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Note on strong $A_\sigma$-summability

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Abstract. The main object of the paper is to study the inclusions $[A_\sigma, p] \subset [A_\sigma, q]$, $[A_\sigma, p] \subset [B_\sigma, q]$ and $[A_\sigma, p] \cap m \subset [B_\sigma, q]$ and to characterize the spaces $M([A_\sigma, p], [A_\sigma, q])$ and $M_0([A_\sigma, p_0], [A_\sigma, q_0])$. Here $[A_\sigma, p]$ and $[A_\sigma, q]$ are certain spaces of strongly summable sequences in terms of an injection $\sigma : N \to N$ and a nonnegative infinite matrix $A$, and $M([A_\sigma, p], [A_\sigma, q])$ is the space of summability factors between $[A_\sigma, p]$ and $[A_\sigma, q]$.

1. Introduction

Let $\sigma$ be a mapping of the set of positive integers into itself. A continuous linear functional $\phi$ on $m$, the space of real bounded sequences, is said to be an invariant mean or a $\sigma$-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all $n$,
(ii) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and
(iii) $\phi((x_{\sigma(n)})) = \phi(x)$ for all $x \in m$.

For certain kinds of mappings $\sigma$, every invariant mean $\phi$ extends the limit functional on the space $c$ of real convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where $V_\sigma$ is the set of bounded sequences all of whose $\sigma$-means are equal [8].

The mappings $\sigma$ are assumed one-to-one and such that $\sigma^k(n) \neq n$ for all positive integers $n$ and $k$, where $\sigma^k(n)$ denotes the $k$th iterate of the mapping $\sigma$ at $n$.

When $\sigma(n) = n + 1$, the $\sigma$-mean are the classical Banach limits on $m$ and $V_\sigma$ is the set of almost convergent sequences [4].

If $x = (x_n)$, set $T x = (T x_n) = (x_{\sigma(n)})$. It can be shown [8] that

$$V_\sigma = \{ x = (x_n) : \lim_{m} t_{mn}(x) = L \text{ for some } L \text{ uniformly in } n \}$$

where $t_{mn}(x) = (x_n + T x_n + \cdots + T^m x_n)/(m + 1)$.

Mursaleen [6] defined strongly $\sigma$-convergent sequences replacing the Banach limits by $\sigma$-means in the following manner. A bounded sequence $x = (x_n)$ is said to be strongly $\sigma$-convergent to a number $L$ if and only if $\lim_{m} m^{-1} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| \to 0$ uniformly in $n$.

The following inequality will be used frequently throughout the paper:

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k})$$

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where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p_k = H$, $C = \max\{1, 2^{H-1}\}$ [5].

Let $p = (p_k)$ be a sequence of strictly positive real numbers and $A = (a_{mk})$ be a nonnegative regular infinite matrix, i.e. $a_{mk} \geq 0$ for all $m, k$ and $x_k \to x$ implies $\sum_k a_{mk} x_k \to x$ as $m \to \infty$. By using a modulus function $f$, we define three sequence spaces as follows [3]:

$$[A_\sigma, f, p] = \left\{ x \in W : \lim_{m} \sum_k a_{mk} f(|x_{\sigma^k(n)} - L|^{p_k}) = 0 \text{ for some } L \right\}$$

uniformly in $n$

$$[A_\sigma, f, p]_0 = \left\{ x \in W : \lim_{m} \sum_k a_{mk} f(|x_{\sigma^k(n)}|^{p_k}) = 0 \text{ uniformly in } n \right\},$$

$$[A_\sigma, f, p]_\infty = \left\{ x \in W : \sup_{m,n} \sum_k a_{mk} f(|x_{\sigma^k(n)}|^{p_k}) < \infty \right\},$$

where $\sum_k$ denotes the summation $k = 1$ to $\infty$ and $W$ denotes all complex valued sequences.

When $f(x) = x$, we have the following sequence spaces:

$$[A_\sigma, p] = \left\{ x \in W : \lim_{m} \sum_k a_{mk} |x_{\sigma^k(n)} - L|^{p_k} = 0 \text{ uniformly in } n \right\}$$

$$[A_\sigma, p]_0 = \left\{ x \in W : \lim_{m} \sum_k a_{mk} |x_{\sigma^k(n)}|^{p_k} = 0 \text{ uniformly in } n \right\},$$

$$[A_\sigma, p]_\infty = \left\{ x \in W : \sup_{m,n} \sum_k a_{mk} |x_{\sigma^k(n)}|^{p_k} < \infty \right\},$$

which we call the spaces of strongly $A_\sigma$-summable, strongly $A_\sigma$-summable to zero and strongly $A_\sigma$-bounded sequences, respectively.

