Math. J., Ibaraki Univ., Vol. 29, 1997

Notes on removable singularities for a certain class of semilinear degenerate elliptic equations

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0. Introduction

Let $N \geq 1$ and let B be an open ball in $\mathbb{R}^N$ with its center being the origin, and set $B' = B \setminus \{0\}$. Let us define

$$(0\text{-}1) \qquad L_\alpha = -\operatorname{div}(|x|^{2\alpha}\nabla\cdot),$$

where $\alpha > -\frac{N-2}{2}$. Assume that $u \in C^2(B')$ satisfies the differential inequality

$$(0\text{-}2) \qquad L_\alpha u + a|x|^{2\beta}|u|^{p-1}u \leq b|x|^{2\gamma}, \qquad \text{in } B',$$

for some $a > 0$ and $b > 0$. Then we will show under some additional conditions that

$$\limsup_{x\to 0} u(x) < +\infty.$$

When $\alpha = \beta = \gamma = 0$, this result is already established by H.Brezis and L.Veron in [BV]. They proved this result by using a comparison principle and a weak maximum principle. Although the operator in this paper is degenerate at the origin, their methods still work under some modifications. In fact we first construct a suitable super-solution, and then we derive a pointwise estimate by a weak muximum principle and Kato's inequality. As a application we will deduce that if $u \in C^2(B')$ satisfies

$$(0\text{-}3) \qquad L_\alpha u + g(x,u) = 0, \qquad \text{in } B',$$

and the function g satisfies the growth condition (1-3) and (1-4), then there is a continuous function in $\mathbb{R}^n$ which coincides with u in B. We shall generalize these in the coming papers [H 1] and [H 2] to a general class of operators of the form

$$(0\text{-}4) \qquad Pu = -\operatorname{div}(A(x)\nabla u) + B(x)Q(u).$$

Received March 4, 1997.

1991*Mathematics Subject Classification.* 35J70, 35J65.

This research was partially supported by Grant-in-Aid for Science Research(No.09640153), Ministry of Education, Science and Culture.

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1. Main results and Counter-examples

Assume that real numbers a, β and γ satisfy

$$\text{(H-1)} \qquad \alpha > -\frac{N-2}{2}.$$

Let us set

$$\text{(1-1)} \qquad p^* = \begin{cases} 1 + 2\dfrac{1-\alpha+\beta}{N+2\alpha-2} = \dfrac{N+2\beta}{N-2+2\alpha}, & \text{if} \quad \alpha < \beta+1 \\ 1, & \text{if} \quad \alpha \geq \beta+1 \end{cases}$$

and assume that

$$\text{(H-2)} \qquad \begin{cases} p \geq p^*, & \text{if} \quad \alpha < \beta+1 \\ p > p^* = 1, & \text{if}\ \alpha \geq \beta+1 \end{cases}$$

Also we assume

$$\text{(H-3)} \qquad \beta \leq \gamma \quad \text{and} \quad \gamma > -\frac{N}{2}.$$

Let us set $u_+ = \max[0, u]$ and $u_- = \max[0, -u]$

THEOREM 1. *Assume that (H-1), (H-2) and (H-3). Assume that* $u \in L^\infty_{loc}(B')$ *satisfies* $L_\alpha u \in L^1_{loc}(B')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in B; u(x) \geq 0\}$

$$\text{(1-2)} \qquad L_\alpha u + a|x|^{2\beta}u^p \leq b|x|^{2\gamma},$$

for some positive constants a *and* b. *Then we have* $u^+ \in L^\infty_{loc}(B)$.

We note that from [H-1] α must be positive if $N \leq 2$. The following is a direct consequence of this Theorem.

COROLLARY 1. *Assume that (H-1), (H-2) and (H-3). Assume* $g(x,u) : B \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function satisfying*

$$\text{(1-3)} \qquad g(x,u) \geq a|x|^{2\beta}u^p - b|x|^{2\gamma}, \qquad \textit{for } u \geq 0$$

$$\text{(1-4)} \qquad g(x,u) \leq -a|x|^{2\beta}|u|^p + b|x|^{2\gamma}, \qquad \textit{for } u \leq 0,$$

for some positive numbers a *and* b.

