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The behaviour of generalized denominator ideals
in anti-integral extensions

SUSUMU ODA* and KEN-ICHI YOSHIDA**

Let $R$ be a Noetherian domain and $R[X]$ a polynomial ring. Let $\alpha$ be an
element of an algebraic field extension $L$ of the quotient field $K$ of $R$ and let
$\pi : R[X] \to R[\alpha]$ be the $R$-algebra homomorphism sending $X$ to $\alpha$. Let $\varphi_{\alpha}(X)$
be the monic minimal polynomial of $\alpha$ over $K$ with $\deg \varphi_{\alpha}(X) = d$ and write

$$\varphi_{\alpha}(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$$

Then $\eta_i$ $(1 \leq i \leq d)$ are uniquely determined by $\alpha$. Let $I_{\eta_i} := R : R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^{d} I_{\eta_i}$, the latter of which is called a generalized denominator ideal of $\alpha$. We say
that $\alpha$ is an anti-integral element if $\ker \pi = I_{[\alpha]} \varphi_{\alpha}(X) R[X]$. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal of $R$ generated by the
coefficients of $f(X)$. For an ideal $J$ of $R[X]$, let $C(J)$ denote the ideal
generated by the coefficients of the elements in $J$. If $\alpha$ is an anti-integral
element, then $C(\ker \pi) = C(I_{[\alpha]} \varphi_{\alpha}(X) R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Let $\bar{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1})$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in \text{Dp}_1(R) := \{ p \in \text{Spec}(R) \mid \text{depth } R_p = 1 \}$, then $\alpha$ is called a super-primitive
element. It is known that a super-primitive element is an anti-integral ele-
ment (cf. [OSY, (1.12)]). It is known that any algebraic element over a Krull
domain $R$ is anti-integral over $R$ (cf. [OSY, (1.13)]). When $\alpha$ is an element in $K$,
$\varphi_{\alpha}(X) = X - \alpha$. Then $I_{[\alpha]} = I_{\alpha} := R : R \alpha$, a denominator ideal of $\alpha \in K$. Put
$J_{\alpha} := J_{[\alpha]}$. So we have $J_{\alpha} = J_{[\alpha]} = I_{[\alpha]}(1, \alpha) = I_{\alpha}(1, \alpha) = I_{\alpha} + \alpha I_{\alpha} = I_{\alpha} + I_{\alpha-1}$.

Unless otherwise specified, we use the above notation, and our general
reference for unexplained technical terms are [M].

Let $\alpha$ be an anti-integral element of degree $d$ over $R$. Consider the following
statements :

(i) $I_{[\alpha]} R[\alpha] = R[\alpha]$ ;
(ii) $R[\alpha]$ is flat over $R$ ;
(iii) $R[\alpha]$ is unramified over $R$.

In [KY], we have shown that the statements above are all equivalent if $d = 1$, i.e., $\alpha \in K$ (cf. [KY, Theorem 12]). It seems natural to ask whether the above
equivalences are valid in the case $d > 1$. Our objective is to investigate the
implications among (i) — (iv).

We begin with the following known result.

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PROPOSITION 1. Let $\alpha$ be an anti-integral element of degree $d$ over $R$. Then the implications $$(i) \Rightarrow (ii), (iii) \Rightarrow (ii) \text{ and } (iii) \Rightarrow (i)$$ hold.

PROOF. The implication $(i) \Rightarrow (ii)$ is seen in [S]. The implications $(iii) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are seen in [KY].

EXAMPLE 2. We construct an example which shows that the implication $(ii) \Rightarrow (i)$ is not always valid. Let $k$ be a field and let $R := k[x,y]$ be a polynomial ring. Let $\alpha$ be an algebraic element whose minimal polynomial is $\varphi_\alpha(X) = X^3 + (1/x)X^2 + (1/y)X + (1/xy) \in k(x,y)[X]$. Then $\alpha$ is anti-integral over $R$ of degree 3. We have $I_{[\alpha]} = xyR$ and hence $I_{[\alpha]}R[\alpha] = xyR[\alpha] \neq R[\alpha]$. It is easy to see that $J_{[\alpha]} = R$. Hence $R[\alpha]$ is flat over $R$ by [OSY, (3.4)].

