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Integrals of Lusin Type
A Survey

YÔTO KUBOTA*

Introduction

There are many integrals of Lusin's type or descriptive definitions of integrals. In this survey the author selects some characteristic integrals of this kind and show the way, as short as possible, to each definition of these integrals.

To extend an integral of Lusin type, it is essential to see how the notion of absolute continuity or generalized absolute continuity of the primitive which is used in defining the integral, could be extended with the corresponding derivative and continuity.

Our main purpose is to discuss inclusion relations between the selected integrals of Lusin type and to show their convergence theorems and integration by parts formulae. In the last section, we consider an abstract treatment for integrals of Lusin type. Some unpublished results are contained, and also some inclusion-relation problems and others are still remained unsolved.

§1. THE N-INTEGRAL AND N*-INTEGRAL.

The origin of integrals of Lusin type is the Newton integral.

DEFINITION 1.1. Let $f : [a, b] \rightarrow R$. If there is a function $F(x)$ such that $F'(x) = f(x)$ at every point of $[a, b]$ then $F$ is a Newton primitive of $f$ on $[a, b]$. The definite $N$-integral of $f$ on $[a, b]$ is $F(b) - F(a)$.

DEFINITION 1.2. If there is a continuous function $F$ such that $F'(x) = f(x)$ n.e., then $f$ is $N^*$-integrable on $[a, b]$. The definite $N^*$-integral of $f$ on $[a, b]$ is $F(b) - F(a)$.

THEOREM 1.1. If $F, G : [a, b] \rightarrow R$ are continuous and $F'(x) = G'(x)$ n.e., then $F$ and $G$ differ by at most a constant.

It follows from Theorem 1.1 that the $N$- and $N^*$-integrals are determined uniquely. Theorem 1.1 comes from the following monotonicity theorem:

THEOREM 1.2. If $F$ is continuous on $[a, b]$ and $F'(x) \geq 0$ n.e., then $F$ is non-decreasing on $[a, b]$.

Generally speaking, an integration process requires a corresponding monotonicity theorem to guarantee the uniqueness of the integral value.

It is well known that the $N^*$-integral is more general than the $N$-integral.

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§2. The Lebesgue integral

Definition 2.1. Let \( f : [a, b] \to R \). If there is an absolutely continuous (AC) function \( F(x) \) such that \( F'(x) = f(x) \) a.e., then \( f \) is Lebesgue integrable on \([a, b]\) and \( \int_a^b f = F(b) - F(a) \).

Theorem 2.1. If \( F \) is a function which is AC on \([a, b]\) and \( F'(x) \geq 0 \) a.e. then \( F \) is non-decreasing on \([a, b]\).

Saks developed the Lebesgue theory using this definition in [39], and F. Riesz gave another definition equivalent to the Lebesgue integral in [36] which is slightly different from the original one. Recently the author [27] also stated a Lebesgue theory by using the new definition which is a version of McShane’s Riemann type definition equivalent to the Lebesgue integral [34].

Let \( \delta(x) \) be a positive function on \( I = [a, b] \). A pair \((x, [u, v])\), where \( x \in I \) and \( [u, v] \subseteq I \), is said to be \( \delta \)-compatible on \( I \) if \( [u, v] \subseteq (x - \delta(x), x + \delta(x)) \).

Note that \( x \) is not necessarily contained in \([u, v]\).

Definition 2.2. Let \( f : I \to R \). A function \( F : I \to R \) is a primitive of \( f \) on \( I \) if for a given \( \varepsilon > 0 \), there exists a positive function \( \delta \) and a nondecreasing function \( \phi \) defined on \( I \) with \( \phi(b) - \phi(a) < \varepsilon \) such that

\[
|F(v) - F(u) - f(x)(v - u)| \leq \phi(v) - \phi(u)
\]

for any \( \delta \)-compatible pair \((x, [u, v])\) on \( I \).

Definition 2.3. A function \( f : I \to R \) is said to be \( L^* \)-integrable on \( I \) if there is a primitive \( F \) of \( f \) on \( I \). The definite \( L^* \)-integral of \( f \) is defined as \( F(b) - F(a) \).

Then we can show the corresponding properties of the \( L^* \)-integral in more elementary and less technical than that of the ordinary Lebesgue theory.

Theorem 2.2. The \( L^* \)-integral is equivalent to the \( L \)-integral on \([a, b]\).

Let define

\[
R[a, b] = \{ f : f \text{ is Riemann integrable on } [a, b] \},
\]

\[
CR[a, b] = \{ f : f \text{ is Cauchy-Riemann integrable on } [a, b] \}
\]

and etc. Then, for fixed \([a, b]\), we have:

Theorem 2.3.

(i) \( N \not\subset R \), (ii) \( R \not\subset N \), (iii) \( N \not\subset L \), (iv) \( CR \not\subset L \).

(v) \( N \not\subset CR \), (vi) \( CR \not\subset N^* \).

It is well known that (i)~(v) hold. The proof of (vi) will be given at the next section.
§3. Special and general Denjoy integrals

Let $F : [a, b] \to R$ and $J$ be a subinterval of $[a, b]$. Let

$$O(F, J) = \sup_{x, y \in J} |F(x) - F(y)|.$$  

DEFINITION 3.1. Let $E \subseteq [a, b]$ and $\{I_k\}$ be a sequence of non-overlapping intervals with end points in $E$. If $\sum O(F, I_k)$ is arbitraly small when $\sum |I_k|$ is sufficiently small then $F$ is AC* on $E$. If $[a, b]$ is the sum of a countable number of sets on each of which $F$ is AC* then $F$ is said to be ACG* on $[a, b]$.

