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On some Properties between Rings and Semigroups

RYŪKI MATSUDA*

§1. On $|\Sigma'(D)|$

Let $\Sigma'(D)$ be the set of all semistar-operations on $D$.
Let $F'(S)$ be the set of nonempty subset $I$ of $q(S) = G$ such that $S + I \subseteq I$.
A mapping $*: A \mapsto A^*$ on $F'(S)$ is called a semistar-operation on $S$ if the following conditions hold for all $a \in G$ and $A, B \in F'(S)$:

1. $(a + A)^* = a + A^*$;
2. $A \subseteq A^*$; if $A \subseteq B$, then $A^* \subseteq B^*$; and
3. $(A^*)^* = A^*$.

Let $\Sigma'(S)$ be the set of all semistar-operations on $S$.

The following (1.1) ~ (1.3) were proved in [16].

(1.1). (1) Let $D$ be a 4-dimensional integrally closed domain with exactly two maximal ideals. Then we have $|\Sigma'(D)| \geq 9$.

(2) Let $D$ be a 4-dimensional integrally closed domain such that $|\Sigma'(D)| \leq 9$. Then $D$ is a Prüfer ring with at most two maximal ideals.

If a commutative ring $R$ has only one maximal ideals, then $R$ is called a local ring.

(1.2). Let $D$ be a 4-dimensional integrally closed domain and $|\Sigma'(D)| \leq 9$. Then $|\Sigma(D)| \leq 2$. Furthermore, if $D$ is not local, then $|\Sigma(D)| = 2$.

(1.3). Let $D$ be a 4-dimensional integrally closed domain which is not local. Then $|\Sigma'(D)| \leq 9$ if and only if the following conditions hold:

1. $D$ is a Prüfer ring with exactly two maximal ideals $M, N$.
2. There exist prime ideals $P_1, P_2$ and $P_3$ of $D$ such that $M \subseteq N \subseteq P_3 \subseteq P_2 \subseteq P_1 \subseteq (0)$.
3. $P_1D_{P_1}, P_2D_{P_2}, P_3D_{P_3}$ are principal.
4. One of $MD_M$ and $ND_N$ is principal and the other is not principal.
5. $|\Sigma(D)| = 2$.

(1.4) EXAMPLE. (1) Let $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ with the lexicographic order $e_1 > e_2 > e_3$, where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. $Ze_3 = H_2$ and $Ze_2 + Ze_3 = H_1$ are the isolated subgroups of $G$.

(2) Let $k$ be a field. For every element $f = \sum a_iX^{n_i}$ of $k[G] = k[X; G]$, we set $u_3(f) = \inf_{i} (g_i)$. Then $u_3$ is a valuation on $q(k[G])$. Let $U_3$ be the

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valuation ring of \( u_3 \) and \( M_3 \) be the maximal ideal of \( U_3 \). Then \( \dim U_3 = 3 \), and \( M_3 = X^{e_3}U_3 \).

(3) Let \( u_2 \) be the natural composed mapping \( q(k[G]) \rightarrow G \cup \{ \infty \} \rightarrow (G/H_2) \cup \{ \infty \} \rightarrow (Ze_1 + Ze_2) \cup \{ \infty \} \). Then \( u_2 \) is a valuation on \( q(k[G]) \). Let \( U_2 \) be the valuation ring, and \( M_2 \) the maximal ideal of \( U_2 \). Then \( M_2 = X^{e_2}U_2 \).

(4) Let \( u_1 \) be the natural composed mapping \( q(k[G]) \rightarrow G \cup \{ \infty \} \rightarrow (G/H_1) \cup \{ \infty \} \rightarrow Ze_1 \cup \{ \infty \} \). Then \( u_1 \) is a valuation on \( q(k[G]) \). Let \( U_1 \) be the valuation ring of \( u_1 \), and \( M_1 \) the maximal ideal of \( U_1 \). Then \( M_1 = X^{e_1}U_1 \).

(5) The residue field \( U_3/M_3 \) is isomorphic onto \( k \). Let \( \phi : U_3 \rightarrow k \) be the natural mapping.

(6) Let \( V_1 \) be the valuation ring with maximal ideal \( M_1 \) of a valuation \( v_1 \) on \( k \). Set \( V = \phi^{-1}(V_1) \). Then \( V \) is a valuation ring of a valuation \( v \) on \( q(k[G]) \). Let \( M' \) be the maximal ideal of \( V \). Then \( M' \) is a principal ideal of \( V \) if and only if \( M_1 \) is a principal ideal of \( V_1 \).

(7) Let \( W_1 \) be the valuation ring of with maximal ideal \( N_1 \) on a valuation \( w_1 \) on \( k \). Let \( W = \phi^{-1}(W_1) \). Then \( W \) is a valuation ring of a valuation \( w \) on \( q(k[G]) \). Let \( N' \) be the maximal ideal of \( W \). Then \( N' \) is a principal ideal if and only if \( N_1 \) is a principal ideal of \( W_1 \).

(8) Let \( v_0 \) be a \( Z \)-valued valuation on a field \( k_0 \), and let \( k = q(k_0[Q]) \). Assume that \( v_1 \) is the natural extension of \( v_0 \) to \( q(k_0[Q]) \). Let \( V_1 \) be the valuation ring of \( v_1 \), and \( M_1 \) be the maximal ideal of \( V_1 \). Then \( v_1 \) is a \( Z \)-valued valuation, and \( M_1 \) is a principal ideal of \( V_1 \). For every element \( f = \sum a_iX^{r_i} \) of \( k_0[Q] \), put \( w_1(f) = \inf_i(r_i) \). Then \( w_1 \) is a valuation on \( q(k_0[Q]) \). Let \( W_1 \) be the valuation ring of \( W_1 \), and let \( N_1 \) be the maximal ideal of \( W_1 \). \( w_1 \) is a \( Q \)-valued valuation, and \( N_1 \) is not a principal ideal of \( W_1 \).

(9) Set \( D = V \cap W \), \( M = M' \cap D \) and \( N = N' \cap D \). Then \( D \) is a 4-dimensional Prufer domain with exactly two maximal ideals \( M \) and \( N \). \( D_M = V \) and \( D_N = W \). Put \( M_3 \cap D = P_3, M_2 \cap D = P_2 \) and \( M_1 \cap D = P_1 \). Then \( D_{P_3} = U_3, D_{P_2} = U_2 \) and \( D_{P_1} = U_1 \). \( M \cap N \not\subseteq P_3 \not\subseteq P_2 \not\subseteq P_1 \not\subseteq (0) \).