A sequence $x$ is called $A_\sigma$-summable if there exists $\lim_m \sum_k a_{mk} x_{\sigma^k(n)} = \Phi(x)$ uniformly in $n$. The sets of $A_\sigma$-summable and $A_\sigma$-summable to zero are denoted, respectively, by $(A_\sigma)$ and $(A_\sigma)_0$.

2. Inclusion relations

**Theorem 1.** Let $0 < q_k \leq p_k$, $(p_k/q_k)$ be bounded and $\|A\| = \sup_m \sum_k a_{mk} < \infty$. Then $[A_\sigma, p] \subset [A_\sigma, q]$.

This inclusion can be proved by using the techniques similar to those used in Theorem 1 of Bilgin [2].

If $c \subset (A_\sigma)$ and $\Phi(x) = \lim x$ for every $x \in c$, we write $c \subset (A_\sigma)$. We have

**Theorem 2.** Let $c \subset (A_\sigma)$ and $\lim p_k/q_k = \infty$. Then $[A_\sigma, p] \notin [A_\sigma, q]$.

**Proof.** Let $\lim p_k/q_k = \infty$ and $c \subset (A_\sigma)$. It is known that if $c \subset (A_\sigma)$ then there is a bounded sequence $y = (y_k)$ consisting of 0's and 1's so that $y \notin (A_\sigma)$. Let $x = (x_k)$, $x_k = \left(\frac{y_k + 1}{3}\right)^{1/q_k}$. Then $x \notin [A_\sigma, q]$ but $x \in [A_\sigma, p]$.

Using the same technique in [1] it is easy to prove the following theorem.
**Theorem 3.** Suppose that $A$ is regular, $0 < r \leq p_k \leq q_k \leq M < \infty$ and $\alpha_k = \sup_m a_{mk} > 0$. Then the inclusion $[A_\sigma, p] \subset [A_\sigma, q]$ holds if and only if

$$\sup_k \frac{p_k - q_k}{\alpha_k p_k q_k} < \infty.$$  

**Theorem 4.** Let $\liminf p_k/q_k > 0$, then $[A_\sigma, q] \cap m \subset [A_\sigma, p]$.

**Proof.** This is routine verification and can be obtained by using standard techniques. \(\blacksquare\)

Then by Theorem 1 and Theorem 4 we have

**Corollary 5.** Let $0 < q_k \leq p_k$, $(p_k/q_k)$ be bounded and $||A|| < \infty$. Then $[A_\sigma, q] \cap m = [A_\sigma, p] \cap m$.

We write $[A_\sigma]$ instead of $[A_\sigma, p]$ if $p_k = 1 \ (k \in \mathbb{N})$. We have

**Theorem 6.** Let $0 < r \leq q_k \leq M < \infty$ and $\alpha_k < \infty$. Then $[A_\sigma, q] \cap m = [A_\sigma] \cap m$.

**Proof.** Let $h = \sup q_k$. Then by Corollary 5 we have $[A_\sigma, q] \cap m = [A_\sigma, h] \cap m$. Now from Theorem 4, we have $[A_\sigma, h] \cap m = [A_\sigma] \cap m$. Hence $[A_\sigma, q] \cap m = [A_\sigma] \cap m$. \(\blacksquare\)

Let $B = (b_{mk})$ be a nonnegative matrix. We write $[A_\sigma, p] \subseteq [B_\sigma, q]$ if $[A_\sigma, p] \subseteq [B_\sigma, q]$ and $[A_\sigma, p]$-lim $x = [B_\sigma, q]$-lim $x$ for every $x \in [A_\sigma, p]$.

**Theorem 7.** Let $\beta_k = \sup_m b_{mk}$. Suppose that $0 < r \leq p_k \leq q_k$, $b_{mk} \neq 0$, implies $a_{mk} \neq 0$ and $b_{mk}/a_{mk} \leq \beta_k/\alpha_k$. If $(\beta_k^{1/q_k} \alpha_k^{-1/p_k})$ is bounded, then $[A_\sigma, p] \subseteq [B_\sigma, q]$.