EXAMPLE 3. ([SOY, §3 Example]). We know that the implication $(ii) \Rightarrow (iii)$ is not always valid as follows. Let $\beta \in L$ satisfy $\beta^d = a \in R$ and $a \notin R^d$ with $[K(\beta) : K] = d$ for $d > 1$. Then $\varphi_\beta(X) = X^d - a$ and $\varphi'_\beta(\beta) = d\beta^{d-1} \notin R[\beta]^d$. Thus $R[\beta]$ is integral and hence flat over $R$ by [OSY, (2.5)] but not unramified over $R$ (cf. [SOY]).

REMARK 4. If $I_{[\alpha]} = R$, then $R[\alpha]$ is integral over $R$ by [OSY]. In this case, $I_{[\alpha]}R[\alpha] = R[\alpha]$. But in general, we cannot say that $R[\alpha]$ is unramified over $R$. In fact, when $I_{[\alpha]}R[\alpha] = R[\alpha]$, $R[\alpha]$ is unramified over $R$ if and only if $\varphi'_\alpha(\alpha)R[\alpha] = R[\alpha]$ (cf. [KY]). But $\varphi'_\alpha(\alpha)$ is not always a unit in $R[\alpha]$ even if $R[\alpha]$ is integral over $R$. So the implication $(i) \Rightarrow (iii)$ does not always hold.

Now we shall give an equivalent condition under which $I_{[\alpha]}R[\alpha] = R[\alpha]$ holds. Let $\alpha$ be an element algebraic over $R$ and put $A = R[\alpha]$. For an ideal $N$ of $R$, put $V(N) := \{ p \in \text{Spec}(R) \mid N \subseteq p \}$. Let $\Delta_{A/R} := \{ p \in \text{Spec}(R) \mid pA = A \}$ and let $\Gamma_{J_{[\alpha]}} := \{ p \in \text{Spec}(R) \mid p + J_{[\alpha]} = R \}$ (See [KY]).

PROPOSITION 5. Assume that $\alpha$ is an anti-integral element over $R$ of degree $d$. Then the following statements are equivalent:

1. $I_{[\alpha]}R[\alpha] = R[\alpha]$;
2. $\sqrt{J_{[\alpha]}} = \sqrt{I_{[\alpha]}}$ and $J_{[\alpha]} = R$.

If (2) holds, then $\text{Im}[\text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)] = \text{Spec}(R) \setminus V(I_{[\alpha]})$, a open subset in $\text{Spec}(R)$.

PROOF. Let $A := R[\alpha]$.

(1) $\Rightarrow$ (2): Since $A$ is flat over $R$ by Proposition 1, we have $J_{[\alpha]} = R$ by [OSY, (2.6)]. Thus $\Delta_{A/R} = V(J_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}} = V(J_{[\alpha]})$ (cf. [KY, Theorem 7]) and hence $I_{[\alpha]} \subseteq J_{[\alpha]}$. So we have $\Delta_{A/R} \subseteq V(I_{[\alpha]})$. Since $I_{[\alpha]}A = A$ by assumption, we have $V(I_{[\alpha]}) \subseteq \Delta_{A/R}$. Hence $V(I_{[\alpha]}) = \Delta_{A/R} = V(J_{[\alpha]})$, which means that $\sqrt{J_{[\alpha]}} = \sqrt{I_{[\alpha]}}$.

(2) $\Rightarrow$ (1): From the assumption, it follows that $\Delta_{A/R} = V(J_{[\alpha]}) \cap \Gamma_{J_{[\alpha]}} = V(J_{[\alpha]}) = V(I_{[\alpha]})$. Hence $I_{[\alpha]}A = A$.

Next assume (2) holds. We know $\text{Im}[\text{Spec}(R[\alpha]) \rightarrow \text{Spec}(R)] = (\text{Spec}(R) \setminus V(J_{[\alpha]})) \cup V(J_{[\alpha]})$ by [KY, Theorem 7]. So $(\text{Spec}(R) \setminus V(J_{[\alpha]})) \cup V(J_{[\alpha]}) = \text{Spec}(R) \setminus V(I_{[\alpha]})$ by (2). Q.E.D.
REMARK 6. Assume that \( \alpha \) is anti-integral over \( R \) of degree \( d \). If \( I_{[\alpha]} R[\alpha] = R[\alpha] \), then \( I_{[\alpha]} \) is an invertible ideal of \( R \) because \( R = J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \cdots, \eta_d) \) by Proposition 5.