(1.5) Example. Let \( k \) be a field and \( y \) an indeterminate. Let \( q(k[y, X^\alpha] \mid \alpha \in R) = K \). Let \( v \) be the \( y \)-adic valuation, and \( V \) the valuation ring of \( v \). Let \( w \) be the natural extension of the identity mapping of \( R \), and \( W \) the valuation ring of \( w \). Set \( D = V \cap W \). Then \( |\Sigma(D)| = 2 \).

For the proof, let \( (a, b) \) and \( (c, d) \in Z \oplus R \). If \( a \leq c \) and \( b \leq d \), then set \( (a, b) \leq (c, d) \). Then \( \leq \) is a partial order on \( Z \oplus R \). Let \( I \) be an ideal \( \neq (0) \) of \( D \). Put \( \chi(I) = \{(w(F), v(F)) \mid 0 \neq F \in I\} \). Then, for non-zero ideal \( I_1 \) and \( I_2 \) of \( D \), \( I_1 \subset I_2 \) if and only if \( \chi(I_1) \subset \chi(I_2) \). If \( \chi(I) \) has a minimum element, then \( I \) is a principal ideal of \( D \). If \( \chi(I) \) does not have a minimum element, then \( \chi(I) \) has an infimum \((\alpha, n)\). Then, \( X^{-\alpha}y^{-n}I = N \), where \( N \) is the center of \( w \) on \( D \). It follows that \( |\Sigma(D)| = 2 \).

§2. On pseudo-radicals

[21, Theorem 4] proved that (3.17) is yes for domains. There is a conjecture
that almost all multiplicative ideal theories hold for $S$ ([13]). In the proof of [21, Theorem 4], [6] is used. In this section we will study a semigroup version of [6].

(2.1). If $M_1, M_2, M_3$ are prime ideals of $S[X]$ with $M_1 \subseteq M_2 \subseteq M_3$, then $M_1 \cap S \subseteq M_3 \cap S$. Also, if $M_1 \cap S = M_2 \cap S \neq \emptyset$ then $M_1 = M_1 + S[X]$. If $M_1 \cap S = \emptyset$, then $M_1$ is a minimal prime ideal of $S[X]$.

The intersection of all prime ideals of $S$ is called the pseudo-radical of $S$.

(2.2). If an extension semigroup $D_2$ is integral over its subsemigroup $D_1$, if $P_1^*$ is the pseudo-dradical of $D_1$, then $P_2^* \cap D_1 = P_1^*$.

(2.3). Let $P^*$ be the pseudo-radical $S$. Then the following statements are equivalent:

1. $P^* \neq \emptyset$.
2. $G$ is a simple semigroup extension of $S$.
3. $G$ is a finite semigroup extension of $S$.

(2.4). (1) Let $S$ be a Noetherian semigroup with the pseudo-radical $P^*$. If $P^* \neq \emptyset$, then $S$ need not be 1-dimensional.

(2) Let $S$ be a Krull semigroup with non-empty pseudo-radical $P^*$. Then $S$ need not be a principal ideal semigroup.

For example, let $k$ be a torsion-free abelian group $\neq \{0\}$, and let $k[X,Y]$ be the polynomial semigroup of $X$ and $Y$ over $k$. Then $k[X,Y]$ is 2-dimensional Noetherian semigroup with non-empty pseudo-radical. $k[X,Y]$ is integrally closed, and a Krull semigroup. The maximal ideal $(X,Y)$ of $k[X,Y]$ is not principal.

(2.5). Assume that $S$ is a valuation semigroup with non-empty pseudo-radical $P^*$. If $P_0$ is a height 1 prime ideal of $S$, then the complete integral closure $S^*$ of $S$ is $SP_0$; and $S^*$ is completely integrally closed.

PROOF. $SP_0$ is a valuation semigroup of rank 1, hence $SP_0$ is completely integrally closed. Let $v$ be the valuation associated with $S$, and let $\Gamma$ be the value group. Put $H = \{v(x) | S \ni x \not\in P_0$ or $S \ni -x \not\in P_0\}$. Then $H \not\subseteq \Gamma$, and $H$ is an isolated subgroup of $\Gamma$, and there is no isolated subgroup $H'$ of $\Gamma$ such that $H \subseteq H' \subseteq \Gamma$.

(2.6). Assume that $S \not\subseteq G$, and let $M$ be the maximal ideal of $S$. Then,

1. The maximal ideal of $S[X]$ is $(M, X)$.
2. $(M, X)$ is not a principal ideal of $S[X]$.
3. $\dim S[X] \geq 2$.

We have finished seeing a semigroup version of [6].

§3. On Krull semigroups

We will study a semigroup version of [21]. The following (3.1)~(3.13) were proved in [16].
(3.1). The followings are equivalent:

(1) $S$ is a Mori semigroup.
(2) For any integral ideal $I$, there exists a finitely generated ideal $J$ such that $J \subseteq I$ and $I^{-1} = J^{-1}$.

(3.2). For any fractional ideal $A$ of a Mori semigroup $S$, there exists a finitely generated fractional ideal $I$ such that $I \subseteq A$ and $A^{-1} = I^{-1}$.

(3.3). Let $V \subseteq q(V)$ be a valuation semigroup which is Mori. Then $V$ is discrete of rank 1.

Let $A$ be an ideal or an extension semigroup of a grading monoid $S$. Then the followings are equivalent:

(1) For any ideal $I, J$ of $S$, $(I \cap J) + A = (I + A) \cap (J + A)$.
(2) For any ideal $I$ of $S$ and for any $s \in S$, $(I : s)S + A = (I + A : s)A$.

If an ideal $A$ of $S$ satisfies the above conditions, then $A$ is called a flat ideal of $S$. If an oversemigroup $A$ of $S$ satisfies the above conditions, then $A$ is called a flat oversemigroup of $S$. If a flat $A$ satisfies $A + M \subseteq A$, then $A$ is called faithfully flat over $S$, where $M$ is the maximal ideal of $S$.

We set $\{I^v \mid I \in F(S)\} = D(S)$ and $\frac{D(S)}{\{(a) \mid a \in q(S)\}} = C(S)$.