DEFINITION 3.2. Let $f : [a, b] \to R$. If there is a continuous, ACG* function $F$ such that $F'(x) = f(x)$ a.e., then $f$ is said to be special Denjoy integrable or $D^*$-integrable on $[a, b]$. The definite $D^*$-integral of $f$ on $[a, b]$ is $F(b) - F(a)$.

Uniqueness of the $D^*$-integral comes from the following theorem:

THEOREM 3.1. If $F$ is continuous, ACG* on $[a, b]$ and $F'(x) \geq 0$ a.e. on $[a, b]$ then $F$ is nondecreasing on $[a, b]$.

Proof of Theorem 2.2 (vi). It is sufficient to prove the relation $R \notin N^*$. To show this, we define $f(x) = 0$ ($x \in C$) and $1$ ($x \in [0, 1] \setminus C$), where $C$ is the Cantor set. Then $f$ is R-integrable on $[0, 1]$ and $G(x) = (R \int_0^x f = x$. Suppose $f$ is $N^*$-integrable on $[0, 1]$. Then there exists a continuous function $F$ such that $F'(x) = f(x)$ n.e. on $[0, 1]$. Since $F$ is continuous and $-\infty < F'(x) < \infty$ n.e. on $[0, 1]$, $F$ is $AC^*$ on $[0, 1]$ [38; p.239] and $(F - G)' = 0$ a.e. Hence we have by Theorem 3.1 that $F = x + \alpha$. This is a contradiction.

THEOREM 3.2. The $D^*$-integral contains the CR-, $N^*$- and L-integrals.

However the $D^*$-integral is not the least general integral which includes CR, $N^*$ and L (see, [26]).

DEFINITION 3.3. A function $F : [a, b] \to R$ is ACG on $[a, b]$ if $[a, b]$ is the sum of a countable number of sets $E_k$ on each of which $F$ is AC.

DEFINITION 3.4. Let $f : [a, b] \to R$. If there is a continuous function $F$ which is ACG on $[a, b]$ and such that $ADF(x) = f(x)$ a.e., where $AD$ is the approximate derivative, then $f$ is Denjoy integrable in the wide sense or $D$-integrable on $[a, b]$. The $D$-integral of $f$ on $[a, b]$ is defined as $F(b) - F(a)$.

Theorem 3.1 is also true for continuous ACG functions and the approximate derivative.

THEOREM 3.3.

(i) If $F$ is continuous and $ADF(x)$ exists n.e. on $[a, b]$ then $ADF(x)$ is $D$-integrable on $[a, b]$ and

$$\int_a^b ADF = F(b) - F(a).$$

(ii) The $D$-integral includes strictly the $D^*$-integral.
THEOREM 3.4. Let \( \{f_n\} \) be a sequence of \( D \)-integrable functions on \([a, b]\). If \( f_n(x) \to f(x) \) a.e. on \([a, b]\) and if \( g(x) \leq f_n(x) \leq h(x) \) a.e. on \([a, b]\) for all \( n \), where \( g, h \) are \( D \)-integrable functions on \([a, b]\), then \( f \) is \( D \)-integrable on \([a, b]\) and
\[
\lim_{n \to \infty} (D) \int_a^b f_n = (D) \int_a^b f.
\]

This convergence theorem is not essential in the Denjoy integral theory itself because the proof is done by using the convergence theorem of Lebesgue and the fact that if \( f \) is \( D \)-integrable and \( f \geq 0 \) a.e. on an interval then \( f \) is \( L \)-integrable there.

P.Y. Lee and his school gave an effective convergence theorem for the general Denjoy integral and a necessary and sufficient condition for a function to be \( D \)-integrable on an interval \([30, 32]\), which will be stated in more general form in the next section.

The following integration by parts formula is due to Saks \([38; p. 246]\).

THEOREM 3.5. If \( f \) is \( D \)-integrable on \([a, b]\) with its primitive \( F \) and if \( g \) is \( VB \) on \([a, b]\) then the function \( fg \) is \( D \)-integrable on \([a, b]\) and
\[
(D) \int_a^b fg = F(b)g(b) - F(a)g(a) - \int_a^b gdF.
\]

Descriptive theory of the Denjoy integrals can be found in \([38, 16, 13, 9]\).

§4. The approximately continuous Denjoy integral

Natural extension of the \( D \)-integral has been obtained by the author \([23]\).

DEFINITION 4.1. A function \( F : [a, b] \to \mathbb{R} \) is said to be \((ACG)\) on \([a, b]\) if \([a, b]\) is the countable union of closed sets on each of which \( F \) is \( AC \).

DEFINITION 4.2. A function \( f : [a, b] \to \mathbb{R} \) is integrable in the approximately continuous Denjoy sense or \( AD \)-integrable if there is an approximately continuous \((ACG)\) function \( F \) such that \( ADF(x) = f(x) \) a.e. The \( AD \)-integral of \( f \) on \([a, b]\) is defined to be \( F(b) - F(a) \).

The integral is well defined because the following theorem holds.

THEOREM 4.1. If \( F \) is approximately continuous, \((ACG)\) and \( ADF(x) \geq 0 \) a.e. on \([a, b]\) then \( F \) is nondecreasing on \([a, b]\).

THEOREM 4.2. The \( AD \)-integral is more general than the \( D \)-integral.

REMARK. We claim in \([23, 1]\) that the \( AD \)-integral includes the approximately continuous Perron integral \((AP)\) defined by Burkill \([1]\). C.M. Lee \([29]\) has pointed out incompleteness of our proof and has given its corrected proof for the first time. Recently Gordon \([17]\) and V. Ene \([9; p. 200]\) remark independently that C.M. Lee’s proof has an error. Therefore the inclusion-relation between the \( AD \)-integral and the \( AP \)-integral is now open (see also §13).
THEOREM 4.3. (Cauchy property) If $f$ is AD-integrable on every interval $[a, c]$ ($a < c < b$), and $\text{ap lim}_{c \to b}(AD) \int_a^c f = A$, then $f$ is AD-integrable on $[a, b]$ and $(AD) \int_a^b f = A$.