PROPOSITION 7. Assume that \( \alpha \) is anti-integral over \( R \) of degree \( d \). Consider the following statements:

1. \( I_{[\alpha]} R[\alpha] = R[\alpha] \);
2. \( I_{[\alpha]} + I_{[\alpha^{-1}]} = R \);
3. \( I_{[\alpha]} = I_{\eta_d} \) and \( J_{\eta_d} = R \).

Then the implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) hold.

PROOF. (1) \( \Rightarrow \) (2): Suppose that \( I_{[\alpha]} + I_{[\alpha^{-1}]} \nsubseteq R \). Take \( p \in \text{Spec}(R) \) such that \( I_{[\alpha]} + I_{[\alpha^{-1}]} \nsubseteq p \). Then \( I_{[\alpha]} R_p = a R_p \) for some \( a \in I_{[\alpha]} \) by Remark 6. Put \( a_0 = a, a \eta_i = a_i \) (\( 1 \leq i \leq d \)). Then we have \( a_0 \alpha^d + a_1 \alpha^{d-1} + \cdots + a_d = 0 \). Since \( I_{[\alpha]}(1, \eta_1, \cdots, \eta_d) = R \) and \( \sqrt{J[a]} = \sqrt{I[a]} \), \( a_d \) is an invertible element in \( R_p \). Moreover we have \( a_0, \cdots, a_d-1 \in \sqrt{J[a]} = \sqrt{I[a]} \). So we can take a large integer \( n \) such that \( (a_d)^n = (-a_0 \alpha^d + \cdots + a_d-1 \alpha)^n = a_b N \alpha^N + \cdots + b_n \alpha^n \) for some \( b_i \in R_p \). Thus we have an algebraic relation:

\[
c N \alpha N + \cdots + c_1 \alpha + \frac{1}{a} = 0
\]

for some \( c_i \in R_p \). This means that \( \alpha \) is a comonic over \( R_p \), i.e., \( \alpha^{-1} \) is integral over \( R_p \). Since \( \alpha \) is anti-integral over \( R \), \( \alpha^{-1} \) is also anti-integral over \( R \) by [KY2, Theorem 6]. Hence \( \alpha^{-1} \) is integral and anti-integral over \( R_p \). So we have \( I_{[\alpha^{-1}]} R_p = R_p \), which contradicts the assumption \( I_{[\alpha^{-1}]} \subseteq p \).

(2) \( \Leftrightarrow \) (3): Since \( I[\alpha^{-1}] = \eta_d I_{[\alpha]} \), (2) implies that \( R = I_{[\alpha]} + I_{[\alpha^{-1}]} = I_{[\alpha]}(1, \eta_d) \). Hence \( I_{[\alpha]} = I_{\eta_d} \) and \( J_{\eta_d} = I_{\eta_d}(1, \eta_d) = R \). The converse implication is easy to see. Q.E.D.

EXAMPLE 8. Let \( R \) be a polynomial ring \( k[a, b] \) over a field \( k \) and let \( \alpha \) be a root of the equation:

\[
\frac{a}{a-1} X^2 + \frac{b}{a(a-1)} X + \frac{a-1}{a} = 0.
\]

Then the minimal polynomial of \( \alpha \) is given by

\[
\varphi_{\alpha}(X) = X^2 + \frac{b}{a^2} X + \left( \frac{a-1}{a} \right)^2.
\]

In this case, \( I_{[\alpha]} = a^2 R \); \( \varphi_{\alpha^{-1}}(X) = X^2 + (b/(a-1)^2)X + (a/(a-1))^2 \) and \( I_{[\alpha^{-1}]} = (a-1)^2 R \). Thus \( I_{[\alpha]} + I_{[\alpha^{-1}]} = R \), \( J_{[\alpha]} = R \) and \( J_{[\alpha]} = a^2(1, b/a^2) R = (a^2, b) R \). Hence \( \text{grade}(J_{[\alpha]}) > 1 \). Therefore \( \sqrt{J_{[\alpha]}} \neq \sqrt{I_{[\alpha]}}, \) which means that \( I_{[\alpha]} R[\alpha] \neq R[\alpha] \). The implication (2) \( \Rightarrow \) (1) is not always valid in Proposition 7.