(3.4). Let $S$ be a Mori semigroup and $T$ an extension semigroup which is flat over $S$. Then there exist natural mappings $\phi : F(S) \to F(T)$, $j : D(S) \to D(T)$ and $\overline{j} : C(S) \to C(T)$.

If $T$ is faithfully flat over $S$, then $\phi$ and $j$ are injective.
If $T$ is an oversemigroup of $S$, then $\phi$, $j$ and $\overline{j}$ are surjective.

(3.5). Let $A$ be a $v$-ideal of a Mori semigroup $S$. Then $A$ is the intersection of a finite number of principal fractional ideals of $S$.

(3.6). Let $S$ be a Mori semigroup and $T$ an extension semigroup which is flat over $S$. If $A$ is a $v$-ideal of $S$, then $A + T$ is a $v$-ideal of $T$.

(3.7). Let $S$ be a Mori semigroup and $T$ an oversemigroup which is flat over $S$. Then $T$ is a Mori semigroup.

(3.8). Let $S$ be a Krull semigroup and $T$ an oversemigroup which is flat over $S$. Then $T$ is a Krull semigroup.

If a fractional ideal $A$ of $S$ satisfies $A^v = A$ and $A + A^{-1} = A$, then $A$ is called strongly divisorial.

We denote the complete integral closure of $S$ by $S^*$.
An element $A$ of $F(S)$ is called an idempotent if $A + A = A$.

(3.9). The followings are equivalent:

(a) $F(S)$ contains a maximum idempotent $E$ under the inclusion.
(b) $F(S)$ contains a completely integrally closed oversemigroup $E$ of $S$.
(c) $S^* \in F(S)$.
(d) There exists a minimum member \( I \) among strongly divisorial fractional ideals of \( S \) under the inclusion.

(3.10). \( S \) is completely integrally closed if and only if \( S \) is a minimum member under the inclusion among the strongly divisorial fractional ideals of \( S \).

The intersection of all prime ideals of a grading monoid \( S \) is called the pseudo-radical of \( S \). Let \( T \) be an oversemigroup of \( S \). Then, if the pseudo-radical of \( S \) is nonempty, then that of \( T \) is also nonempty.

(3.11). Let the pseudo-radical of \( S \) be nonempty. Then

1. If \( V \) is a valuation oversemigroup of \( S \), then \( V^* \subseteq F(V) \).
2. If \( \mathcal{S} \) is the integral closure of \( S \), then \( (\mathcal{S})^* \subseteq F(\mathcal{S}) \).

For \( n \in \mathbb{N} \) and a subset \( I \) of \( S \), set \( nI = I + \cdots + I = \{a_1 + \cdots + a_n \mid a_i \in I\} \).

(3.12). Let \( S \) be a Mori semigroup, \( M \) a maximal ideal of \( S \) and \( \dim S = 1 \). Let \( x \in S \). Then,

1. There exists an \( n \in \mathbb{N} \) such that \( nM \subseteq (x) \).
2. \( M^{-1} \nsubseteq S \).

(3.13). Let \( S \) be a Mori semigroup, \( M \) a maximal ideal of \( S \) and \( \dim S > 1 \). Then either \( M^{-1} = S \) or \( M \) is strongly divisorial.

Let \( L(S) \) be the set of strongly divisorial fractional ideals of \( S \).

(3.14). Let \( S \) be a Mori semigroup and \( T \) an extension semigroup of \( S \) which is flat over \( S \). Then \( j(L(S)) \subseteq L(T) \). If \( T \) is an oversemigroup, then \( j(L(S)) = L(T) \).

**Proof.** Let \( A \in L(S) \). Then \( A + T \in D(T) \) by (3.6), and \( A^{-1} + T = (A + T)^{-1} \) by (3.4). Therefore \( A + T \in L(T) \).

(3.15). Let \( S \) be a Mori semigroup and \( P \) a prime ideal of \( S \). If \( \text{ht}(P) = 1 \), then \( P \) is divisorial. If \( \text{ht}(P) \geq 2 \), then \( P \) is either strongly divisorial or \( P^{-1} = S \).

**Proof.** For every prime ideal \( P \) of \( S \), \( S_P \) is a Mori semigroup and \( S_P \) is a faithfully flat oversemigroup of \( S \). If \( \text{ht}(P) = 1 \), then \( S_P \) is 1-dimensional. By (3.12), \( (P + S_P)^{-1} \nsubseteq S_P \). Hence \( P + S_P \) is a divisorial ideal of \( S_P \). It follows that \( P \) is a divisorial ideal of \( S \). Assume that \( \text{ht}(P) \geq 2 \) and \( P^{-1} \nsubseteq S \). Then \( \dim(S_P) \geq 2 \). Since \( j : D(S) \to D(S_P) \) is injective, \( P^{-1} + S_P \nsubseteq S_P \). By (3.13), \( P + S_P \) is a strongly divisorial ideal of \( S_P \). By (3.14), \( P \) is a strongly divisorial ideal of \( S \).

(3.16) ([5]). If \( S \) is a Mori semigroup which is completely integrally closed, then \( S \) is a Krull semigroup.

We denote \( q(S) \) by \( G \). And for every subset \( I \) of \( G \), we denote \((S : I)_G \) by \( S : I \).

Next let \( S \) be a 1-dimensional integrally closed Mori semigroup. We will study if \( S \) becomes a discrete valuation semigroup of rank 1. If \( S = S^* \), then \( S \) is a discrete valuation semigroup of rank 1. Suppose that \( S \nsubseteq S^* \). Let \( M \) be
the maximal ideal of $S$, and let $\{V_\lambda|\lambda\}$ be the set of valuation oversemigroups of $S$ with $V_\lambda \subseteq F$ for every $\lambda$. The proof of (3.11)(1) shows that $M + V_\lambda^* \subseteq V_\lambda$ for each $\lambda$. Hence $M + \cap_\lambda V_\lambda^* \subseteq S$. Let $x \in \cap_\lambda V_\lambda^*$ and let $p \in M$. Then $p + nx \in p + \cap_\lambda V_\lambda^* \subseteq S$ for each $n \in \mathbb{N}$. Hence $x \in S^*$. Therefore $\cap_\lambda V_\lambda^* = S^*$ and $M \subset S : S^* \subset S$. It follows that $M = S : S^*$. If $M + M^{-1} = S$, then $M$ is a finitely generated ideal of $S$. It follows that $S$ is a Noetherian semigroup, and hence $S^* = S = S$; a contradiction. Therefore $M + M^{-1} = M$. It follows that $S^* \subset M^{-1} = M : M = (S : S^*) : (S : S^*)$.

