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引用
Mathematical Journal of Ibaraki University, 30: 1-7

発行日
1998

URL
http://hdl.handle.net/10109/3063

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Some Theorems for Semigroups

RYŪKI MATSUDA*

Dedicated to Professor Hasumi
on his retirement from Ibaraki University

In Zariski-Samuel[10], Theorems 30 and 31 of Ch.IV, and Theorems 7 and 14 of Ch.V are, without saying, important theorems in commutative ring theory. And so is Theorem A of Appendix 2 of Gilmer[2]. The aim of this paper is to prove these theorems for semigroups. We will prove all these theorems for semigroups except [10,Ch.IV,Theorem 31]. We will prove [10,Ch.IV,Theorem 31] for semigroups of dimensions 1 and 2.

A subsemigroup $S_0$ of a torsion-free abelian (additive) group is called a g-monoid. Explicitly, semigroups we mentioned above mean g-monoids. Many propositions for commutative rings are known to hold for g-monoids. The author conjectures that almost all propositions in multiplicative ideal theory hold for g-monoids (cf.[5]).

Let $Z_0$ be the non-negative integers. A g-monoid $S$ of the form $\sum_1^n Z_0 s_i$ for a finite number of elements $s_1, \ldots, s_n$ of $S$ is called a finitely generated semigroup. Let $P$ be a prime ideal of a g-monoid $S$. Then the height $ht(P)$ of $P$ is naturally defined. If there exist no prime ideals $\not\subseteq P$, then $ht(P) = 1$.

The semigroup ring of a g-monoid $S$ over a commutative ring $R$ is denoted by $R[X;S]$, where $X$ is a symbol.

**Lemma 1.** Let $I$ be a proper ideal in a finitely generated semigroup $S$. If $I$ is $r$-generated, that is, generated by $r$ elements of $S$, then every prime ideal $P$ minimal among containing $I$ has height at most $r$.

**Proof.** Let $k$ be a field. Then the semigroup ring $k[X;S]$ of $S$ over $k$ is a Noetherian ring, and $Pk[X;S]$ is a prime ideal minimal among containing $Ik[X;S]$. Therefore $ht(Pk[X;S]) \leq r$. Hence $ht(P) \leq r$.

If every ideal of a g-monoid $S$ is finitely generated, $S$ is called a Noetherian semigroup.

**Lemma 2.** Let $S$ be a Noetherian semigroup such that the unit group $H$ of $S$ is a free abelian group. Let $I$ be a proper ideal in $S$ generated by $r$ elements $a_1, \ldots, a_r$. If the maximal ideal $M$ of $S$ is a prime ideal minimal among containing $I$, then $ht(M) \leq r$.

**Proof.** Let $\{u_\lambda \mid \lambda \}$ be a free generators of $H$. By Lemma 1, we may assume that $|\Lambda| = \infty$. Let $p_1, \ldots, p_l$ be the set of non-associated irreducible
elements of $S$. The quotient group $G$ of $S$ equals to $\sum_{\Lambda} u_\lambda + \sum_i \mathbb{Z}p_i$. There exist a subset $\Lambda_1$ and a finite subset \{\lambda_1, \ldots, \lambda_n\} of $\Lambda$ such that $G = \sum_{\Lambda_1} u_\lambda + (\sum_i \mathbb{Z}u_\lambda_i + \sum_i \mathbb{Z}p_i)$. Let $k$ be a field. $M_k[X;S]$ is a prime ideal minimal among containing $Ik[X;S]$. Set $S_1 = \sum_i \mathbb{Z}u_\lambda + \sum_i \mathbb{Z}p_i$, $D = k[X;\sum_{\Lambda_1} u_\lambda]$, and let $K$ be the quotient field $q(D)$ of $D$. We see that $k[X;S] = D[X;S_1]$, and $MK[X;S_1]$ is a prime ideal minimal among containing $(a_1, \ldots, a_r)$ $K[X;S_1]$. It follows that $ht(MK[X;S_1]) < r$. Hence $ht(M) < r$. Therefore $ht(M) < r$.

**Theorem 1.** Let $I$ be a proper ideal of a Noetherian semigroup $S$. If $I$ is generated by $r$ elements, then every prime ideal minimal among containing $I$ has height at most $r$.