THEOREM 4.4. (Harnack property) Let $E$ be a closed set in $[a, b]$ and $\{I_k\} = \{[a_k, b_k]\}$ be the sequence of contiguous closed intervals of $E$ with respect to $[a, b]$ and let $f$ be a function which is L-integrable on $E$ and AD-integrable on each $I_k$. Suppose that the following conditions are satisfied:

(i) $\sum_k |(AD) \int_{I_k} f| < \infty$.

(ii) If $x \in E$ is a limit point of $\{I_k\}$, then there exists a set $E_x$ which has unit density at $x$ and contains all the end points of $I_k$ in a sufficiently small neighbourhood of $x$, such that $\lim_{k \to \infty} O(AD, f, E_x \cap I_k) = 0$, where $O(AD, f, E_x \cap I_k)$ means the oscillation of the AD-primitive of $f$ on $E_x \cap I_k$.

Then $f$ is AD-integrable on $[a, b]$ and we have

$$(AD) \int_a^b f = (L) \int_E f + \sum_k (AD) \int_{I_k} f.$$


Let $T$ be a functional by which there corresponds to each closed interval $J$ contained in $[a, b]$ a linear space of measurable functions defined on $J$, and to each $f$ of this class a real number. The class of functions will be denoted by $K(T, J)$ and the real number associated with $f$ by $T(f, J)$.

DEFINITION 4.3. A functional $T$ is termed an approximately continuous integral if the following conditions are satisfied:

(i) If $f \in K(T, [a, b])$ then $f \in K(T, J)$ for all $J \subset [a, b]$.

(ii) If $J_1$ and $J_2$ are abutting intervals and if $f \in K(T, J_1) \cap K(T, J_2)$ then $f \in K(T, J_1 \cup J_2)$ and

$$T(f, J_1 \cup J_2) = T(f, J_1) + T(f, J_2).$$

(iii) If $f \in K(T, J)$ where $J = [c, d]$, then $F(x) = T(f, [c, x])(c < x \leq d)$ is approximately continuous on $J$.

The Cauchy (C) and Harnack (H) properties of an approximately continuous integral $T$ are given by Theorem 4.3 and Theorem 4.4 respectively in which the AD-integral is replaced by $T$.

Then we can characterize the AD-integral as follows [24]:

THEOREM 4.5. The AD-integral is the least general approximately continuous integral which satisfies the properties (C) and (H).

Recently S. Bambang and P.Y. Lee [30] have established the following general convergence theorem for the AD-integral.
THEOREM 4.6. Let $f_n$ be AD-integrable on $[a,b]$ with primitive $F_n$ for every $n$. If the sequences of functions $\{f_n\}$ and $\{F_n\}$ satisfy the following conditions:

(i) $f_n$ converges to $f$ a.e. on $[a,b]$.

(ii) $\{F_n\}$ is (ACG) on $[a,b]$ uniformly in $n$, i.e., there is a sequence of closed sets $\{X_i\}$ such that $[a,b] = \bigcup X_i$ and $\{F_n\}$ is AC on $X_i$ uniformly in $n$.

(iii) $F_n$ converges to $F$ for every $x \in [a,b]$ where $F$ is approximately continuous on $[a,b]$.

then $f$ is AD-integrable on $[a,b]$ with the primitive $F$.

They also remarked that Cauchy and Harnack properties for the AD-integral can be obtained as corollaries of this convergence theorem.

The following interesting theorem is due to T.Y. Lee and P.Y. Lee [32].

THEOREM 4.7. In order that a function $f : [a,b] \rightarrow \mathbb{R}$ be AD-integrable, it is necessary and sufficient that there exists an increasing sequence $\{X_n\}$ of closed sets whose union is $[a,b]$ such that

(i) $f$ is $L$-integrable on each $X_n$;

(ii) the sequence of primitives $\{F_n\}$ of $\{f\chi_{X_n}\}$ is (ACG) uniformly on $[a,b]$;

(iii) $F_n \rightarrow F$ at every point of $[a,b]$, where $F$ is approximately continuous on $[a,b]$.

We can show the following:

THEOREM 4.8. Let $f$ be AD-integrable function on $[a,b]$ with the primitive $F$, $g$ be VB on $[a,b]$, and $G(x) = (L) \int_a^x g$. If $F$ is $D^*$-integrable on $[a,b]$ then $fG$ is AD-integrable on $[a,b]$ and

$$\int_a^b fG = F(b)G(b) - \int_a^b Fg.$$

§5. Ellis’ theory of integrals of Lusin type

Ellis [5] has established a comprehensive theory of integration of Lusin type as follows:

DEFINITION 5.1. A class of functions $P$ which are defined on $[a,b]$ and approximately derivable a.e on $[a,b]$, is called a class of primitives if it has the following properties:

(1) The class $P$ is a linear class.

(2) If $F, G \in P$ and $ADF(x) = ADG(x)$ a.e. then they differ by a constant.

(3) If $F, G \in P$ and if $ADF(x) \leq ADG(x)$ a.e. then $F(b) - F(a) \leq G(b) - G(a)$.

(4) If $F \in P$ and if $G \in D$ (the class of all primitives of $D$-integrable functions) and if $ADF(x) = ADG(x)$ a.e. then they differ by a constant.