For every ideal $I$ of $S$, it is easy to see that $I : I$ is an idempotent element of $F(S)$. Therefore $M^{-1}$ is an idempotent. By the proof of (3.9), $S^*$ is the maximum idempotent element of $F(S)$. Hence $M^{-1} = S^*$. Now let $J \in D(S^*)$. Then $J \in F(S)$, and we have

$$J = S^* : (S^* : J) = M^{-1} : (M^{-1} : J) = M^{-1} : (M + J)^{-1}$$

$$= (M + (M + J)^{-1})^{-1}.$$ Hence $J \in D(S)$. Namely $D(S^*) \subset D(S)$. Therefore $S^*$ is a Mori semigroup. Next $S^*$ is a completely integrally closed semigroup. Hence $S^*$ is a Krull semigroup, and $S^* = W_1 \cap \cdots \cap W_n$ for discrete valuation semigroups $W_i$ of rank 1.

(3.17) PROBLEM. Assume that $S$ is a 1-dimensional integrally closed Mori semigroup. Is $S$ a discrete valuation semigroup of rank 1?

(3.18). Let $S$ be a Krull semigroup and $T$ an oversemigroup which is flat over $S$. Then the mapping $j : C(S) \rightarrow C(T)$ is bijective if and only if every divisorial prime ideal $P$ of $S$ such that $P + T = T$ is principal.

(3.19). Let $S$ be a Krull semigroup and $T$ an oversemigroup which is flat over $S$. If $S$ is a UFS, then $T$ is a UFS. If every divisorial prime ideal $P$ of $S$ such that $P + T = T$ is principal and if $T$ is a UFS, then $S$ is a UFS.

$S$ is called quasi-coherent if $I, J \in f(S)$ implies $I : J \in f(S)$. If $S$ is coherent, then $S$ is quasi-coherent.

(3.20). Assume that $S$ is a quasi-coherent Mori semigroup. Then $\bar{S} = S^*$.

(3.21). Assume that $S$ is a quasi-coherent integrally closed Mori semigroup. Then $S$ is a Krull semigroup.

(3.22). Let $T = S[X]$ be the polynomial semigroup of an indeterminate $X$ over $S$. Then there exist natural mappings $j : D(S) \rightarrow D(T)$ and $\bar{j} : C(S) \rightarrow C(T)$. $j$ is injective.

Let $I$ be an ideal of $S$. For every $n \in \mathbb{N}$, put $B_n = (nI)^{-1}$. Then $S(I) = \cup_n B_n$ is an oversemigroup of $S$. $S(I)$ is called the ideal transform of $I$ with respect to $S$.

(3.23). Let $S$ be a Krull semigroup, and $I$ an ideal of $S$. Then $S(I)$ is a Krull semigroup.

(3.24). Let $A$ be an ideal of $S$ and $B = A^v$.

(1) If $B + S(B) = S(B)$, then $S(A)$ is over $S$ and $S(A) = \cap S_P$ ($P$ is a prime ideal with $P \not\supset B$).
(2) If $S(A)$ is flat over $S$ and if $A^{-1} \in f(S)$, then $B + S(B) = S(B)$.

(3.25). If $A$ is invertible, then $S(A)$ is flat over $S$.

(3.26). Let $S$ be a Mori semigroup, $A$ an ideal of $S$ and $B = A^v$. Then $S(A)$ is flat over $S$ if and only if $B + S(B) = S(B)$.

**Proof.** We may assume that $A = (f_1, \ldots, f_n)$ is finitely generated. We have $S(A) = \cap_i S_{f_i}$. Set $S(A) = T$. Then $T_{f_i} = S_{f_i}$ for every $i$. Let $T(A + T)$ be the ideal transform of $A + T$ with respect to $T$. Then

$$T(A + T) = \cap_i T_{f_i} = \cap_i S_{f_i} = T.$$  

Therefore $T : (A + T) = T$. We confer (5.6).

$$B + T = j(B) = j(A^v) = (A + T)^v = T.$$  

Hence $B + S(B) = S(B)$.

(3.27). If $S$ is a 1-dimensional Mori semigroup. Then, for each ideal $A$ of $S$, $S(A) = G$.

§4. On reflexive semigroups

We will study a semigroup version of [22]. Let $S$ be a Mori semigroup. If every ideal generated by two elements is divisorial, then $S$ is called an $M$-semigroup.

(4.1)([23] AND [24]). Let $S \subset G$ be a Mori semigroup and $M$ the maximal ideal of $S$. Then the following conditions are equivalent:

1) $\dim S = 1$ and $M^{-1}$ is 2-generated.

2) $S$ is reflexive.

3) Each 2-generated ideal of $S$ is a $v$-ideal.

(4.2) (A PART OF [17, PROPOSITION 18]). Let $S$ be integrally closed. Then the followings are equivalent:

1) $S$ is a valuation semigroup.

2) Each 2-generated ideal of $S$ is divisorial.

The dimension of every $M$-semigroup is $\leq 1$ ([24, Theorem 1]).

(4.3). Let $S$ be a Mori semigroup of dimension 1 with maximal ideal $M$. If $M^{-1}$ is 2-generated, then $S$ is Noetherian and reflexive.

Because (4.1) implies that $S$ is reflexive. If $I_1 \subset I_2 \subset I_3 \subset \cdots$ is an ascending chain of ideals of $S$, since each $I_j$ is divisorial, there exists $n$ such that $I_n = I_{n+1} = \cdots$. 
Let $S$ be a Mori semigroup of dimension 1 with maximal ideal $M$. Then $M$ is divisorial.

The proof follows by (3.12)(2).

Let $\Sigma$ be a family $\neq \emptyset$ of ideals of $S$ which is additively closed. Then the subset $S_\Sigma = \{x \in G \mid x + I \subseteq S \text{ for some } I \in \Sigma\}$ of $G$ is an oversemigroup of $S$. $S_\Sigma$ is called the generalized quotient semigroup of $S$ with respect to $\Sigma$.

(4.5). (1) Let $I$ be an ideal of $S$. Put $\Sigma = \{I, 2I, 3I, \cdots\}$. Then $S_\Sigma = \bigcup_1^\infty (nI)^{-1}$ is the ideal transform $S(I)$ of $I$.