**Proof.** We may assume that $P$ equals to the maximal ideal $M$ of $S$. Let \{u_\lambda \mid \Lambda\} be a maximal independent subset of the unit group $H$ of $S$. By Lemma 1, we may assume that $|\Lambda| = \infty$. Let $p_1, \ldots, p_l$ be the set of all non-associated irreducible elements of $S$. We may assume that $M$ is a prime ideal minimal among containing $(p_1, \ldots, p_r)$. Set $S_1 = \sum_{\Lambda} \mathbb{Z}u_\lambda + \sum_i \mathbb{Z}p_i$. Then $S$ is integral over $S_1$, and the maximal ideal $M_1$ of $S_1$ is a prime ideal minimal among containing $(p_1, \ldots, p_r)S_1$. By Lemma 2, $ht(M_1) < r$. Hence $ht(M) < r$.

Let $S$ be a g-monoid, and $Y$ a non-empty set. Assume that, for each $s \in S$ and $x \in Y$, an element $s + x \in Y$ is determined, and, for each $s_1$ and $s_2$ of $S$, $(s_1 + s_2) + x = s_1 + (s_2 + x)$, and $0 + x = x$. Then $Y$ is called an $S$-module.

**Theorem 2.** Let $S$ be an integrally closed semigroup, $G$ its quotient group, $L$ an extension torsion-free abelian group of $G$ with $(L:G) = n < \infty$, and $T$ the integral closure of $S$ in $L$. Then there exists a complete representative system $x_1, \ldots, x_n$ of $L$ modulo $G$ such that $T$ is contained in the $S$-module $\bigcup_i (S + x_i)$.

**Proof.** Let $k$ be a field of characteristic 0. Then $k[X;S]$ is an integrally closed domain, the quotient field $F$ of $k[X;T]$ is a finite separable algebraic extension of the quotient field $K$ of $k[X;S]$, and $k[X;T]$ is the integral closure of $k[X;S]$ in $F$. Hence there exists a basis \{\phi_1, \ldots, \phi_m\} of $F$ over $K$ such that $k[X;T]$ is contained in the $k[X;S]$-module $\sum_i k[X;S] \phi_i$. Therefore there exist elements $\alpha_1, \ldots, \alpha_l$ of $L$ such that $T \subset \bigcup_i (S + \alpha_i)$. Let $\{x_1, \ldots, x_n\}$ be a complete representative system of $L$ modulo $G$. We may assume that $\{x_1, \ldots, x_n\} \subset \{\alpha_1, \ldots, \alpha_l\}$. For each $i$, $\alpha_i$ is contained in $x_j + G$ for some $j$. Therefore $\alpha_i - x_j + a_i \in S$ for some $a_i \in S$. Put $s = a_1 + \ldots + a_l$. Then we see that $S + \alpha_i \subset S + x_j - s$. It follows that $T \subset \bigcup_i (S + x_i - s)$.

**Proposition 1.** Let $S$ be a Noetherian semigroup, and $Y$ be a finitely generated $S$-module. Then $Y$ satisfies the a.c.c. on submodules.

**Proof.** It is sufficient to consider the case where $Y$ is a cyclic module $S+x$. Suppose that $Y_1 \subset Y_2 \subset \cdots$ be an ascending chain of submodules of $Y$. For each $i$, let $A_i = \{a \in S \mid a + x \in Y_i\}$. Then $A_1 \subset A_2 \subset \cdots$ is an ascending chain of ideals of $S$. There is an $n$ such that $A_i = A_n$ for each $i > n$. Then $Y_i = Y_n$ for each $i > n$.

Theorem 2 and Proposition 1 imply the following,
COROLLARY 1. The assumptions being the same as in Theorem 2, let us furthermore assume that \( S \) is Noetherian. Then \( T \) is a finitely generated \( S \)-module and is a Noetherian semigroup.

**Lemma 3.** Let \( S \) be a Noetherian semigroup, \( A \) and \( B \) be two ideals of \( S \) such that \( A \neq S \). Then \( A = A :_S B \) if and only if \( B \) is contained in no prime ideal of \( A \).