Ellis makes correspond to each primitive class $P$ an integral of Lusin type, and defines its integral as follows:
DEFINITION 5.2. A function \( f(x) \) is integrable on \([a,b]\) if there is a function \( F \in \mathcal{P} \) with \( ADF(x) = f(x) \) a.e. The definite integral of \( f \) on \([a,b]\) is \( F(b) - F(a) \).

The uniqueness of the integral comes from condition (2), and condition (4) implies that the Ellis integral is compatible to the D-integral.

The following fundamental properties are valid:

THEOREM 5.1.

(i) The class of integrable functions on \([a,b]\) is a linear class and the integral is a linear functional on it.
(ii) If \( f \) is integrable on \([a,b]\) then \( f \) is necessarily measurable.
(iii) If \( f \) is integrable on \([a,b]\) and if there is a Lebesgue integrable function \( g \) such that \( f \leq g \) a.e. then \( f \) is Lebesgue integrable.
(iv) Let \( f \) and \( g \) be integrable on \([a,b]\) and \( F, G \) be primitives of \( f, g \) respectively. Suppose that \( F(x)G(x) \) belongs to \( \mathcal{P} \) and \( F(x)g(x) \) is integrable on \([a,b]\). Then \( f(x)G(x) \) is integrable on \([a,b]\) and

\[
\int_a^b fG = F(b)G(b) - F(a)G(a) - \int_a^b Fg.
\]

THEOREM 5.2. Any linear subclass of the class of all functions which are Darboux continuous and (ACG) on \([a,b]\) is a \( \mathcal{P} \) class.

Note that the class of all Darboux continuous functions is not necessarily linear. To prove Theorem 5.2, the following is essential:

THEOREM 5.3. If \( F : [a,b] \to \mathbb{R} \) is Darboux continuous, (ACG) on \([a,b]\), and if \( ADF(x) \geq 0 \) a.e. then \( F \) is non-decreasing on \([a,b]\).

Since a function that is approximately continuous on an interval is necessarily Darboux continuous, the AD-integral is an example of Ellis’ integral.

\section{6. The proximally continuous Denjoy integral}

Sarkhel and De [40] begin their integral theory by modifying the notion of dispersion point of sets as follows:

DEFINITION 6.1. A set \( E \) is said to be sparse at \( x \) in the right if there exists, for every \( \epsilon > 0 \), a \( k > 0 \) such that every interval \((a,b) \subset (x, x + k)\) with \( a - x < k(b - x) \) contains at least one point \( y \) such that \( |E \cap (x, y)| < \epsilon(y - x) \).

The family of sets sparse at \( x \) on the right is denoted by \( S(x+) \) (similarly \( S(x-) \)), and \( E \) is said to be sparse at \( x \) if \( E \subset S(x) = S(x+) \cup S(x-) \).

DEFINITION 6.2. Let \( f : X \to \mathbb{R} \) be given, and let define

\[
\overline{f}(x) = \inf\{r \in \mathbb{R} : y \in X : f(y) > r \in S(x)\},
\]
\[
\underline{f}(x) = \sup\{r \in \mathbb{R} : y \in X : f(y) < r \in S(x)\}.
\]

If \( \overline{f}(x) = \underline{f}(x) = f(x) \) then \( f \) is said to be proximally continuous at \( x \).
Note that proximally continuity implies Darboux continuity.

Given $E \subset X$, $f : X \to R$, and $r > 0$, let

$$V(f, E, r) = \sup \sum |f(b_k) - f(a_k)|,$$

where $\{(a_k, b_k)\}$ are pairwise disjoint open intervals whose end points belong to $E$ and $\sum(b_k - a_k) < r$. Define $V(f, E, 0) = \inf\{V(f, E, r) : r > 0\}$. Then $f$ is AC on $E$ if and only if $V(f, E, 0) = 0$.

DEFINITION 6.3. If for an $\varepsilon > 0$ there is a sequence of sets $\{E_n\}$ whose union is $E_0$ with $E \setminus E_0$ being at most countable, such that $V(f, E_n, 0) < \varepsilon$ for every $n$ then $f$ is said to be $\varepsilon$-PAC on $E$. If $f$ is $\varepsilon$-PAC on $E$ for every $\varepsilon > 0$ then $f$ is said to be PAC on $E$. If $E = \bigcup E_k$ and $f$ is PAC on each $E_k$ then $f$ is PACG on $E$.

THEOREM 6.1.

(i) If $f$ is PACG on $E$ then $f$ satisfies Lusin’s condition $(N)$.

(ii) If $f$ is PACG and measurable on $E$ then $f$ is approximately derivable a.e. on $E$.

(iii) If $F_1, F_2 : [a, b] \to R$ are PACG, proximally continuous and $AD(F_1 - F_2) = 0$ a.e. then $F_1 - F_2$ is a constant.

DEFINITION 6.4. A function $f : [a, b] \to R$ is said to be PD-integrable, if there is a function $F : [a, b] \to R$ such that $F$ is proximally continuous, PACG on $[a, b]$ and $ADF(x) = f(x)$ a.e.. The increment $F(b) - F(a)$ is the definite PD-integral of $f$ on $[a, b]$.

Theorem 6.2. The PD-integral is more general than the AD-integral.

There are examples of functions on an interval which are not AD-integrable but PD-integrable [40; p.37].

§7. An integral with basis

One of the problems in the theory of the trigonometric series

$$\frac{1}{2}a_0 + \sum (a_n \cos nx + b_n \sin nx)$$

is that of suitably defining a trigonometric integral with the property that, if the series converges everywhere to a function $f$, then $f$ is necessarily integrable and $a_n, b_n$ are given in the usual Fourier form. The problem has been solved by, for instance, Marcinkiewicz and Zygmund, Burkill, James and Jeffery. The integrals introduced by them are called, respectively, MZ, SCP, $P^2$, $J$-integral [33, 2, 20, 22].