(2) Let $\Pi$ be a family $\neq \emptyset$ of prime ideals of $S$. Let $\Sigma$ be the set of ideals $I$ of $S$ such that $I \notin P$ for each $P \in \Pi$. Then $S_\Sigma = \cap \{S_P \mid P \in \Pi\}$.

(3) Let $T$ be an oversemigroup of $S$ which is flat over $S$. Then $T$ is a generalized quotient semigroup.

For a fractional ideal $A$ of $S$, put $A_\Sigma = \{x \in G \mid x + I \subseteq A \text{ for some } I \in \Sigma\}$. Then $A_\Sigma$ is a fractional ideal of $S_\Sigma$, and $A + S_\Sigma \subseteq A_\Sigma$.

(4.6). Let $A, B, A_1, \cdots, A_m$ be ideals of $S$. Then,

1. $(\cap_i A_i) + B_\Sigma \subseteq \cap_i (A_i + B_\Sigma) \subseteq \cap_i (A_i + B)_\Sigma$.
2. $A_\Sigma + B_\Sigma \subseteq (A + B)_\Sigma$.
3. $(A : B)_\Sigma \subseteq A_\Sigma : B_\Sigma$.
4. If $B$ is finitely generated, then $A_\Sigma : B_\Sigma = (A : B)_\Sigma = A_\Sigma : (B + S_\Sigma)$.

(4.7). Let $S$ be a Mori semigroup, and let $T$ be the generalized quotient semigroup of $S$ with respect to $\Sigma$. Then there exists natural mapping $j$ of $D(S)$ onto $D(T)$. Also there exists natural mapping $\tilde{j}$ of $C(S)$ onto $C(T)$. Let $\nu$ (resp. $\nu'$) be the $\nu$-operation on $S$ (resp. $\nu'$). Then, for each $A \in F(S)$, we have $j(A^\nu) = (A_\Sigma)^{\nu'} = (A + T)^{\nu'} = (A^\nu)_\Sigma$.

(4.8). Let $T$ be the generalized quotient semigroup of $S$ with respect to $\Sigma$.

1. If $S$ is a Mori semigroup, then $T$ is a Mori semigroup.
2. If $S$ is a Krull semigroup, then $T$ is a Krul semigroup.
3. If $S$ is a UFS, then $T$ is a UFS.

For the proof, if $S$ is a Krull semigroup, then $D(S)$ is a group. Since $j(D(S)) = D(T)$, $D(T)$ is a group, and $T$ is completely integrally closed.

(4.9). Let $T$ be an oversemigroup of $S$, which is flat over $S$. If $S$ is Noetherian reflexive, then $T$ is Noetherian reflexive.

Proof. By (3.7), $T$ is a Mori semigroup. Let $\nu$ (resp. $\nu'$) be the $\nu$-operation on $S$ (resp. $T$). Let $A'$ be an ideal of $T$. Put $A' \cap S = A$. The proof of (4.7) shows that $(A + T)^{\nu'} = (A')^{\nu'}$. Since $A^\nu = A$, $(A + T)^{\nu'} = A + T$ by (3.6). Hence $(A')^{\nu'} = A + T \subseteq A'$. Therefore $T$ is reflexive.

(4.10). Let $S$ be a reflexive semigroup with maximal ideal $M$. If $M$ is principal, then $S$ is a valuation semigroup.

Proof. Set $M = (a)$. Assume that $\tilde{S} \nsubseteq S$. Take $x \in \tilde{S} - S$. Then $S : S[x] \subseteq M$. Since $S[x] \in F(S)$, we have $S[x] \supseteq (-a)$. Hence $M + S[x] = S[x]$.
Since $x$ is integral over $S$, we have $S[x] = \bigcup_{i=1}^{n} (f_i + S)$ for some $f_i \in S[x]$. We may assume that $f_1 = m_1 + f_2, f_2 = m_2 + f_3, f_3 = m_3 + f_4, \cdots$ for some $m_i \in M$. Thus we have $f_j = m_j + f_k$ for some $j > k$. We may assume $k = 1$. Adding these $j$ equations, we have $0 = m_1 + \cdots + m_j$; a contradiction. Therefore $S$ is integrally closed. By (4.2), $S$ is a valuation semigroup.

Let $S$ be a reflexive semigroup with maximal ideal $M$. If $M$ is invertible, then $M$ is principal. By (4.10), $S$ is a valuation semigroup.

(4.11). Let $S$ be a reflexive semigroup with maximal ideal $M$. Assume that $M + M^{-1} = M$. Then,

(1) $T = M^{-1}$ is an oversemigroup of $S$ with $T \supsetneq S$.
(2) There exist no oversemigroups $T'$ such that $S \subsetneq T' \subsetneq T$.
(3) $T \subset \bar{S} \subsetneq G$.
(4) For each $\alpha \in T - S$, $T = S \cup (\alpha + S)$ and $2\alpha \in S$.

If $\bar{S} = G$, then $S = G$; a contradiction. If $2\alpha \in \alpha + S$, then $\alpha \in S$; a contradiction. Hence $2\alpha \in S$.

(4.12). In (4.11), let $A'$ and $B'$ be distinct ideals of $T$ such that $A' \cap S = B' \cap S$. Then $A' \cap B' \subset S$.

**Proof.** Suppose that $A' \cap B' \not\subset S$. Choose $\alpha \in (A' \cap B') - S$. By (4.11)(2), $T = S[\alpha]$. We may assume that $A' \not\subset B'$. Let $x \in A' - B'$. Then $x = s + n\alpha$ for $s \in S$ and $n \in N$. Hence $x \in B'$; a contradiction.

(4.13). In (4.11), let $M_1$ be the maximal ideal of $T$. Then,

(1) $M_1 \cap S = M$.
(2) $M$ is an ideal of $T$.
(3) There exist no ideals $J$ of $T$ such that $M_1 \supsetneq J \supsetneq M$.

**Proof.** By (4.11)(3), $T$ is integral over $S$. Hence $M_1 \cap S = M$. $M_1 + (S : M_1) \subset M_1 + (S : M) = M_1$. Therefore $M_1 + (S : M_1) \subset M$. Note that $F(T) \subset F(S) = D(S)$. It follows that $S : M_1$ is an ideal of $T$. Since $M + T = M + M^{-1} = M$, $M$ is an ideal of $T$.