Assume that $u \in C^2(B')$ *satisfies*

$$\text{(1-5)} \qquad Lu + g(x,u) = 0, \qquad \textit{in } D'(B').$$

Then we have $u \in L^\infty_{loc}(B)$.

PROOF OF COROLLARY 1. From Theorem 1 we have $u_+ \in L^\infty_{loc}(B)$. And similarly $u_- \in L^\infty_{loc}(B)$. Thus $u \in L^\infty_{loc}(B)$.

In the rest of this subsection we construct examples showing that in certain respects Theorem 1 gives best possible results.

COUNTER-EXAMPLES. We consider a function u of the form

(1-6) $$u(x) = c|x|^{-M}, \qquad \text{for } M, c > 0.$$

If u solves the following equation for some c and M, then u becomes a counter-example which is clearly unbounded at the origin.

(1-7)
$$\begin{aligned} &L_\alpha u + a|x|^{2\beta}u^p \\ &= c|x|^{2\alpha-M-2}[M(N+2\alpha-2-M) + c^{p-1}a|x|^{2(1-\alpha+\beta)-M(p-1)}] \\ &= 0. \end{aligned}$$

Then it follows

(1-8) $$p = 1+\frac{2(1-\alpha+\beta)}{M}, \quad M(N+2\alpha-2-M)+c^{p-1}a = 0, \quad \text{for some } c > 0.$$

Here we note that the second condition is equivalent to the enequality

(1-9) $$M > N + 2\alpha - 2.$$

PROPOSITION 1. *For the validity of Theorem 1, the assumptions [H-1] and [H-2] are necessary.*

PROOF OF PROPOSITION 1. As for [H-1] and [H-2], We see that they are necessary by the examples listed below: Now we classify the cases.

1: $-\frac{N-2}{2} < \alpha < \beta + 1$. From (1-8) and (1-9) u is a counter-example for a suitable c provided $1 < p < p^*$.

2: $\alpha \leq -\frac{N-2}{2}, \alpha < \beta+1$. In this case u becomes a counter-example provided $M > 0$. Namely there exists a counter-example when $1 < p < \infty$.

3: $\alpha \leq -\frac{N-2}{2}, \alpha = \beta + 1$. In this case we have $p = 1$. Hence if $M > 0$ and $a = M(M - N - 2\alpha + 2)$, u becomes a counter-example.

4: $-\frac{N-2}{2} < \alpha, \alpha > \beta + 1$. As in the case 1, u becomes a counter-example when $1 + \frac{2(1-\alpha+\beta)}{N+2\alpha-2} < p < 1$.

5: $\alpha \leq -\frac{N-2}{2}, \alpha > \beta + 1$. As in the case 4, u becomes a counter-example when $-\infty < p < 1$.

6: $p = 1$. Then the operator is linear. Hence a fundamental solution exists and becomes a counter-example. (See also 3).

As for [H-3], we have the following:

PROPOSITION 2. *For the validity of Theorem 1, $\beta \leq \gamma$ is necessary if $\alpha \geq \gamma + 1$. If $\alpha < 1$, then $\beta < \gamma + p(\alpha + \frac{N-2}{2})$ is necessary.*

PROOF OF PROPOSITION 2. First we assume that $\alpha \neq 1$. Then $u = |x|^{-M}$ becomes a counter-example for a sufficiently small $M > 0$, if $\alpha \geq \gamma+1$. If $\alpha = 1$, then $-\log|x|$ will do. Lastly we assume that $\alpha < 1$. Then $u = |x|^{-(N+2\alpha-2)}$ satisfies $L_\alpha u = 0$ in Ω'. Hence $\beta < \gamma + p(\alpha + \frac{N-2}{2})$ is necessary to avoid u.

2. Lemmas

Theorem 1 will be proved in a chain of auxiliary lemmas.