The proof is an analogy for semigroups of the proof of [10, Ch.IV, Theorem 11].

**Lemma 4.** Let \( S \) be a Noetherian integrally closed semigroup, and \( P \) be a maximal ideal of \( S \). If \( P \) is a prime ideal of a principal ideal \( (y) \), then \( P \) is a principal ideal.

**Proof.** We have \((y) :_S P \nsubseteq (y)\) by Lemma 3. Choose an element \( x \) of \((y) :_S P \) such that \( x \notin (y) \). Then \( x - y \notin P \), and \( x - y \notin S \). Suppose that \( P + P^{-1} \nsubseteq S \). Then we have \( P = P + P^{-1} \). Set \( P = (x_1, \ldots, x_n) \), and put \( z = x - y \). Then \( z + x_i \in P \) for each \( i \). We may assume that \( z + x_1 = x_2 + r_1, z + x_2 = x_3 + r_2, \ldots, z + x_k = x_1 + r_k \) for some \( k \geq 1 \) and for some elements \( r_1, \ldots, r_k \) of \( S \). Then we have \( kx = r_1 + \ldots + r_k \in S \). Hence \( x - y = z \in S \); a contradiction. We have proved that \( P + P^{-1} = S \). There exist \( p \in P \) and \( q \in P^{-1} \) such that \( p + q = 0 \). Then we have \( P = (p) \).

**Lemma 5.** Let \( A \) be a proper ideal of a \( g \)-monoid \( S \) admitting an irredundant primary representation \( A = \bigcap_i Q_i \). Let \( T \) be an additive system of \( S \), and suppose that, for \( 1 \leq i \leq r \), we have \( Q_i \cap T = \emptyset \), and that, for \( r + 1 \leq j \leq n \), we have \( Q_j \cap T \neq \emptyset \). Then \( AS_T = \bigcap_i (Q_i S_T) \) is an irredundant primary representation of \( AS_T \).

The proof is straightforward.

**THEOREM 3.** In an integrally closed Noetherian semigroup \( S \), each prime ideal \( P \) of any proper principal ideal \( (y) \) has height 1.

**Proof.** \( S_P \) is a Noetherian integrally closed semigroup, and \( PS_P \) is a maximal ideal of \( S_P \) which is a prime ideal of a principal ideal \( yS_P \) by Lemma 5. Lemma 4 implies that \( PS_P \) is a principal ideal. Therefore \( PS_P \) has height 1. Hence \( P \) has height 1.

Let \( P \) be a prime ideal of a \( g \)-monoid \( S \). If \( P \) is the only \( P \)-primary ideal of \( S \), then \( P \) is called unbranched.

**THEOREM 4.** Let \( V \) be a nontrivial valuation semigroup with quotient group \( L \), and suppose that \( V \) is of the form \( G \cup M \), where \( G \) is a group and \( M \) is the maximal ideal of \( V \). Let \( S \) be a \( g \)-monoid which is a proper subsemigroup of \( G \), and let \( S_1 = S \cup M \).

(1) \( S_1 \) is a \( g \)-monoid and \( M \) is the conductor of \( S_1 \) in \( V \). Therefore, \( S_1 \) and \( V \) have the same complete integral closure. In particular, \( S_1 \) is not completely integrally closed.
(2) The integral closure of $S_1$ is $\bar{S} \cup M$, where $\bar{S}$ is the integral closure of $S$ in $G$.

(3) Each ideal of $S_1$ compares with $M$ under $\subset$.

(4) The set of ideals of $S_1$ containing $M$ is $M$ and $\{A_\alpha \cup M \mid \alpha\}$ where $\{A_\alpha \mid \alpha\}$ is the set of ideals of $S$. If $A_\alpha \cup M = A_\beta \cup M$, then $A_\alpha = A_\beta$. $M$ is a prime ideal of $S_1$. Further, $A_\alpha$ is maximal, prime, or $P_\alpha$-primary in $S$ if and only if $A_\alpha \cup M$ is, respectively, maximal, prime, or $(P_\alpha \cup M)$-primary in $S_1$. If $T_\alpha$ is a generating set for $A_\alpha$ as an ideal of $S$, then $T_\alpha$ is also a generating set for $A_\alpha \cup M$ as an ideal of $S_1$.