(4.12) implies that there exist no ideals $J$ of $T$ such that $M_1 \supsetneq J \supsetneq M$.

(4.14). In (4.11), assume that $M_1 \supsetneq M$. Then,

(1) $S : M_1 = M_1$.
(2) $2M_1 \subset M$.
(3) If $M_1$ is not a principal ideal of $B$, then $M + M_1 = 2M_1$.
(4) If $M_1$ is a principal ideal of $B$, then $M = 2M_1$.

**Proof.** $M + M_1 \subset M + T = M$. Hence $M \subset S : M_1$. Therefore either $S : M_1 = M$ or $S : M_1 = M_1$. If $S : M_1 = M$, then $M_1 = M^{-1} = T$; a contradiction. Hence $S : M_1 = M_1$.

$2M_1 = M_1 + (S : M_1) \subset S$. Since $0 \not\in 2M_1$, we have $2M_1 \subset M$.

It follows that $(M_1 : M) + M_1 + M \subset S$. Hence $(M_1 : M) + M_1 \subset T$. Assume that $M_1$ is not a principal ideal of $T$. Then $(M_1 : M) + M_1 \subset M_1$. Hence $M_1 : M = M_1 : M_1$. Since $M_1 = S : M_1$, we have $S : (M_1 + M) = S : (M_1 + M_1)$. Therefore $M_1 + M = 2M_1$. 

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Assume that $M_1$ is a principal ideal of $T$. If $(M_1 : M) + M_1 \subset M_1$, we have $M = M_1$; a contradiction. Therefore $(M_1 : M) + M_1 = T$. We have $0 = \alpha + (-\alpha)$ for $\alpha \in M_1$ and $-\alpha \in M_1 : M$. Then $M \subset \alpha + M_1$ and $M_1 = \alpha + T$. It follows that $M \subset \alpha + M_1 = \alpha + \alpha + T = M_1 + M_1 \subset M$; hence $2M_1 = M$.

(4.15). In (4.14), if $M_1$ is a principal ideal of $T$, then $T$ is a reflexive valuation semigroup, and $T = S$.

**Proof.** We have $M_1 = \alpha + T$ for some $\alpha \in M_1$.

Let $J$ be an ideal of $T$. Then $M = 2M_1 = 2\alpha + T$ and $J + M = J + 2\alpha + T = 2\alpha + J$. We have

$$T : (T : J) = (S : M) : ((S : M) : J) = (S : (S : (J + M))) : M = (J + M) : M = (2\alpha + J) : (2\alpha + T) = J : T = J.$$ 

Therefore $T$ is reflexive. By (4.10), $T$ is a valuation semigroup.

(4.16). In (4.11), $M$ is not a cancellation ideal of $S$.

**Proof.** If $M$ is a cancellation ideal of $S$, then $M = (a)$ is principal ([16, (8.2)]). $M + M^{-1} = S$; a contradiction.

§5. On cancellation ideals

[2] posed the problem whether a cancellation ideal of a local domain is principal. The following (5.1)～(5.3) were proved in [16].

(5.1). Every cancellation ideal of a grading monoid $S$ is principal.

[2] holds for $S$([25, Proposition 5]). For example,

(5.2). All ideals of $S$ are cancellation ideals if and only if $S = G$ or $S$ is a discrete valuation semigroup of rank 1.

(5.3). (1) Assume that $S$ has a unique maximal ideal $M$ and satisfies the ascending chain condition on principal ideals. If $M$ is a cancellation ideal, then $S$ is a discrete valuation semigroup of rank 1.

(2) Assume that a valuation semigroup $S$ has a unique maximal ideal $M$ which is principal. Then $S$ need not be a discrete valuation semigroup of rank 1.

Every faithfully flat ideal of $S$ is principal (cf. [14, Remark 3, (1)]).

(5.4). Let $A$ be an ideal of $S$ or an oversemigroup of $S$. If $A$ is faithfully flat over $S$, then $A$ is a principal ideal of $S$. If $A$ is an oversemigroup, then $A = S$.

**Proof.** Choose $A \ni a \not\in A + M$. Let $a' \in A$. Then $a + a' \in ((a) + A) \cap ((a) + A) = (a \cap (a')) + A$. Hence $a + a' = c + a''$ for some $c \in (a) \cap (a')$. $c = a + s = a' + s'$ for $s, s' \in S$. $a = s' + a''$. By the choice of $a$, $s'$ is a unit of $S$. Then $a' \in (a)$. Therefore $A = (a)$. If $A$ is an oversemigroup, then $a + a \in (a)$ implies $a \in S$.

As a ring version of (5.4),
(5.5). Let $D$ be a local domain with maximal ideal $M$. Let $E$ be an overring of $D$. If $E$ is faithfully flat over $D$, then $E = D$.

This seems to have been known.

Let $R$ be a commutative ring with zero-divisors, and $q(R)$ be the total quotient ring of $R$. A non-zero-divisor of $R$ is called a regular element of $R$. If an ideal $I$ of $R$ contains at least one regular element, then $I$ is called a regular ideal of $R$. If every regular ideal of $R$ is generated by regular elements, then $R$ is called a Marot ring. Let $P$ be a prime ideal of $R$. Then the overring $\{x \in q(R) \mid sx \in R$ for some $s \in R - P\}$ of $R$ is denoted by $R_{[P]}$. In the proof of the following (5.6), we confer [8, Section 4].

(5.6). Let $A$ be a Marot ring. Let $T$ be an overring of $A$ which is faithfully flat over $A$. Then $T = A$.

**Proof.** Let $M$ be a regular maximal ideal of $A$. Then $T \supseteq MT$. Let $N$ be a maximal ideal of $T$ with $N \supseteq MT$. Then $T_{[N]} = A_{[M]}$. Since $TA_{[M]}$ is flat over $A_{[M]}$, either $TA_{[M]} = MTA_{[M]}$ or $A_{[M]} \supset T$. Suppose that $TA_{[M]} = MTA_{[M]}$. Then $A_{[M]} = MA_{[M]}$; a contradiction. Therefore $A_{[M]} \supset T$. Hence $T = A$.

(5.6) seems to have been known.

§6. On units of semigroup rings

The following (6.1) ~ (6.5) were proved in [10].

(6.1). Assume that $R$ is indecomposable. Let $\sum_{1}^{n} a_i X^{\alpha_i}$ be the canonical form of a nonzero element $f$ of $R[X; S]$.