LEMMA 2-1. *Let Ω be an open set of $\mathbb{R}^N \setminus \{0\}$. Assume that $u \in L^1_{loc}(\Omega)$ and $L_\alpha u \in L^1_{loc}(\Omega)$. Then we have*

$$(2\text{-}1) \qquad L_\alpha u^+ \leq (L_\alpha u) sgn^+ u, \qquad \text{in } D'(\Omega),$$

$$\text{where} \qquad sgn^+ u = \begin{cases} 1, & \text{for } u > 0, \\ 1/2, & \text{for } u = 0, \\ 0, & \text{for } u < 0. \end{cases}$$

PROOF. This follows from Kato's inequality. Namely for $u \in L^1_{loc}(\Omega)$ and $\Delta u \in L^1_{loc}(\Omega)$, it holds that

$$(2\text{-}2) \qquad \Delta|u| \geq (\Delta u) sgn\, u \qquad \text{in } D'(\Omega),$$

where

$$sgn\, u = \begin{cases} 1, & \text{for } u > 0, \\ 0, & \text{for } u = 0, \\ -1, & \text{for } u < 0. \end{cases}$$

As is well-known, this inequality holds for second order elliptic operators of the divergence form. In fact, let us set

$$(2\text{-}3) \qquad M(x, \partial_x) = \partial_{x_j}(a_{jk}(x)\partial_{x_k} \cdot), \qquad a_{jk}(x) \in C^2(\Omega).$$

The we have, for $u, M(x, \partial_x)u \in L^1_{loc}(\Omega)$,

$$(2\text{-}4) \qquad M(x, \partial_x)|u| \geq (M(x, \partial_x)\, u) sgn\, u \qquad \text{in } D'(\Omega),$$

Since L_α is elliptic in $\mathbb{R}^n \setminus \{0\}$ and $2u^+ = |u| + u$, we get the desired inequality. For the detailed proof of Kato's inequality see [K].

LEMMA 2-2. *Assume that $-\frac{N-2}{2} < \alpha$, and let us set $p \geq 1 + 2\frac{1-\alpha+\beta}{N+2\alpha-2} = \frac{N+2\beta}{N-2+2\alpha}$. Assume that $f \in L^1_{loc}(B)$ and $|x|^{2\beta} \cdot |u|^p \in L^1_{loc}(B)$, and assume that*

$$(2\text{-}5) \qquad L_\alpha u \leq f, \qquad \text{in } D'(B').$$

Then we have

$$(2\text{-}6) \qquad L_\alpha u \leq f, \qquad \text{in } D'(B).$$

PROOF. We take a sequence of nonnegative smooth functions $\eta_n \in C^\infty(\mathbb{R}^N)$ such that for some positive number C

$$0 \leq \eta_n(x) \leq 1, \qquad |\eta_n^k(x)| \leq Cn^k, \qquad \text{for } x \in \mathbb{R}^N,$$

$$(2\text{-}7) \qquad \eta_n(x) = \begin{cases} 1, & \text{for } |x| \geq \dfrac{1}{n}, \\ 0, & \text{for } |x| \leq \dfrac{1}{2n}, \end{cases}$$

Then we have for a test function $\varphi \in D(\Omega)$,

(2-8)
$$\langle L_\alpha u, \eta_n \varphi\rangle = \int_\Omega u L_\alpha(\eta_n\varphi)\,dx \le \int_\Omega f\eta_n\varphi\,dx.$$

We also have

(2-9)
$$\begin{aligned}|uL_\alpha(\eta_n\varphi)| \le{}& |u||\operatorname{div}[|x|^{2\alpha}(\varphi\nabla\eta_n + \eta_n\nabla\varphi)]| \\ &+ |x|^{2\alpha}|u|(\varphi|\Delta\eta_n| + 2|\nabla\eta_n|\cdot|\nabla\varphi|) \\ &+ 2\alpha|x|^{2\alpha-2}|u|(|x\cdot\nabla\eta_n|\varphi + |x\cdot\nabla\varphi|\eta_n).\end{aligned}$$