Next under the condition \( I_{[\alpha]} R[\alpha] = R[\alpha] \), we shall give rise to a condition which is equivalent to the statement that \( R[\alpha] \) is unramified over \( R \).
THEOREM 9. Assume that $\alpha$ is anti-integral over $R$ of degree $d$ and that $I[\alpha] R[\alpha] = R[\alpha]$. Then $R[\alpha]\eta_1, \ldots, \eta_d$ is unramified over $R[\eta_1, \ldots, \eta_d]$ if and only if $R[\alpha]$ is unramified over $R$.

PROOF. ($\Rightarrow$): Take $P \in \text{Spec}(R[\alpha])$ and put $p = P \cap R$. Since $I[\alpha] R[\alpha] = R[\alpha]$, we have $I[\alpha] \not\subseteq p$. Thus $\eta_1, \ldots, \eta_d \in R_p$. Since $R[\alpha]\eta_1, \ldots, \eta_d$ is unramified over $R[\eta_1, \ldots, \eta_d]$, it follows that $R_p[\alpha] = R_p[\alpha][\eta_1, \ldots, \eta_d]$ is unramified over $R[\eta_1, \ldots, \eta_d]$. Hence $I[\alpha] R_p[\alpha][\eta_1, \ldots, \eta_d] = I[\alpha] R_p[\alpha][\eta_1, \ldots, \eta_d]$ by [KY, Theorem 8]. Note here that $\alpha$ is anti-integral over $R_p$ of degree $d$. Hence $\varphi' \alpha(\alpha)$ is a unit in $R_p[\alpha][\eta_1, \ldots, \eta_d] = R_p[\alpha]$ because $I[\alpha] \not\subseteq p$. Therefore $R_p[\alpha]$ is unramified over $R_p$. So we conclude that $R[\alpha]$ is unramified over $R$.

($\Leftarrow$): Since $I[\alpha] R[\alpha] = R[\alpha]$, we have only to show that $\varphi' \alpha(\alpha)$ is a unit in $R[\alpha]\eta_1, \ldots, \eta_d$ by [KY, Theorem 8]. Take $P' \in \text{Spec}(R[\alpha]\eta_1, \ldots, \eta_d)$, and put $P := P' \cap R$. Since $I[\alpha] R[\alpha] = R[\alpha]$, we have $p \not\subseteq I[\alpha]$. Hence $\eta_1, \ldots, \eta_d \in R_p$ and $R_p[\alpha]\eta_1, \ldots, \eta_d = R_p[\alpha]$. Since $R[\alpha]$ is unramified over $R$ by the assumption, it follows that $\varphi' \alpha(\alpha)^{-1} \in R_p[\alpha]$, which means that $\varphi' \alpha(\alpha)$ is a unit in $R_p[\alpha]$. Therefore we conclude that $\varphi' \alpha(\alpha)$ is a unit in $R[\alpha]\eta_1, \ldots, \eta_d$.

Q. E. D.

REMARK 10. Under the same assumption as above, $R[\alpha]\eta_1, \ldots, \eta_d$ is unramified over $R[\eta_1, \ldots, \eta_d]$ if and only if $\varphi' \alpha(\alpha)$ is a unit in $R[\alpha]\eta_1, \ldots, \eta_d$. (See Remark 4.)

THEOREM 11. Assume that $\alpha$ is anti-integral over $R$ of degree $d$. Consider the following statements:

(1) $I[\alpha] R[\alpha] = R[\alpha]$;
(2) $I[\alpha] R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$;
(3) $R[\eta_1, \ldots, \eta_d]$ is flat over $R$.

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold.

PROOF. (1) $\Rightarrow$ (2): The assumption $I[\alpha] R[\alpha] = R[\alpha]$ yields $I[\alpha] R[\alpha]\eta_1, \ldots, \eta_d = R[\alpha]\eta_1, \ldots, \eta_d$. Since $\alpha$ is integral over $R[\eta_1, \ldots, \eta_d]$ with a monic relation $\varphi_\alpha(X) \in R[\eta_1, \ldots, \eta_d][X]$. So we have $I[\alpha] R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$.

(2) $\Rightarrow$ (3): Since $I[\alpha] R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$ by the assumption, we have $I[\eta_i] R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$ for each $i$. Thus $I[\eta_1] \cdots I[\eta_d] R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d]$. Hence $R[\eta_1, \ldots, \eta_d]$ is flat over $R$ by [S, Proposition 1]. Q.E.D.