The author [25] also defined an integral of Lusin type which solves the above problem and is less general than these integrals.

Let $[a, b]$ be compact interval and let $P$ be a measurable subset of $[a, b]$ with $a \in P, b \in P$ and $m(P) = b - a$. then we call such a set $P$ basis of $[a, b]$.

Let $F$ be a $D^*$-integrable function in a neighbourhood of the point $x$. If

$$\lim_{h \to 0} (D^*) \int_x^{x+h} F = F(x)$$

then $F$ is PACG on $[a, b]$.
Integrals of Lusin type

then $F$ is said to be $C$-continuous at $x$. The symmetric $C$-derivative of $F$ at $x$, $SCDF(x)$, is defined to be

$$
\lim_{h \to 0} \frac{(D^*) \int_{x}^{x+h} F - (D^*) \int_{x-h}^{x} F}{h^2}.
$$

THEOREM 7.1. If $F$ is $D^*$-integrable on $[a,b]$, $C$-continuous on a basis $P$ and $SCDF(x) \geq 0$ everywhere on $(a,b)$ then $F$ is nondecreasing on $P$.

DEFINITION 7.1. Let $f : [a,b] \to R$ and $P$ be a basis of $[a,b]$. If there exists a $D^*$-integrable function $F(x)$ which is $C$-continuous at each point of $P$ and $SCDF(x) = f(x)$ on $(a,b)$ then $f$ is said to be integrable with respect to the basis $P$ or $B(P)$-integrable on $[a,b]$. The definite integral of $f$ on $[a,b]$ is $F(b) - F(a)$.

The $B(P)$-integral is uniquely determined by Theorem 7.1. This integral does not have additive property of intervals, and does not include the Lebesgue integral. However it has a powerful property for solving the trigonometric series problem stated above.

THEOREM 7.2. Let

$$
\frac{1}{2} a_0 + \sum (a_n \cos nx + b_n \sin nx)
$$

be a trigonometric series converging everywhere to a function $f(x)$ and let $P$ be the set of points at which the series

$$
\frac{1}{2} a_0 x + \sum (a_n \sin nx - b_n \cos nx)/n
$$

converges. Then, for any $c \in P$, $f(x)$ is $B(P_1)$-integrable on $[c, c + 2\pi]$, where $P_1 = P \cap [c + 2\pi]$, and we have

$$
a_n = \frac{1}{\pi} B(P_1) \int_{c}^{c+2\pi} f(t) \cos nt dt \ (n = 0, 1, 2, \cdots),
$$

$$
b_n = \frac{1}{\pi} B(P_1) \int_{c}^{c+2\pi} f(t) \sin nt dt \ (n = 1, 2, \cdots).
$$

We note that the series (2) converges a.e. and that the limit function is $C$-continuous at each point of $P$.

THEOREM 7.3. The $B(P)$-integral is less general than the integrals $MZ$, $SCP$, $P^2$ and $J$.

For another interesting Lusin type like integral $J$, see [22; pp.27-32].
§8. The Foran integral

J. Foran [11] gave an integral of Lusin's type by generalizing the notion of absolute continuity $AC$ as follows:

**Definition 8.1.** Given set $E$ and a natural number $N$, a function $F$ is said to be $A(N)$ on $E$ if every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\{I_k\}$ is any sequence of nonoverlapping intervals with $E \cap I_k \neq \emptyset$ and $\sum |I_k| < \delta$ then there exist intervals $J_{kn}, n = 1, 2, \ldots, N$ such that

$$B(F; E \cap \bigcup_{k=1}^{N} I_k) \subset \bigcup_{k=1}^{N} \left( I_k \times J_{kn} \right), \quad \sum_{k=1}^{N} \sum_{n=1}^{N} |J_{kn}| < \varepsilon,$$

where $B(F; A)$ is the graph of $F$ on $A$.

Note that $F$ is $A(N)$ on $E$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\{I_k\}$ is a sequence of nonoverlapping intervals with $E \cap I_k \neq \emptyset$ and $\sum |I_k| < \delta$ then there exist the sets $E_{kn}(n = 1, 2, \ldots, N)$ such that

$$\bigcup_{n=1}^{N} E_{kn} = E \cap I_k, \quad \sum_{k=1}^{N} \sum_{n=1}^{N} O(F, E_{kn}) < \varepsilon.$$

The class $\mathcal{F}$ consists of all continuous functions $F$ defined on $I$ for which there exist a sequence of sets $E_k$ and natural numbers $N_k$ such that $I = \bigcup E_k$ and $F$ is $A(N_k)$ on $E$.

**Theorem 8.1.**

(i) If $F$ is continuous, $A(N)$ on $E$ and if $N = 1$ or $E$ is an interval, then $F$ is AC on $E$.

(ii) If $F$ is $A(N)$ on $E$ and $Z$ is a set of Lebesgue measure 0 contained in $E$, then $m(F(Z)) = 0$.

(iii) If $F$ is $A(N_1)$ on $E$ and $G$ is $A(N_2)$ on $E$ then every linear combination of $F$ and $G$ is $A(N_1 \cdot N_2)$ on $E$.

It follows from Theorem 8.1 that $\mathcal{F}$ is a linear class of functions each of which satisfies Lusin's condition $(N)$.

Let $D_{ap}$ be the class of functions defined on $I$ each of which is approximately derivable almost everywhere. There is a function in $\mathcal{F}$ that is not in $D_{ap}$ [6; pp.202–204].

**Definition 8.2.** A function $f$ defined on $I$ is said to be $FD$-integrable on $I$ if there is a function $F \in \mathcal{F} \cap D_{ap}$ and such that $ADF(x) = f(x)$ a.e. The definite integral of $f$ on $I$ is the increment $F(I)$.