The we have

(2-10)
$$\begin{aligned}&\left|\int_\Omega uL_\alpha(\eta_n\varphi)\,dx\right| \le \int_\Omega |f\eta_n\varphi|\,dx \\ &+C\left[\int_{|x|\le 1/n}(n|x|+1)n|x|^{2\alpha-1}|u|\,dx\right. \\ &\left.+\int_{\operatorname{supp}\varphi}(|x|+1)|x|^{2\alpha-1}|u|\,dx\right]\end{aligned}$$

and

(2-11)
$$\begin{aligned}&\int_{|x|\le 1/n}(n|x|+1)n|x|^{2\alpha-1}|u|\,dx \le 2\int_{|x|\le 1/n} n|x|^{2\alpha-1}|u|\,dx \\ &\le 2n\left[\int_{|x|\le 1/n}|u|^p|x|^{\beta 2}\,dx\right]^{1/p}\cdot\left[\int_{|x|\le 1/n}|x|^{(2\alpha-1-\frac{2\beta}{p})p'}\,dx\right]^{1/p'} \\ &= \int_{|x|\le 1/n}|u|^p|x|^{\beta 2}\,dx\Bigg]^{1/p}\cdot\left(\frac{1}{n}\right)^{N+2\alpha-2-\frac{2\beta+N}{p}} \to 0, \\ &\quad\text{as } n\to+\infty.\end{aligned}$$

Here we note that $p \ge \frac{N+2\beta}{N+2\alpha-2}$ and so

(2-12)
$$N + 2\alpha - 2 - \frac{2\beta+N}{p} \ge 0.$$

Then by the dominated convergence theorem we find

(2-13)
$$\int_\Omega uL_\alpha\,dx \le \int_\Omega f\varphi\,dx.$$

This proves the assertion.

LEMMA 2-3. *Assume that $u \in L^\infty_{loc}(B')$ satisfies $Lu \in L^1_{loc}(B')$ in the distribution sense. Assume that (H-2) and $-\frac{N-2}{2} < \alpha \neq \beta + 1$. Moreover we assume that for almost all $x \in \{x \in B; u(x) \geq 0\}$*

(2-14) $$L_\alpha u + a|x|^{2\beta}u^p \leq b|x|^{2\gamma},$$

for some positive constants a and b. Then we have, for some positive number C,

(2-15) $$u(x) \leq \frac{C}{|x|^A}, \qquad \text{for } x \text{ with } |x| \leq 1/2,$$

where $A = \max[A_1, A_2]$,

(2-16) $$A_1 = 2\frac{1-\alpha+\beta}{p-1} \quad A_2 = 2(\beta-\gamma)\frac{1}{p}.$$

In particular if $\alpha < \beta + 1$ and $p = p^*$, then we have

(2-17) $$A_1 = N - 2 + 2\alpha \quad \text{and} \quad A_2 = 2(\beta-\gamma)\frac{N-2+2\alpha}{N+2\alpha}.$$

REMARK. If we also assume (H-1) in this Lemma, we see that $A_2 \leq 0$. In addition, if $\alpha > \beta + 1$ then we see that u is bounded.

PROOF. Let δ satisfy

(2-18) $$\delta + 2 = p\delta.$$

By virtue of (H-2), we see that δ is positive. Let $x_0 \in \mathbb{R}^N$, with $0 < |x_0| < 1/2$. For $R = |x_0|/2$ we set

(2-19) $$G = \{x \in \mathbb{R}^N; |x - x_0| < R\}$$

and define for $r = |x - x_0|$

(2-20) $$v(x) = \lambda(R^2 - r^2)^{-\delta} + \mu, \quad x \in G.$$

Now we determine constants λ and mu so that v satisfies

(2-21) $$L_\alpha v + a|x|^{2\beta}v^p \geq b|x|^{2\gamma}, \qquad \text{in } G.$$

Then

(2-22)
$$\begin{aligned} L_\alpha v &= -(|x|^{2\alpha}\Delta + 2\alpha|x|^{2\alpha-2}x\cdot\nabla)v \\ &= -|x|^{2\alpha}\left(\frac{\partial^2 v}{\partial r^2} + (N-1)\frac{1}{r}\frac{\partial v}{\partial r}\right) - 2\alpha|x|^{2\alpha-2}x\cdot\nabla v \\ &= -2\lambda\delta|x|^{2\alpha-2}(R^2-r^2)^{-\delta-2}\times \\ &\quad [|x|^2(NR^2 + (2\delta+2-N)r^2) + 2\alpha x\cdot(x-x_0)(R^2-r^2)] \\ &\geq -2\lambda\delta C_1(R^2-r^2)^{-\delta-2}R^{2\alpha+2}, \end{aligned}$$

where C is a positive number independent of x_0, x, and R . Therefore we get

(2-23)
$$L_\alpha v + a|x|^{2\beta} v^p \geq$$
$$- 2\lambda\delta C_1 (R^2 - r^2)^{-\delta-2} R^{2\alpha+2} + a|x|^{2\beta}[\mu^p + \lambda^p (R^2 - r^2)^{p\delta}].$$

