( S \) is cancellative and \( T \) is a submonoid of the quotient group of \( S \), then an element \( t \in T \) is almost integral over \( S \) if there exist \( s \in S \).
such that \( s + nt \in S \) for each positive integer \( n \). The set \( S^* \) of all \( t \in T \) that are almost integral over \( S \) is called the complete integral closure of \( S \) in \( T \). It is a submonoid of \( T \) containing \( S \), and \( S \) is completely integrally closed in \( T \), if \( S = S^* \).

**Lemma 2.7 (cf. [2, page 152]).** Let \( T \) be a cancellative monoid. Let \( S \) be a submonoid of \( T \), and let \( R \) be a commutative ring.

1. \( X^t \in R[X; T] \) is integral over \( R[X; S] \) if and only if \( t \) is integral over \( S \).
2. If \( R \) is an integral domain and \( T \) is a submonoid of the quotient group of \( S \), then considered as an element of the quotient field of \( R[X; S] \), an element \( X^t \) is almost integral over \( R[X; S] \) if and only if \( t \) is almost integral over \( S \).

The following example shows that if \( R \) is not integrally closed, \( C(R) = C(\bar{R}) \) need not hold. Therefore, \( C(R) \) is not necessarily the intersection of all rank one valuation overrings of \( R \).

**Example.** Let \( \Gamma_0 \) be a monoid in the set of nonnegative real numbers, and \( \Gamma = \Gamma_0 - \Gamma_0 \) be the generated additive subgroup of \( R \). Then by definition,

\[
\begin{align*}
\overline{\Gamma}_0 &= \{ \gamma \in \Gamma : 2N_0 \in N, N_0 \gamma \in \Gamma_0 \} \\
\Gamma^*_0 &= \{ \gamma \in \Gamma : 2N_0 \in N, N \gamma + \gamma_0 \in \Gamma_0 \} \\
(\overline{\Gamma}_0)^* &= \{ \gamma \in \Gamma : 2N_0 \in N, 2N_0(2N \gamma + \gamma_0) \in \Gamma_0 \}
\end{align*}
\]

Let \( \Gamma_0 = \langle \gamma_1, \gamma + \gamma_1, 2(2 \gamma + \gamma_1), \ldots \rangle \) be a monoid generated by \( \gamma_1 \) and \( i(i \gamma + \gamma_1) \) for all positive integers \( i \).

Choosing \( \gamma_1 = 1 \) and \( \gamma = \sqrt{2} \), we have

\[ \Gamma_0 = \langle 1, \gamma + 1, 2(2 \gamma + 1), \ldots, N(N \gamma + 1), \ldots \rangle. \]

It is easy to show that each generator is not in the monoid generated by the earlier generators.

Claim: \( \Gamma_0^* \neq (\overline{\Gamma}_0)^* \).

In the notation of (2) and (3), we prove the assertion in two steps:

Step 1: \( \gamma \notin \Gamma_0^* \). If \( \gamma \in \Gamma_0^* \), then using (2), there must be a \( \gamma_0 \) such that for all \( N \in N, N \gamma + \gamma_0 \in \Gamma_0 \). Putting \( \gamma_0 = a + b \gamma \) with \( a, b \in N_0 \) this implies

\[ N \gamma + \gamma_0 = (N + b) \gamma + a = \sum_{i=1}^{r} N_i (N \gamma + 1) \text{ with } N_i, r \in N. \]

Since \( \gamma \) and 1 are \( \mathbb{Q} \)-linearly independent, we get

\[ a = \sum_{i=1}^{r} N_i \text{ and } N + b = \sum_{i=1}^{r} N_i^2. \]

From the first equation, we see \( N_i \leq a \). From the second it follows \( N + b = \sum N_i^2 \leq \sum N_i a \leq a^2 \). Consequently, \( N \leq a^2 - b \) for all \( N \in N \). This is impossible.
Step 2: $\gamma \in (\Gamma_0)^*$. By definition, we have

$$N(N\gamma + 1) \in \Gamma_0$$

for all $N \in \mathbb{N}$.

If $R = \mathbb{Q}[X;\Gamma_0]$ is a semigroup ring of $\Gamma_0$ over $\mathbb{Q}$, then by (3),

$$x^\sqrt{2} \in \mathbb{Q}[X;(\Gamma_0)^*] \setminus \mathbb{Q}[X;\Gamma_0^*].$$

This implies by Lemma 2.7, $C(R) \neq C(\bar{R})$.

ACKNOWLEDGEMENT. I would like to acknowledge Prof. Dr. W.-D. Geyer, my advisor, for the devoted and selfless way in which he has laboured to make the best out of me and who has kindled my initial interest in Krull's conjectures.

I would like also to thank Prof. Matsuda, the manager and the referee of the journal for their patience and valuable suggestions in writing this paper.

References