(5) If $Q$ is $P$-primary in $S_1$, where $P \subseteq M$, then $Q$ and $P$ are ideals of $V$, and $Q$ is $P$-primary in $V$. If $M$, as an ideal of $V$, is unbranched, then $M$ is also unbranched as an ideal of $S_1$.

(6) $\dim S_1 = \dim S + \dim V$.

(7) If $N$ is an additive system in $S$, then $(S_1)N = S_N \cup M$. If $P$ is prime in $S_1$ and if $P \subseteq M$, then $(S_1)P = V_P$ so that $(S_1)P$ is a valuation semigroup.

(8) $S_1$ is a valuation semigroup on $L$ if and only if $S$ is a valuation semigroup on $G$.

(9) The valuative dimension of $S_1$ is equal to $k + \dim V$, where $k$ is the maximal dimension of a valuation semigroup on $G$ containing $S$. ($k$ may be infinite.)

(10) The finitely generated ideals of $S_1$ which properly contain $M$ are those of the form $A_\alpha \cup M$, where $A_\alpha$ is a finitely generated ideal of $S$. Any finitely generated ideal $A$ of $S_1$ contained in $M$ can be obtained as follows: let $W$ be a finitely generated $S$-submodule of $G$, let $m \subseteq M$, and set $A = (W + m) \cup (M + m)$.

(11) $S_1$ is Noetherian if and only if $V$ is Noetherian, $S$ is a group, and $(G : S) < \infty$.

Almost all of the proof of Theorem 4 is an analogy of the proof of [2, Appendix 2, Theorem A]. The sufficiency of (11): $V$ is the integral closure of $S_1$. By the Mori-Nagata theorem for semigroups[6], $V$ is Noetherian.

Let $S$ be a 1-dimensional Noetherian semigroup, and $M$ be a maximal ideal of $S$. Then, clearly, $M$ is a prime ideal minimal among containing a 1-generated ideal of $S$.

Let $S$ be a g-monoid, and $x_1, \ldots, x_n$ be elements of an extension semigroup of $S$. Then $S + \sum_i \mathbb{Z}x_i$ is called the semigroup generated by $x_1, \ldots, x_n$ over $S$, and is denoted by $S[x_1, \ldots, x_n]$.

**Theorem 5.** Let $S$ be a 2-dimensional Noetherian semigroup, and $M$ be its maximal ideal. Then $M$ is a prime ideal minimal among containing a 2-generated ideal of $S$.

**Proof.** Assume that $M$ is $l$-generated. Assume that $l \geq 3$, and that our assertion holds for $l - 1$. Suppose that $M$ is not a prime ideal minimal among containing a 2-generated ideal of $S$. We will derive a contradiction.

Let $M = (x_1, \ldots, x_l)$. We may assume that $\{x_1, \ldots, x_l\}$ is a complete representative system of irreducible elements of $S$. Let $H$ be the unit group of $S$. 

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At first, we will show that, for each \( j \), \( x_j \) is not integral over the semigroup \( H[x_\alpha \mid \alpha \neq j] \) generated by \( \{x_\alpha \mid 1 \leq \alpha \leq l, \alpha \neq j\} \) over \( H \). Thus, suppose, for instance, that \( x_i \) is integral over \( H[x_1, \ldots, x_{i-1}] \). Then \( S_1 = H[x_1, \ldots, x_{i-1}] \) is a 2-dimensional Noetherian semigroup. And the maximal ideal \( M_1 \) of \( S_1 \) is generated by \( x_1, \ldots, x_{i-1} \). Therefore \( M_1 \) is a prime ideal minimal among containing a 2-generated ideal \( (y_1, y_2)S_1 \) of \( S_1 \). Since \( S \) is integral over \( S_1 \), we see that \( M \) is a prime ideal minimal among containing \( (y_1, y_2)S \); a contradiction.