PROPOSITION 12. Assume that $\alpha$ is anti-integral over $R$. If $R[\alpha]$ is flat over $R$, then $R[\eta_1, \ldots, \eta_d]$ is flat over $R$.

PROOF. Since $R[\alpha]$ is flat over $R$, we have $J[\alpha] = I[\alpha](1, \eta_1, \ldots, \eta_d) R = R$ by [OSY, (3.4)]. Hence $(1, \eta_1, \ldots, \eta_d) R$ is an invertible ideal of $R$. Localizing at $p \in \text{Spec}(R)$, we can assume that $(1, \eta_1, \ldots, \eta_d) R_p = (1/a) R_p$ for some $a \in R$. Hence $R[\eta_1, \ldots, \eta_d]_p = R_p[1/a]$, which is flat over $R$. Thus $R[\eta_1, \ldots, \eta_d]$ is flat over $R$. Q.E.D.

EXAMPLE 13. In Theorem 11, the implication (2) $\Rightarrow$ (1) does not hold and in Proposition 9, the converse implication does not hold. Let $R := k[x,y]$ be
a polynomial ring over a field \( k \). Let \( \alpha \) be an element satisfying the minimal polynomial \( \varphi_\alpha(X) := X^2 + (1/x)X + 1/y \in k(x, y)[X] \). Then \( \alpha \) is anti-integral over \( R \). It is easy to see that \( I_\alpha = xyR \) and \( J_\alpha = (x, y)R \). Hence \( R[\alpha] \) is not flat over \( R \) (cf. [OSY, (3.4)]). But both \( 1/x \) and \( 1/y \) are flat elements over \( R \). Thus \( R[1/x, 1/y] \) is flat over \( R \). In this case, \( I_\alpha R[1/x, 1/y] = R[1/x, 1/y] \) but \( I_\alpha R[\alpha] \neq R[\alpha] \). Indeed, \( J_\alpha \neq R \) and \( I_\alpha R[\alpha] \subseteq J_\alpha R[\alpha] \neq R[\alpha] \).

Now we investigate the relationship between \( R[\alpha] \) and \( R[\eta_1, \ldots, \eta_d] \). In the rest of this paper, let \( \overline{R} \) denote the integral closure of \( R \) in \( K \). Then \( \overline{R} \) is a Krull domain (cf. [M]).

**Theorem 14.** \( R[\alpha] \) is integral over \( R \) if and only if \( R[\eta_1, \ldots, \eta_d] \) is integral over \( R \).

**Proof.** (\( \Rightarrow \)): Since \( \overline{R} \) is a Krull domain, \( \alpha \) is anti-integral over \( \overline{R} \) (cf. [OSY]). By the assumption, \( \alpha \) is integral over \( R \), and hence \( \alpha \) is integral over \( \overline{R} \). Thus \( \varphi_\alpha(X) \in \overline{R}[X] \) (cf. [OSY]). So we have \( \eta_1, \ldots, \eta_d \) are integral over \( R \). Therefore \( R[\eta_1, \ldots, \eta_d] \) is integral over \( R \).

(\( \Leftarrow \)): Note that \( \alpha \) satisfies the monic relation \( \varphi_\alpha(X) = 0 \) over \( R[\eta_1, \ldots, \eta_d] \). Since \( R[\eta_1, \ldots, \eta_d] \) is integral over \( R \), \( \alpha \) is integral over \( R \). Q. E. D.

**Theorem 15.** Assume that \( \alpha \) is super-primitive over \( R \). If \( I_\alpha R[\alpha] = R[\alpha] \), then \( R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha] \).

**Proof.** Since \( I_\alpha R[\alpha] = R[\alpha] \), \( R[\alpha] \) is flat over \( R \) by Proposition 1. Take \( P \in \text{Dp}_1(R[\alpha]):= \{ P \in \text{Spec}(R) \mid \text{depth} \cap P = 1 \} \) and put \( p = P \cap R \). Then \( p \in \text{Dp}_1(R) \). Since \( \alpha \) is super-primitive over \( R \), \( J_\alpha R_p = I_\alpha(1, \eta_1, \ldots, \eta_d) \cap R_p = R_p \). So \( I_\alpha R_p \) is invertible ideal, and hence \( I_\alpha R_p \) is a principal ideal of \( R_p \). Thus there exists an element \( a \in I_\alpha \) such that \( I_\alpha R_p = aR_p \). Since \( I_\alpha R[\alpha] = R[\alpha] \), \( aR_\alpha \) is principal ideal of \( R_\alpha \). So \( I_\alpha \subseteq I_\alpha \). Since \( I_\alpha \subseteq I_\alpha \), we have \( a\eta_1 = b \) for some \( b \in R_\alpha \). Hence \( \eta_1 \in R_\alpha \subseteq R[\alpha] \) because \( a \) is a unit in \( R_\alpha \). Therefore \( \eta_1 \in \bigcap R[\alpha] \subseteq R[\alpha] \) which means that \( R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha] \). Q. E. D.