Puttig $\mu = C_2 R^{2\frac{\gamma-\beta}{p}}$ for sufficiently small number C_2, we get

(2-24)
$$L_\alpha v + a|x|^{2\beta} v^p \geq$$
$$\geq b|x|^{2\gamma} + \lambda (R^2 - r^2)^{-\delta-2}[aR^{2\beta}\lambda^{p-1} - 2\delta R^{2\alpha+2}], \text{in } G$$

where we used $|x_0| = R/2$ and $R \leq |x| \leq 3R$ in G. Lastly we put

(2-25)
$$\lambda = C_3 R^{-2\frac{1-\alpha+\beta}{p-1}}, \qquad C_3 = \left(\frac{2\delta C_2}{a}\right)^{\frac{1}{p-1}},$$

then we get the desired inequality (2-21). By virtue of (1-2), (2-21) and Lemma 2-1, we have

(2-26)
$$-L_\alpha (u - v)_+ \geq -L_\alpha (u - v) \cdot sgn^+(u - v)$$
$$\geq a|x|^{2\beta}(u^p - v^p) \cdot sgn^+(u - v) \geq 0 \text{ in } D'(G).$$

Since $(u - v)_+ = 0$ near ∂D, it follows from a usual maximum principle that

(2-27)
$$u(x_0) \leq v(x_0) = \lambda R^{-\delta} + \mu = C_3 R^{-A_1} + C_2 R^{-A_2},$$

and this proves the assertion.

In the next we deal with the excluded case $\alpha = \beta + 1$.

LEMMA 2-4.
Assume that $-\frac{N-2}{2} < \alpha \leq \beta + 1$. *For an arbitrary* $\varepsilon > 0$, *we set*

$$q = \frac{N + 2\beta + 2\varepsilon}{N + 2\alpha - 2} = p^* + \frac{2\varepsilon}{N + 2\alpha - 2}.$$

Assume that $u \in L^\infty_{loc}(B')$ *satisfies* $L_\alpha u \in L^1_{loc}(B')$*in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in B; u(x) \geq 0\}$

(2-27)
$$Lu + a|x|^{2\beta} u^q \leq b|x|^{2\gamma},$$

for some positive constants a and b. Then we have:

(2-28)
$$u(x) \leq \frac{C}{|x|^A}, \qquad \textit{for } x \textit{ with } |x| \leq 1/2,$$

where $A = \max[A_1, A_2]$,

$$A_1 = (1 - \alpha + \beta)\frac{N + 2\alpha - 2}{1 - \alpha + \beta + \varepsilon}, \quad A_2 = 2(\beta - \gamma)\frac{1}{q}.$$

In particular if $\alpha = \beta + 1$, *we have* $A_1 = 0$.

PROOF. Since $q > 1$, the proof is done in a similar way.

LEMMA 2-5. *Assume that (H-1) and (H-2). Assume that* $u \in L^{\infty}_{loc}(B')$ *satisfies* $L_\alpha u \in L^1_{loc}(B')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in B; u(x) \geq 0\}$

$$L_\alpha u + a|x|^{2\beta}u^p \leq b|x|^{2\gamma}, \tag{1-2}$$

for some positive constants a and b. Then we have

$$|x|^{2\beta}(u^+)^p \in L^1_{loc}(B).$$

PROOF. From Lemma 2-4 u_+ is bounded if $\alpha = \beta + 1$. Hence the assertion is clear in this case. Hence we assume that $\alpha \neq \beta + 1$. We note that we can assume that $p = p^*$ if $\alpha < \beta + 1$ and $p = 1 + s$ for some $s > 0$ if $\alpha > \beta + 1$. Then we get

$$L_\alpha u_+ + a|x|^{2\beta}(u_+)^p \leq B|x|^{2\gamma}, \quad \text{in} D'(B'). \tag{2-29}$$