The uniqueness of the integral follows from the following theorem [38; p.286].
Theorem 8.2. If a continuous function $F$ satisfies Lusin's condition (N) on $I$ and $F'(x)$ is nonnegative at almost every point $x$ where $F'(x)$ exists then $F$ is nondecreasing on $I$.

Theorem 8.3. The $FD$-integral is more general than the $D$-integral.

There exists a function of $F$ which is not in $ACG \cap C$. An exact example is given by V. Ene [6].

Theorem 8.4. Let $F, g : [a, b] \to R$ be such that $F \in BV$ and $g$ is $FD$-integrable on $[a, b]$. Then $Fg$ is $FD$-integrable, and denoting by $G$ the $FD$-indefinite integral of $g$, we have

$$\int_a^b Fg = (G(b)F(b) - G(a)F(a) - (S) \int_a^b GdF).$$

Recently Fu [14] defined an integral of Lusin's type which includes both $FD$- and $AD$-integrals.

The class $AF$ consists of all approximately continuous functions $F$ defined on $I$ for which there exist a sequence of sets $E_k$ and natural numbers $N_k$ such that $I = \bigcup E_k$ and $F$ is $A(N_k)$ on $E_k$.

Definition 8.3. A function $f$ defined on $I$ is $AF$-integrable on $I$ if there is a function $F \in AF \cap D_{ap}$ and such that $ADF(x) = f(x)$ a.e. The definite integral of $f$ on $I$ is the increment of $F$ on $I$.

Uniqueness of the $AF$-integral is obtained from the theorem of O'Malley [35]:

Theorem 8.5. If $F$ satisfies the following conditions:

(i) $F$ is Baire class 1.

(ii) $\text{ap lim sup}_{t \to x_-} F(t) \leq F(x) \leq \text{ap lim sup}_{t \to x_+} F(t)$.

(iii) The interior of the set $F\{x : ADF(x) \leq 0\}$ is empty.

then $F$ is nondecreasing on $I$, where $\overline{ADF}$ means the upper right approximate derivate.

Let $F \in AF \cap D_{ap}$ and $ADF(x) = f(x)$ a.e. Then $F$ satisfies clearly conditions (i) and (ii). We set, for any $\epsilon > 0$, $G(x) = F(x) + \epsilon x$. Then $G$ meets (i), (ii) and (iii), and hence $G$ is nondecreasing on $I$ by Theorem 8.5. It follows that $F$ is also so.

Theorem 8.6. The $AF$-integral includes strictly the Foran- and $AD$-integrals.

By the way, we state here a general monotonicity theorem established by Bruckner [3].

Theorem 8.7. Let $P$ be a function-theoretic property that satisfies the following two conditions:

(i) Any continuous $BV$ function that satisfies property $P$ on $[a, b]$ is nondecreasing.
(ii) Any Darboux function in Baire class one which satisfies property \( \mathcal{P} \) on \([a, b]\) is BVG on \([a, b]\).

Then any Darboux function in Baire class one which satisfies property \( \mathcal{P} \) on \([a, b]\) is continuous and nondecreasing on \([a, b]\).

This theorem has wide applications.

§9. The sparse integral

Ka. Iseki has defined several integration processes [18; pp.226–227]. Here we take up only the sparse integral [19].

A set which is expressible as the sum of a finite number of closed intervals is termed figure. A figure \( W \) is said to pertain to a set \( E \) if \( E \) contains all end points of the component intervals of \( W \).

**Definition 9.1.** A figure \( W \) is said to be sparse, if every component interval of \( W \) is shorter than every open interval contiguous to \( W \). Each closed interval and the empty set are sparse. For the more exact definition, see V. Ene [9; p.111].

Given a function \( F \) and a figure, let write \( F(W) = \sum F(K) \) and \( F^*(W) = \sum |F(K)| \) where \( K \) ranges over the components of \( W \).

**Definition 9.2.** A function \( F \) is called sparsely continuous on a set \( E \) if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |W| < \delta \) implies \( |F(W)| < \varepsilon \) for every sparse figure \( W \) pertaining to \( E \), and we write \( F \in SC(E) \).

In this definition, \( F(W) \) may be replaced by \( F^*(W) \). Also the family \( SC(E) \) is a linear class. Further, if \( F \in SC(E) \) and if \( F \in C(\overline{E}) \) then \( F \in SC(\overline{E}) \).

**Theorem 9.1.**

(i) If \( F \in SC(I) \) where \( I \) is a closed interval then \( F \in AC(I) \).

(ii) If \( F \in SC(E) \) then \( |F(S)| = 0 \) for every closed set \( S \subset E \) with \( |S| = 0 \).

However we do not know whether a sparsely continuous function on a measurable set is necessarily approximately derivable at almost all points.

**Definition 9.3.** A function \( F \) is said to be SCG on a set \( E \) if \( F \in C(E) \) and there is a sequence of sets \( \{E_n\} \) such that \( E = \bigcup E_n \) and \( F \in SC(E_n) \) \((n = 1, 2, \cdots)\).

Then we have: (i) The family \( SCG(E) \) is a linear class. (ii) A function which is \( SCG \) on a closed set maps every closed null subset onto a null set.

**Theorem 9.2.** A continuous function on a closed set \( S \) is SCG on \( S \) if and only if every closed subset of \( S \) contains a portion on which the function is SC.

**Definition 9.4.** A function \( f(x) \) is said to be sparsely integrable on a closed interval \( I \) if there is an SCG function \( F(x) \) on \( I \) and if \( F \) is approximately
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The definite integral of $f$ on $I$ is defined as the increment $F(I)$.