**Corollary 15.1.** Assume that \( \alpha \) is super-primitive over \( R \). Then \( I_\alpha R[\alpha] = R[\alpha] \) if and only if \( R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha] \).

**Proof.** (\( \Rightarrow \)): Since \( I_\alpha R[\alpha] = R[\alpha] \), \( I_\alpha R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d] \) and \( R[\eta_1, \ldots, \eta_d] \) is flat over \( R \) by Theorem 11. By Theorem 15, we have \( R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha] \).

(\( \Leftarrow \)): Let \( I_{\eta_1, \ldots, \eta_d} := \bigcap_{i=1}^d I_{\eta_i}^{R[\eta_1, \ldots, \eta_d]} \), where \( I_{\eta_i}^{R[\eta_1, \ldots, \eta_d]} := \{ b \in R[\eta_1, \ldots, \eta_d] \mid b\eta_i \in R[\eta_1, \ldots, \eta_d] \} \). Since \( \alpha \) is anti-integral over \( R \), \( \alpha \) is also anti-integral over \( R[\eta_1, \ldots, \eta_d] \). Since \( R[\eta_1, \ldots, \eta_d] \) is flat over \( R \), we have \( I_{\eta_1, \ldots, \eta_d}^{R[\eta_1, \ldots, \eta_d]} = I_{\eta_1, \ldots, \eta_d} R[\eta_1, \ldots, \eta_d] \). Since \( R[\alpha] \) is integral over \( R[\eta_1, \ldots, \eta_d] \) with the integral dependence \( \varphi_\alpha(X) = 0 \), we have \( I_{\eta_1}^{R[\eta_1, \ldots, \eta_d]} = R[\eta_1, \ldots, \eta_d] \). So \( I_{\eta_1, \ldots, \eta_d} R[\eta_1, \ldots, \eta_d] = R[\eta_1, \ldots, \eta_d] \). Hence \( I_{\alpha} R[\alpha] = R[\alpha] \) because \( R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha] \). Q. E. D.

**Corollary 15.2.** Assume that \( \alpha \) is super-primitive over \( R \). If \( I_\alpha R[\alpha] = R[\alpha] \), then \( R[\alpha] \cap K = R[\eta_1, \ldots, \eta_d] \).
PROOF. Note that $K(\eta_1, \ldots, \eta_d) = K$. By Theorem 15, $R[\eta_1, \ldots, \eta_d] \subseteq R[\alpha]$. Since $R[\alpha]$ is integral over $R[\eta_1, \ldots, \eta_d]$ and $\alpha$ is anti-integral over $R[\eta_1, \ldots, \eta_d]$, $R[\alpha]$ is free over $R[\eta_1, \ldots, \eta_d]$ of rank $d$. Hence $R[\alpha] = R[\eta_1, \ldots, \eta_d] + R[\eta_1, \ldots, \eta_d]\alpha + \ldots + R[\eta_1, \ldots, \eta_d]\alpha^{d-1} \subseteq K + K\alpha + \ldots + K\alpha^{d-1}$. Hence $R[\alpha] \cap K = R[\eta_1, \ldots, \eta_d]$. Q.E.D.

THEOREM 16. Assume that $\alpha$ is super-primitive over $R$ of degree $d$. Then $R[\alpha] \cap K = R$ if and only if $R[\alpha] \cap R[\eta_1, \ldots, \eta_d] = R$.