Next, let \( y_1, \ldots, y_{l-1} \) be \( l-1 \) distinct members in \( \{x_1, \ldots, x_l\} \). We will show that \( (y_1, \ldots, y_{l-1}) \) is a \( ht \) 1 prime ideal of \( S \). Thus, suppose, for instance, that the ideal \( I = (x_1, \ldots, x_{l-1}) \) is not prime. Then any prime ideal which contains \( I \) is equal to \( H[x_\alpha \mid \alpha \neq i] \). Therefore, suppose, for instance, that \( x_i \) is integral over \( H[x_1, \ldots, x_{i-1}] \); a contradiction.

Next, we will show that, for each \( 1 < i < j < k < l \), there exist natural numbers \( a_i \) and \( a_j \) such that \( a_i x_i + a_j x_j \) belongs to the semigroup \( H[x_\alpha \mid 1 \leq \alpha \leq k, \alpha \neq i, j] \) over \( H \). We depend on the induction on \( k \) from \( l \). Thus, assume that \( k = l \). Let \( P = (x_\alpha \mid \alpha \neq i) \) be the ideal of \( S \) generated by \( \{x_\alpha \mid 1 \leq \alpha \leq l, \alpha \neq i\} \). Then \( P \) is a prime ideal of \( S \), and \( S_P \) is a 1-dimensional Noetherian semigroup. Hence \( PS_P \) is a prime ideal minimal among containing the ideal \( x_i S_P \). Therefore, for each \( j \), \( a_j x_j \) belongs to \( x_i S_P \) for some natural number \( a_j \). It follows that \( a_i x_i + a_j x_j \in (x_1) \) for some \( a_i \in \mathbb{Z}_0 \). \( x_i \) is not a unit of \( S \), \( x_j \) is not integral over \( H[x_\alpha \mid \alpha \neq j] \), and \( x_i \) is not integral over \( H[x_\alpha \mid \alpha \neq i] \). Therefore \( a_i > 0 \), and moreover \( a_i x_i + a_j x_j \in H[x_\alpha \mid \alpha \neq i, j] \).

Next, assume that \( 1 < i < j < k < l \), and that the assertion holds for \( k+1 \). There exist natural numbers \( a_i \) and \( a_j \) such that \( a_i x_i + a_j x_j \in H[x_\alpha \mid 1 \leq \alpha \leq k, \alpha \neq i, j] \). Suppose that \( a_i x_i + a_j x_j \notin H[x_\alpha \mid 1 \leq \alpha \leq k, \alpha \neq i, j] \). Then we have \( a_i x_i + a_j x_j = a_{k+1} x_{k+1} + \sum_{\alpha \neq i, j} b_\alpha x_\alpha + h_1 \) \((a_{k+1} > 0, \; b_\alpha \geq 0, \; h_1 \in H)\), where \( \sum_{\alpha \neq i, j} \) means that \( \alpha \) ranges over \( 1 \leq \alpha \leq k \) with \( \alpha \neq i, j \). Also we have \( c_i x_i + c_{k+1} x_{k+1} = \sum_{\beta \neq i} d_\beta x_\beta + h_2 \) \((c_i, c_{k+1} > 0; \; d_\beta \geq 0, \; h_2 \in H)\). We may assume that \( a_{k+1} = c_{k+1} \). Since \( x_i \) is not integral over \( H[x_\alpha \mid 1 \leq \alpha \leq k, \alpha \neq i] \), we have \( a'_i x_i + a'_j x_j = \sum_{\alpha \neq i, j} b'_\alpha x_\alpha + h'(a'_i, a'_j > 0; \; b'_\alpha \geq 0, \; h' \in H) \).

Now if we apply the above result for \( i = 2, j = 3, k = 3 \), we have \( a_2 x_2 + a_3 x_3 \in H[x_1] \) for natural numbers \( a_2, a_3 \). Therefore \( x_1 \) is integral over \( H[x_2, x_3] \); a contradiction.

The motive of this paper, first, was to prove all such propositions in [10] and in [2] for \( g \)-monoids \( S \) that have meaning for \( S \). All these propositions in [10] except [10, Theorems 30 and 31 of Ch.IV, Theorems 7 and 14 of Ch.V] have been proved for \( g \)-monoids. However, since they have not been published, we will state them briefly, for convenience. A semigroup version of [3] is under preparation in [4]. The only such propositions in [2] that is not contained in [3] and has meaning for \( S \) is [2, Appendix 2, Theorem A].