1. $f \in V(R[X; S])$ if and only if $\sum a_i = 1$, $a_k \equiv 1(N)$ and $\alpha_i \equiv 0(N)$ for all $i \neq k$.

2. $f \in W(R[X; S])$ if and only if $\sum a_i = 1$, the coefficient of degree 0 is 1 modulo $N$, and the other coefficients are 0 modulo $N$.

(6.2). (1) Assume that $R$ is reduced and indecomposable. Then we have $V(R[X; S]) = X^H$ and $W(R[X; S]) = 1$.

(2) $W(R[X; S]) = 1$ if and only if $R$ is reduced and indecomposable.

(6.3). Assume that $R$ is reduced and indecomposable. Then the following is equivalent:

1. $U(R[X; S])$ is free modulo torsion;

2. $U(R)$ is free modulo torsion and $H$ is free.

(6.4). Assume that $R$ is indecomposable and not reduced. Then $W(R[X; S])$ is not a finitely generated free abelian group.

(6.5). Assume that $R$ is indecomposable. Then the following is equivalent:

1. $U(R[X; S])$ is a finitely generated free abelian group modulo torsion.
(2) \(U(R)\) is a finitely generated free abelian group modulo torsion, \(H\) is a finitely generated free abelian group, and \(N = M\).

(6.6). Let \(e_1, \ldots, e_n\) be nonzero idempotents of \(R\) such that \(e_1 + \cdots + e_n = 1\) and \(e_i e_j = 0\) for \(i \neq j\), where \(n \geq 2\). Set
\[H \oplus \cdots \oplus H \equiv \{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in H\} \supset D = \{(\alpha, \ldots, \alpha) \mid \alpha \in H\}.
Then,
(1) \(U(R[X;S]) \equiv U(Re_1[X;S]) \oplus \cdots \oplus U(Re_n[X;S])\).
(2) \(V(R[X;S]) \equiv V(Re_1[X;S]) \oplus \cdots \oplus V(Re_n[X;S])\).
(3) \(U(R[X;S]) \equiv \frac{H \oplus \cdots \oplus H}{D} \oplus \cdots \oplus \frac{U(Re_1[X;S])}{D}\).
(4) \(U(R[X;S])\) is free modulo torsion if and only if, for each \(i\), \(U(Re_i[X;S])\) is free modulo torsion.
(5) \(U(R[X;S])\) is a finitely generated free abelian group modulo torsion if and only if, for each \(i\), \(U(Re_i[X;S])\) is a finitely generated free abelian group modulo torsion.

(6) \(W(R[X;S]) \equiv \frac{H \oplus \cdots \oplus H}{D} \oplus W(Re_1[X;S]) \oplus \cdots \oplus W(Re_n[X;S])\).

(7) \(H\) has a free complement in \(V(R[X;S])\) if and only if \(\frac{H \oplus \cdots \oplus H}{D}\) is free and, for each \(i\), \(H\) has a free complement in \(V(Re_i[X;S])\).
(8) \(H\) has a free complement in \(V(R[X;S])\) if and only if \(V(R[X;S])\) is free.

Proof (6) For each \(i\), there is a natural isomorphism of \(V(Re_i[X;S])\) on to \(X^H e_i \otimes W(Re_i[X;S])\). Hence there exists a natural isomorphism \(\sigma\) of \(V(R[X;S])\) onto \(X^H e_1 \otimes \cdots \otimes X^H e_n \otimes W(Re_1[X;S]) \oplus \cdots \otimes W(Re_n[X;S])\). Then the image \(\sigma(X^H)\) is \(\{(X^{\alpha_1}, \ldots, X^{\alpha_n}, e_1, \ldots, e_n) \mid \alpha \in H\}\). Hence
\[V(R[X;S]) \equiv \frac{X^H e_1 \otimes \cdots \otimes X^H e_n}{D} \oplus W(Re_1[X;S]) \oplus \cdots \oplus W(Re_n[X;S]).\]

(8) Put \(K_2 = \{(0, \alpha, 0, \cdots, 0) \mid \alpha \in H\}\), \(\cdots\), \(K_n = \{(0, 0, 0, \cdots, \alpha) \mid \alpha \in H\}\). Then \(H \oplus \cdots \oplus H = D \oplus K_2 \oplus \cdots \oplus K_n\). Hence \(\frac{H \oplus \cdots \oplus H}{D} \cong K_2 \oplus \cdots \oplus K_n\).
If \(H\) has a free complement in \(V(R[X;S])\), then \(K_2\) is free by (7). Hence \(H\) is free. Therefore \(V(R[X;S])\) is free.

The semigroup version posed in [12] of the Karpilovsky’s problem [11, Chapter 7, Problem 9] reduces to the following,

(6.7) PROBLEM. Assume that \(R\) is indecomposable. Find necessary and sufficient conditions for \(H\) to have a free complement in \(V(R[X;S])\).

§7. On \(r\)-GCD rings

D.D. Anderson posed fifteen problems at the problem session at the Midwest-Great Planes Commutative Algebra Workshop held at The University of Missouri-Columbia, June 21-22, 1991. The eighth problem of them is the following:
If $a$ and $b$ are regular elements of a ring $R$ and $a$ and $b$ have an LCM, then they have a GCD. However, even in an integral domain, $a$ and $b$ may have a GCD but not an LCM. But in an integral domain, every pair of elements has a GCD if and only if every pair of elements has an LCM. Let $R$ be a ring in which every pair of regular elements has a GCD; does every pair of regular elements have an LCM? What if we add ACC on regular principal ideals?

Let $R$ be a commutative ring $\mathfrak{R}$. Let $d$ and $e$ be elements of $R$. If $e = dx$ for some element $x$ of $R$, then $d$ is called a divisor of $e$. In this case, $e$ is called a multiple of $d$.

Let $a$ and $b$ be elements of $R$, and let $d$ be a common divisor of $a$ and $b$. If every common divisor $d'$ of $a$ and $b$ is a divisor of $d$, then $d$ is called a greatest common divisor (for short, GCD) of $a$ and $b$. Let $e$ be a common multiple of $a$ and $b$. If every common multiple $e'$ of $a$ and $b$ is a multiple of $e$, then $e$ is called a least common multiple (for short, LCM) of $a$ and $b$.