**Proof.** Let $x \in [A_\sigma, p]$. Then there exists $K$ such that $\alpha_k |x^{\sigma^k(n)} - L|^{p_k} < K$ for all $k$ and $n$. Since $0 < r \leq p_k$, there exists a number $K_0$ so that

$$\alpha_k^{1/p_k} |x^{\sigma^k(n)} - L| < K^{1/p_k} < K_0.$$  

Since $(\beta_k^{1/q_k} \alpha_k^{-1/p_k})$ is bounded, then by (3) we have

$$\sum_k b_{mk}|x^{\sigma^k(n)} - L|^{q_k} = \sum_k a_{mk}|x^{\sigma^k(n)} - L|^{p_k} \frac{b_{mk}}{a_{mk}} |x^{\sigma^k(n)} - L|^{q_k - p_k} \leq \sum_k a_{mk}|x^{\sigma^k(n)} - L|^{p_k} \frac{\beta_k}{\alpha_k} |x^{\sigma^k(n)} - L|^{q_k - p_k} = \sum_k a_{mk}|x^{\sigma^k(n)} - L|^{p_k} (\beta_k^{1/q_k} \alpha_k^{-1/p_k})^{q_k} (\alpha_k^{1/p_k} |x^{\sigma^k(n)} - L|)^{q_k - p_k} \to 0 \ (m \to \infty) \text{ uniformly in } n.$$  

Hence $[A_\sigma, p] \subseteq [B_\sigma, q]$. \(\blacksquare\)

For $A = B$ we have the following result:
**Corollary 8.** Suppose that $0 < r \leq p_k \leq q_k$. Then (2) implies $[A_\sigma, p] \subseteq [A_\sigma, q]$.

By the above inclusion relations we deduced;

**Result 9.** Let $q_k \leq p_k$, $(p_k/q_k)$ be bounded, $||A|| < \infty$ and $(A_\sigma)_0 \subseteq (B_\sigma)_0$. Then $[A_\sigma, q] \cap m = [B_\sigma, p] \cap m$.

3. Summability factors

In this section we shall characterize sequences belonging to the spaces $M([A_\sigma, p], [A_\sigma, q])$ and $M_0([A_\sigma, p], [A_\sigma, q], [A_\sigma, q])$, which are defined as follows:

$$M([A_\sigma, p], [A_\sigma, q]) = \{ \lambda = (\lambda_k) : \lambda x \in [A_\sigma, q] \text{ for all } x \in [A_\sigma, p] \}$$

and

$$M_0([A_\sigma, p], [A_\sigma, q], [A_\sigma, q]) = \{ \lambda = (\lambda_k) : \lambda x \in [A_\sigma, q] \text{ for all } x \in [A_\sigma, p] \}.$$ We call a sequence $\lambda = (\lambda_k)$ belonging to one of the above spaces a *summability factor*. We have

**Theorem 10.** Suppose that $0 < q_k < p_k$, $r = \inf_k (q_k/p_k) > 0$ and $h = \sup_k (q_k/p_k) < 1$. If conditions

$$\sup_{m,n} \sum_k a_m k^{\lambda_k} |x_{\sigma_k(n)}|^{\frac{1}{1-r}} < \infty$$

and

$$\sup_{m,n} \sum_k a_m k^{\lambda_k} |x_{\sigma_k(n)}|^{\frac{1}{h-1}} < \infty$$

are fulfilled, then $\lambda \in M_0([A_\sigma, p], [A_\sigma, q], [A_\sigma, q])$.

**Proof.** Let $x \in [A_\sigma, p]$. Write $t^n_k = |x_{\sigma_k(n)}|^{p_k}$ and consider $r_k = q_k/p_k$. Then

$$\lim_m \sum_k a_m k^{\lambda_k} |x_{\sigma_k(n)}|^{p_k} = \lim_m \sum_k a_m t^n_k = 0 \text{ uniformly in } n.$$ Define $U^n_k = t^n_k$ and $V_n = 0$ if $t^n_k \geq 1$, and $U^n_k = 0$ and $V^n_k = t^n_k$ if $0 \leq t^n_k < 1$. 