PROOF. The implication ($\Rightarrow$) is obvious because $R[\eta_1, \ldots, \eta_d] \subseteq K$. We must the converse implication. Take $a \in \bigcap_{i=1}^{d-1} I_n_i$. Then $a\eta_d = -(a\alpha^d + a\eta_1\alpha^{d-1} + \ldots + a\eta_{d-1}\alpha) \in R[\alpha]$. Thus we have $(\bigcap_{i=1}^{d-1} I_n_i)\eta_d \subseteq R[\alpha] \cap R[\eta_1, \ldots, \eta_d] = R$, which implies that $\bigcap_{i=1}^{d-1} I_n_i \subseteq I_{\eta_d}$. Hence $\alpha$ is exclusive by [OY2, Proposition 4]. Q.E.D.

PROPOSITION 17. If there exists $p \in \text{Spec}(R)$ such that $R[\eta_1, \ldots, \eta_d]_p$ is integral over $R_p$ and $R[\eta_1, \ldots, \eta_d]_p \neq R_p$, then $\alpha$ is not anti-integral over $R$.

PROOF. Suppose that $\alpha$ is anti-integral over $R$. Then $\alpha$ is anti-integral over $R_p$. Since $R[\eta_1, \ldots, \eta_d]_p$ is integral over $R_p$, it follows that $\alpha$ is integral over $R_p$ by Theorem 14. Thus $\varphi_\alpha(X) \in R_p[X]$ (cf. [OSY,(2.2)]), which contradicts the assumption $R[\eta_1, \ldots, \eta_d]_p \neq R_p$. Q.E.D.

Let $C(R/R)$ denote the conductor between $R$ and $\overline{R}$.

PROPOSITION 18. Assume that $\overline{R}$ is a finite $R$-module. If there exists a prime divisor $p$ of $I[\alpha]$ such that $p \supseteq I[\alpha] : R C(R/R)$, then $\alpha$ is not anti-integral over $R$.

PROOF. Localizing at $p$, we may assume that $(R,p)$ is a local ring with $p \in \text{Dp}_1(R)$. We have $C(\overline{R}/R) \subseteq I[\alpha] \subseteq I_n_i$. Thus $\eta_i \in C(\overline{R}/R)^{-1} = \overline{R}$ for $1 \leq i \leq d$. But since $p \supseteq I[\alpha]$, we have $R[\eta_1, \ldots, \eta_d] \neq R$. Our conclusion follows from Proposition 17. Q.E.D.

THEOREM 19. If grade $(I[\alpha] + C(\overline{R}/R)) > 1$, then $\alpha$ is anti-integral over $R$.

PROOF. We have only to show that $\alpha$ is anti-integral over $R_p$ for each $p \in \text{Dp}_1(R)$ because $I[\alpha]$ is a divisorial ideal of $R$(cf. [OSY]). From the assumption, either $I[\alpha] \not\subseteq p$ or $C(\overline{R}/R) \not\subseteq p$. If $I[\alpha] \not\subseteq p$, then $\varphi_\alpha(X) \in R_p[X]$. If $C(\overline{R}/R) \not\subseteq p$, then $R_p = \overline{R}_p$, which is a Krull domain. Hence $\alpha$ is anti-integral over $R$ by [OSY,(1.13)]. Q.E.D.

We close this paper by giving the following result.

PROPOSITION 20. Assume that $\alpha$ is anti-integral over $R$ of degree $d$. Let $\beta_1, \ldots, \beta_n$ be elements in $K$, each of which is super-primitive over $R$. If $R[\alpha]$ is flat over $R$, then $\beta_1, \ldots, \beta_n$ are super-primitive over $R[\alpha]$.

PROOF. Note that $I[\beta_i] = I[\beta_i] := R : R \beta_i$ and $J[\beta_i] = I[\beta_i] + \beta_i I[\beta_i] = I[\beta_i] + I[\beta_i]^{-1}$ because $\beta_i \in K$. Since $\beta_i$ is super-primitive over $R$, we have grade $(I[\beta_i] + I[\beta_i]^{-1}) = \text{grade}(J[\beta_i]) > 1$ by definition. Let $T_{\beta_i} := R[\alpha] : R \beta_i$. Since $R[\alpha]$ is flat over $R$,
we have $I_{\beta_i} + I_{\beta_i^{-1}} \subseteq T_{\beta_i} + T_{\beta_i^{-1}}$, and hence we have \( \text{grade}(T_{\beta_i} + T_{\beta_i^{-1}}) > 1 \). Thus $\beta_i$ is super-primitive over $R[\alpha]$ by [SOY,(2.5)]. Q.E.D.

References