Non-zerodivisors of $R$ is called regular elements of $R$. If every pair of regular elements of $R$ has a GCD, then we will call $R$ an r-GCD ring ("r" means "regular"). If every pair of regular elements of $R$ has an LCM, then we will call $R$ an r-LCM ring. If $R$ satisfies ascending chain condition on regular principal ideals, then $R$ is said to satisfy a.c.c.r.p.

The eighth problem of D.D. Anderson is

(7.1). Is an r-GCD ring an r-LCM ring? What if we add a.c.c.r.p? 

(7.2). If $a$ and $b$ are regular elements of $R$ and $a$ and $b$ have an LCM, then they have a GCD. However, even in an integral domain, $a$ and $b$ may have a GCD but not an LCM. But in an integral domain, every pair of elements has a GCD, if and only if every pair of elements has an LCM.

Let $D = k[X_2, X_3, X_4, \ldots]$ where $k$ is a field and $X$ an indeterminate. Then GCD($X_2, X_3$) = 1, but LCM($X_2, X_3$) does not exist.

(7.3). If $R$ is an r-LCM ring, then $R$ is an r-GCD ring.

Let $a, b \in R$. If there exists a unit $u$ of $R$, such that $a = bu$, then $a$ and $b$ are called associated.

Let $a \in R$. Assume that $a$ is not 0, $a$ is not a unit, and if $a = bc$, then $b$ or $c$ is a unit of $R$. Then $a$ is called an irreducible element of $R$.

If every regular element $a$ of $R$ is expressible as $a = p_1p_2 \cdots p_n$ for irreducible elements $p_i$ uniquely up to associates and order, then $R$ is called an r-UFR.

(7.4). $R$ is an r-UFR if and only if $R$ is an r-GCD ring and $R$ satisfies a.c.c.r.p.

PROOF. The sufficiency: Then each regular element $a$ of $R$ is a product of irreducible elements. Suppose that $a = p_1 \cdots p_n = q_1 \cdots q_m$, where every $p_i$ and $q_j$ are irreducible elements. We will show by induction on $n$ that $n = m$ and, under a suitable order, every $p_i$ and $q_i$ are associated. If $q_1$ and $p_i$ are associated for some $i$, then the proof is over. Suppose the contrary. Then GCD($p_1, q_1$) = 1. Then GCD($p_1q_2 \cdots q_m, q_1q_2 \cdots q_m$) = $q_2 \cdots q_m$. Since $p_1$ is a common divisor of
$p_1 q_2 \cdots q_m$ and $a$, we have $q_2 \cdots q_m = p_1 r$ for some $r \in R$. Then $p_2 \cdots p_n = q_1 r$.
By induction, $q_1$ is associated with some $p_j$; a contradiction.

(7.5). Let $R$ be an r-UFR, $a$ a regular element of $R$ and $x$ a nilpotent of $R$. Then $x = ar$ for some $r \in R$.

Proof Let $a = p_1 \cdots p_n$ be an irreducible decomposition of $a$. We have $x^m = 0$ for some $m$. Then $(p_1 + x)^m = p_1 x_1$ for some $x_1 \in R$. Hence $p_1 + x = p_1 x_2$ for some $x_2 \in R$. Hence $x = p_1 x_3$ for some $x_3 \in R$. We will rely on induction on $n$. We have $x = p_1 \cdots p_n r_1$ for some $r_1 \in R$. Then $r_1$ is a nilpotent of $R$. Hence $r_1 = p_n r_2$ for some $r_2 \in R$. Then $x = p_1 \cdots p_n r_2$.

(7.6) EXAMPLE. Let $D$ be a Dedekind domain, \{${P_0, P_\lambda \mid \lambda}$\} the set of distinct maximal ideals of $D$, $M = \sum_{\lambda} D_{P_\lambda}$ a $D$-module of the restricted direct sum, $R = D \oplus M$ the semidirect product of $D$ and the $D$-module $M$. Then $R$ is an r-LCM ring.

Proof. Let $a \in D$ be a regular irreducible element of $R$. Then $aD = P_0^e$ with $e \geq 1$. Hence $R$ has only one regular irreducible element $a$ up to associates.

(7.7) EXAMPLE. Let $D = \mathbb{Z}[\sqrt{-5}]$ be the ring of algebraic integers of $\mathbb{Q}(\sqrt{-5})$, $P_1 = (2, 1 + \sqrt{-5})$, $P_2 = (3, 1 - \sqrt{-5})$; \{${P_\lambda \mid \lambda}$\} be the other maximal ideals of $D$; $M = \sum_{\lambda} D_{P_\lambda}$; $R = D \oplus M$. Then $R$ is not an r-GCD ring.

Proof. $2D = P_2^2$, $(1 - \sqrt{-5})D = P_1 P_2$, $(2 + \sqrt{-5})D = P_2^2$, $1 - \sqrt{-5}$, $2 + \sqrt{-5}$ are regular irreducible elements which are not associated each other. $-2(2 + \sqrt{-5}) = (1 - \sqrt{-5})^2$.

(7.8) EXAMPLE. Let $D$ be a Dedekind domain, \{${P, Q, P_\lambda \mid \lambda}$\} the distinct maximal ideals, $M = \sum \oplus_{\lambda} D_{P_\lambda}$, $R = D \oplus M$. If $R$ is an r-UFR, then $R$ is an r-LCM ring.

Proof. Let $x \in D$ be a regular irreducible element of $R$. If $x$ is the only regular irreducible element of $R$ (up to associates), $R$ is an r-LCM ring. Let $y \in D$ be a regular irreducible element of $R$ which is not associated with $x$. $x = P^{e_1} Q^{e_1}$, $y = P^{e_2} Q^{e_2}$. Then we have $P^{e_3} = x_3, Q^{e_3} = y_3$ for some $e_3, e_3 \geq 1$ and $x_3, y_3 \in D$. We may assume that $x_3$ and $y_3$ are irreducible elements of $R$. If $x_3$ and $y_3$ are only regular irreducible elements of $R$ (up to associates), $R$ is an r-LCM ring. Suppose that $a \in D$ is a regular irreducible element of $R$ which is not associated with any of $x_3$ and $y_3$. $a = P^{e_4} Q^{e_4}$, where $e_4 < e_3, e_4 < e_3$. Then $xy = a P^{e_4 - e_4} Q^{e_3 - e_4}$. Hence $R$ is not an r-UFR.

References

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