For a test function $\phi \in D(B)$, we have

(2-30)
$$\begin{aligned}&\langle L_\alpha u_+ + a|x|^{2\beta}(u_+)^p, \eta_n\varphi\rangle \\ &= \int_B [uL_\alpha(\eta_n\varphi) + a|x|^{2\beta}(u_+)^p\eta_n\varphi)]\,dx \leq \int_B b|x|^{2\gamma}\eta_n\varphi\,dx.\end{aligned}$$

Here $\{\eta_n\}_{n=1}^{\infty}$ is a sequence of nonnegative smooth functions defined in Lemma 2-2.

In view of Lemma 2-3, we see

(2-31)
$$\begin{aligned}&\left|\int_B uL_\alpha(\eta_n\varphi)\,dx\right| \\ &\leq C\left[\int_{|x|\leq 1/n}(n|x|+1)n|x|^{2\alpha-1}|u|\,dx + \int_{\text{supp}\varphi}(|x|+1)|x|^{2\alpha-1}|u|\,dx\right]\end{aligned}$$

and

(2-32)
$$\begin{aligned}&\int_{|x|\leq 1/n}(n|x|+1)n|x|^{2\alpha-1}|u|\,dx \leq 2\int_{|x|\leq 1/n} n|x|^{2\alpha-1}|u|\,dx \\ &\leq 2nC\int_{|x|\leq 1/n}|x|^{-A+2\alpha-1}\,dx = O(1)\end{aligned}$$

similarly

$$\int_{\text{supp}\varphi}(|x|+1)|x|^{2\alpha-1}|u|\,dx < +\infty.$$

Then by taking $n \to \infty$ we see that $|x|^{2\beta}(u_+)^p \in L^1_{loc}(B)$.

PROOF OF THEOREM 1. Let us take a μ so that $\mu = (b/a)^{1/p}$. Then we have, as in the proof of Lemma 2-3,

$$L_\alpha(u-\mu)_+ + a|x|^{2\beta} sgn^+(u-\mu)(u^p - \mu^p) \le 0, \qquad \text{in } D'(B'). \tag{2-33}$$

Hence we have from Lemma 2-2

$$L_\alpha(u-\mu)_+ + a|x|^{2\beta} sgn^+(u-\mu)(u^p - \mu^p) \le 0, \qquad \text{in } D'(B). \tag{2-34}$$

Now we assume that $mu \ge \sup_{1/2<|x|<3/4} u(x)$, then we shall see

$$u(x) \le \mu, \qquad \text{for } |x| < 1/2. \tag{2-35}$$

Let us set

$$\phi = \begin{cases} (u-\mu)^+ & \text{if } |x| < 3/4, \\ 0 & \text{otherwise .} \end{cases} \tag{2-36}$$

Then we can approximate ϕ by a sequence of smooth functions $\phi_m \in D(B)$. It is easy to see that

$$\begin{aligned} &\phi_m \to \phi \qquad \text{in } B \quad \text{(a.e.)} \\ &|x|^{2\beta}\phi_m^p \to |x|^{2\beta}\phi^p \qquad \text{in } L^1(B), \\ &L_\alpha \phi_m \to L_\alpha \phi \quad \text{in } D'(B). \end{aligned} \tag{2-37}$$

Hence we have

$$L_\alpha \phi_m + a|x|^{2\beta} sgn^+(u-\mu)(u^p - \mu^p) \le o(1). \tag{2-36}$$

Since $\operatorname{supp}\phi$ and $\operatorname{supp}\phi_m \subset \frac{3}{4}\Omega$, we have

$$0 \le \int_{\frac{3}{4}\Omega \cap \{u>\mu\}} |x|^{2\beta}|u^p - \mu^p|\, dx \le o(1). \tag{2-37}$$

Therefore we get $\limsup_{x\to 0}(u-\mu)_+ = 0$. This proves the assertion.

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