**Theorem 9.3.** The sparse integral contains strictly the $D$-integral.

We do not know general inclusion relation between the sparse integral and the Foran integral. For integrals which extend both Foran and sparse integrals, see V. Ene [9; pp.210–211].

§10. An integral theory of C.M.Lee

C.M. Lee [29] developed a comprehensive integral theory of Lusin type.

**Definition 10.1.** A function defined on $I$ is said to be upper closed monotone or $uCM$ in $I$ if the function is monotone increasing in the closed interval $[c, d]$ whenever it is so on any open interval $(c, d) \subset I$.

Every Darboux function on $I$ is $uCM$ on $I$.

**Definition 10.2.** A function $F$ on $I$ is said to be $lAC$ on a set $E \subset I$ if each $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum |F(b_k) - F(a_k)| > -\varepsilon$ for each finite set $\{[a_k, b_k]\}$ of nonoverlapping intervals with end points in $E$ and $\sum (b_k - a_k) < \delta$.

**Definition 10.3.** A function $F$ on $I$ is said to be $(lACG)$ on a set $E$ if $E$ is the union of countably many closed sets on each of which $F$ is $(lAC)$.

By definition,

\[ lCM = \{F : -F \in uCM\}, \quad CM = lCM \cup uCM, \]
\[ (uACG) = \{F : -F \in (lACG)\}, \quad (ACG) = (lACG) \cup (uACG). \]

**Theorem 10.1.** Let $F : I \rightarrow R$. If $F \in uCM \cap (lACG)$ on $I$ and $\overline{DF}(x) \geq 0$ a.e., then $F$ is nondecreasing on $I$.

A class of functions is termed upper semilinear space if it is closed under linear combinations with non-negative constants. For an upper semilinear space $uL$ contained in $uCM$, the set $L = \{F : F, -F \in uL\}$ is a linear space.

**Definition 10.4.** A function $f$ on $I$ is said to be $LDG$-integrable on $I$ if there is a function $F \in L \cap (ACG)$ such that $ADF(x) = f(x)$ a.e. The definite integral $f$ on $I$ is the increment of $F$ on $I$.

Let denote, for the fixed interval,

$C = \{F: F \text{ is continuous}\}, \quad C_{ap} = \{F: F \text{ is approximately continuous}\}$.

Then each class is the linear one contained in $uCM$. Hence we obtain the $CDG$-integral and the $C_{ap}DG$-integral respectively. The $D$-integral is the $CDG$-integral and the $AD$-integral is the $C_{ap}DG$-integral.

§11. An integral theory of V. Ene and G. Ene

In their theory [8], the notions $CM$, $B_1^*$ and Foran's conditions $(M)$, $[M]$ play an essential role.
DEFINITION 11.1. A function defined on \([a, b]\) is \(B_1^\ast\) if every closed set has a portion on which the restriction of the function is continuous.

DEFINITION 11.2. A continuous function \(F\) satisfies Foran’s condition \((M)\) if it is \(AC\) on any set on which it is \(VB\). A function \(F\) satisfies condition \([M]\) on a closed set \(E\) if \(F\) is \(AC\) on each closed subset of \(E\) on which it is continuous and \(VB\).

THEOREM 11.1. Let \(F\) be a function which satisfies \(CM\), \(B_1^\ast\) and \([M]\) on a closed interval. If \(ADF(x) \geq 0\) a.e., where \(ADF(x)\) exists (finite or infinite), then \(F\) is nondecreasing on \([a, b]\).

To prove the theorem we need the following lemmas.

LEMMA 1. Suppose that \(F\) is continuous and satisfies \((M)\) on \([a, b]\). Then \(F\) is approximately derivable at every point of a set with positive measure. If \(P = \{x : \infty \geq ADF(x) \geq 0\}\) and \(F(a) < F(b)\) then \(|F(P)| \geq F(b) - F(a)\). Also if \(N = \{x : -\infty \leq ADF(x) \leq 0\}\) and \(F(a) > F(b)\) then \(|F(N)| \geq F(a) - F(b)\).

PROOF. We prove only the essential part. Suppose that \(F(a) < F(b)\). For each \(y \in [F(a), F(b)]\), let \(x_y = \min\{x \in [a, b]: F(x) \geq y\}\) and \(E = \{x_y : y \in [F(a), F(b)]\}\).

Then \(F\) is nondecreasing on \(E\) and \(F(E) = [F(a), F(b)]\). Since \(F\) satisfies \((M)\), it is \(AC\) on \(E\) and hence fulfils \((N)\) on \(E\). Therefore \(|E| > 0\) and \(F\) is approximately derivable almost everywhere on \(E\) with \(ADF(x) \geq 0\) a.e. on \(E\). Hence \(|F(P)| \geq |F(E)| = F(b) - F(a)\). The other cases are similar.

The first part is an extension of Banach’s theorem [38; p.286]: if \(F \in C \cap (N)\) on \([a, b]\) then \(F\) is derivable at every point of a set of positive measure.

COROLLARY 11.1. If \(F\) is continuous and satisfies \((M)\) on \([a, b]\) then \(F\) is approximately derivable almost everywhere on \([a, b]\).

PROOF. The function \(F\) is approximately derivable at every point of a set of positive measure by Lemma 1. It follows from the latter half of Theorem 10 [38; p.295] that \(F\) is approximately derivable almost everywhere on \([a, b]\).

LEMMA 2. If a function \(F\) is continuous and satisfies condition \((M)\) on \([a, b]\) and if \(ADF(x) \geq 0\) a.e., where \(ADF(x)\) exists (finite or infinite) then \(F\) is nondecreasing on \([a, b]\).