so \((U^n_k)^{r_k} \leq (U^n_k)^{h}\) and \((V^n_k)^{r_k} \leq (V^n_k)^{r}\). By Hölder's inequality we obtain
\[
\sum_k a_{mk} |(\lambda x)_{\sigma^*_k}(n)|^{q_k} = \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)|^{q_k} (t^n_k)^{r_k}
\]
\[
= \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)|^{q_k} (U^n_k)^{r_k} + \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)|^{q_k} (V^n_k)^{r_k}
\]
\[
\leq \sum_k (a_{mk} U^n_k)^{h} (a_{mk})^{1-h} |\lambda_{\sigma^*_k}(n)|^{q_k} + \sum_k (a_{mk} V^n_k)^{r} (a_{mk})^{1-r} |\lambda_{\sigma^*_k}(n)|^{q_k}
\]
\[
\leq \left( \sum_k a_{mk} U^n_k \right)^h \left( \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)|^{q_k} \right)^{1-h} + \left( \sum_k a_{mk} V^n_k \right)^r \left( \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)|^{q_k} \right)^{1-r}
\]
\[
= \left( \sum_k a_{mk} t^n_k \right)^h \left( \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)|^{q_k} \right)^{1-h} + \left( \sum_k a_{mk} t^n_k \right)^r \left( \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)|^{q_k} \right)^{1-r}
\]

Now it follows by (6) that conditions (4) and (5) are sufficient for \(\lambda \in M([A_\sigma, p], [A_\sigma, q])\). This completes the proof. \(\blacksquare\)

Note that \(e = (1, 1, 1, \ldots) \in [A_\sigma, p]\) implies \(M([A_\sigma, p], [A_\sigma, q]) \subset [A_\sigma, q]\).

**Theorem 11.** Suppose that \(0 < q_k < p_k, \sup_k q_k < \infty, r = \inf_k (q_k/p_k) > 0\)
and \(h = \sup_k (q_k/p_k) < 1\). Let \(\lambda \in [A_\sigma, q]\). Then conditions (4) and (5) are sufficient for \(\lambda \in M([A_\sigma, p], [A_\sigma, q])\).

**Proof.** Let \(\lambda \in [A_\sigma, q]\) and \(x \in [A_\sigma, p]\). Then there exist some numbers \(L'\)
and \(L\) such that

\[
[A_\sigma, q]-\lim \lambda = L' \quad \text{and} \quad [A_\sigma, p]-\lim x = L.
\]

It follows from (1) that
\[
\sum_k a_{mk} |\lambda_{\sigma^*_k}(n)x_{\sigma^*_k}(n) - LL'|^{q_k} \leq C \sum_k a_{mk} |\lambda_{\sigma^*_k}(n)(x_{\sigma^*_k}(n) - L)|^{q_k} + C \sup_k |L|^{q_k} \sum_k a_{mk} |\lambda_{\sigma^*_k}(n) - L'|^{q_k}.
\]

Since \((x_k - L) \in [A_\sigma, p]_0\), it follows by Theorem 10 and by condition (7) that conditions (4) and (5) are sufficient for \(\lambda \in M([A_\sigma, p], [A_\sigma, q])\). \(\blacksquare\)

**Theorem 12.** Suppose that \(0 < r \leq p_k \leq q_k < \infty\). If \(\lambda \in [A_\sigma, q]\) and

\[
\sup_k (|\lambda_{\sigma^*_k}(n)|^{p_k} - |\lambda_{\sigma^*_k}(n)|^{p_k}) < \infty,
\]

\(\blacksquare\)
then $\lambda \in M([A_{\sigma},p],[A_{\sigma},q])$.

**PROOF.** Condition (7) is fulfilled for every $\lambda \in [A_{\sigma},q]$ and $x \in [A_{\sigma},p]$. So it follows from (1), (3), (7) and (8) that

$$
\sum_{k} a_{mk}|\lambda \sigma^k(n)x_{\sigma^k(n)} - LL'|^{q_k}
\leq C \sum_{k} a_{mk}|\lambda \sigma^k(n)|^{q_k}|x_{\sigma^k(n)} - L|^{|q_k} + C \sup_{k} |L|^{q_k} \sum_{k} a_{mk}|\lambda \sigma^k(n) - L'|^{q_k}
\leq C \sum_{k} a_{mk} |x_{\sigma^k(n)} - L|^{p_k} |\lambda \sigma^k(n)|^{\frac{p_k-q_k}{p_k}} \alpha_k^{1/p_k} |\lambda \sigma^k(n) - L|^{|q_k-p_k}
+ C \sup_{k} |L|^{q_k} \sum_{k} a_{mk}|\lambda \sigma^k(n) - L'|^{q_k}
\rightarrow 0 \ (m \rightarrow \infty) \ \text{uniformly in} \ n.
$$

This completes the proof. \[\square\]

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**References**