Let \( Y \) be an \( S \)-module. Let \( Y = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_r \) be submodules of \( Y \). If, for each \( i \), there exist no submodules \( Y' \) such that \( Y_i \supseteq Y' \supseteq Y_{i+1} \), then the chain of submodules of \( Y \) is called a composition series of length \( r \).

**Proposition 2.** If an \( S \)-module \( Y \) has one composition series of length \( r \),
then every composition series of $Y$ has length $r$.

The proof of Proposition 2 appears on [8].

**Proposition 3.** If $S$ is a Noetherian semigroup, then so is any semigroup $S[x_1, \cdots, x_n]$ generated by a finite number of elements $x_1, \cdots, x_n$ over $S$.

**Proof.** Let $X_1, \cdots, X_n$ be indeterminates. Then [9, Theorem 69'] shows that $S[X_1, \cdots, X_n]$ is a Noetherian semigroup. Let $I$ be an ideal of $S[x_1, \cdots, x_n]$. Set $J = \{s + \sum k_i X_i | \text{each } k_i \geq 0, s + \sum k_i x_i \in I\}$. Then $J$ is a finitely generated ideal of $S[X_1, \cdots, X_n]$. It follows that $I$ is a finitely generated ideal of $S[x_1, \cdots, x_n]$.

**Proposition 4.** Let $S$ be a Noetherian semigroup. Then every proper ideal $I$ of $S$ admits an irredundant primary representation $I = \bigcap_1^n Q_i$. The set of prime ideals $\sqrt{Q_1}, \cdots, \sqrt{Q_n}$ are uniquely determined by $I$.

The proof of Proposition 4 appears on [7, (1.1) Proposition] and [8].

**Proposition 5.** Let $S$ be a Noetherian semigroup and $I$ a proper ideal of $S$. Then $\bigcap_1^\infty nI = \emptyset$.

The proof of Proposition 5 appears on [9, Proposition 77].

**Proposition 6.** Let $S$ be a $g$-monoid, $T$ an extension semigroup which is integral over $S$. If $P$ is a prime ideal of $S$, then there exists a unique prime ideal $Q$ of $T$ lying over $P$.

**Proof.** Let $\Sigma$ be the family of all ideals $J$ of $T$ so that $J \cap (S - P) = \emptyset$. Then $\Sigma$ is not empty. Let $Q$ be a maximal member of $\Sigma$ under the inclusion relation. Then $Q$ is a prime ideal of $T$, and lies over $P$.

**Proposition 7.** Let $S$ be a $g$-monoid, and $S'$ an extension semigroup of $S$ which is integral over $S$. If $P$ and $Q$ are prime ideals in $S$ such that $Q \subseteq P$, and if $P'$ is a prime ideal of $S'$ lying over $P$, then there exists a prime ideal $Q'$ of $S'$ uniquely, contained in $P'$ and lying over $Q$.

**Proof.** There exists a prime ideal $Q'$ of $S'$ uniquely which lies over $Q$ by Proposition 6. Let $x \in Q'$. We have $nx \in S$ for some $n > 0$. Then $nx \in Q \subseteq P \subseteq P'$. It follows that $Q' \subseteq P'$.

**Proposition 8.** Let $V$ be a discrete valuation semigroup of rank 1, and $L$ an extension torsion-free abelian group of the quotient group $G$ of $V$ with $(L : G) < \infty$. Then the integral closure $W$ of $V$ in $L$ is a discrete valuation semigroup of rank 1.

For the proof, choose elements $x_1, \cdots, x_n$ in $W$ such that the quotient group of $V[x_1, \cdots, x_n]$ is $L$. $V[x_1, \cdots, x_n]$ is a Noetherian semigroup by Proposition 3. Then $W$ is a 1-dimensional Krull semigroup by [6]. Hence $W$ is a discrete valuation semigroup of rank 1.

**Question.** If $P$ is a prime ideal of height $n$ in a Noetherian semigroup $S$, then is $P$ a prime ideal minimal among containing an $n$-generated ideal of $S$?
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