PROOF. Suppose that there exist \(c, d \in [a, b]\), \((c < d)\), with \(F(d) < F(c)\). Let \(N_0 = \{x \in [c, d]: ADF(x) = 0\}\), \(N_- = \{x \in [c, d]: -\infty \leq ADF(x) < 0\}\), and \(N = N_0 \cup N_-\). First we have, by Lemma 9.2, [38, p.290], \(|F(N_0)| = 0\). Also by Theorem 10.8 [38; p.237], \(F\) is \(VBG\) on \(N_-\), and since \(F\) satisfies \((M)\), it is \(ACG\) on \(N_-\). On the other hand, \(F\) is approximately derivable almost everywhere on \([a, b]\) by Corollary 11.1 and \(ADF(x) \geq 0\). Hence \(|N_-| = 0\) and \(|F(N_-)| = 0\). It follows from Lemma 1 that \(0 = |F(N)| \geq |F(N_-)| = F(c) - F(d)\). Hence \(F(d) > F(c)\), which is a contradiction.

PROOF OF THEOREM 11.1. Let \(U\) be the interior of the set of points at which \(F\) is continuous. Since \(F \in B^\ast_1\), \(U\) contains an open interval at least.
Suppose $U \neq (a, b)$, and let $\{I_n\}$ be the component intervals of $U$. Note that $C \cap [M] = (M)$. Since $F \in CM$, it follows from Lemma 2 and Corollary 11.1 that $F$ is nondecreasing in $I_n$ for every $n$. Hence $P = [a, b] - U$ has no isolated points and is perfect. Since $F \in B_1^*$, there exists an open interval $I$ such that $F'|P \cap I$ is continuous. Hence $F$ is continuous on $I$ which is a contradiction for $I$ contains points of $P$. Therefore $U = (a, b)$. By Lemma 2, $F$ is nondecreasing on $(a, b)$ and so on $[a, b]$.

**Definition 11.3.** Let $f$ be a function on $[a, b]$ and let $S$ be a linear subclass of $CM \cap [M] \cap B_1^* \cap Dap$ on $[a, b]$. If there is a function $F \in S$ and $ADF(x) = f(x)$ a.e., then $f$ is said to be $D(S)$-integrable on $[a, b]$. The definite integral of $f$ on $[a, b]$ is defined to be $F(b) - F(a)$.

It is easy to see by Baire's category theorem that $B_1^*$ is equivalent to $[CG]$ on $[a, b]$, that is, $[a, b] = \bigcup E_n$ ($E_n$: closed set) and the restricted function to $E_n$ is continuous. Since $CM \cap (ACG) \subset CM \cap B_1^* \cap Dap$ on $[a, b]$, we have:

**Theorem 11.2.** The $D(S)$-integral is more general than Lee's SDG-integral.

§12. Abstract theory for integrals of Lusin type

Let $f, g : [a, b] \to R$, $\alpha, \beta \in R$ and $x \in I = [a, b]$. A functional $T : (f, x) \to R$ is called the abstract derivative of $f$ at $x$ if it satisfies the conditions:

(D1) If $f$ is approximately derivable at $x$ then $T(f, x) = ADf(x)$.

(D2) $T(\alpha f + \beta g, x) = \alpha T(f, x) + \beta T(g, x)$.

We also use $D_{ab}f(x)$ in place of $T(f, x)$.

A function $F : E \to R$, where $E \subset [a, b]$, is said to be absolutely continuous in abstract sense on $E$, $F \in AC_{ab}(E)$, if the following conditions are satisfied:

(AC1) If $F \in AC_{ab}(E)$ then $AC_{ab}(E')$ for any $E' \subset E$.

(AC2) If $F, G \in AC_{ab}(E)$ then $\alpha F + \beta G \in AC_{ab}(E')$ for any $\alpha, \beta \in R$.

(AC3) If $F \in AC(E)$ then $F \in AC_{ab}(E)$, where $F \in AC(E)$ means that $F$ is absolutely continuous on $E$.

(AC4) If $F \in AC_{ab}(E)$ for any closed set $E$ then $D_{ab}F(x)$ exists a.e. on $E$.

(AC5) If $F$ is $uCM$ on $[c, d]$ and is nondecreasing on each complementary interval of a closed set $E \subset (c, d)$, and if $F \in AC_{ab}(E)$ and $D_{ab}F(x) \geq 0$ a.e. on $E$ then $F$ is nondecreasing on $(c, d)$.

A function $F : I \to R$ is $ACG_{ab}(I)$ if $I$ is the sum of a countable number of closed sets $E_k$ such that $F \in AC_{ab}(E_k)$ for each $k$.

**Theorem 12.1.** If $F \in uCM \cup ACG_{ab}(I)$ and if $D_{ab}F(x) \geq 0$ a.e., then $F$ is nondecreasing on $I$.

To prove the theorem, it is convenient to use the following lemma by Romanovski [37].

**Lemma.** Let $\mathcal{F}$ be a family of open subintervals of $I_0 = (a, b)$ which satisfies the following conditions:

1. If $(p, q) \in \mathcal{F}$ and $(q, r) \in \mathcal{F}$ then $(p, r) \in \mathcal{F}$.
(2) If \( I \in \mathcal{F} \) then \( J \in \mathcal{F} \) for any open subinterval \( J \) of \( I \).

(3) If \( J \in \mathcal{F} \) for any \( J \) with \( \overline{J} \subset I \) then \( I \in \mathcal{F} \).

(4) If \( \mathcal{F}_1 \) is a subfamily of \( \mathcal{F} \) such that \( \mathcal{F}_1 \) does not cover \( I_0 \), then there is an \( I \in \mathcal{F} \) such that \( \mathcal{F}_1 \) does not cover \( I \).

Then \( I_0 \in \mathcal{F} \).

PROOF. We have \( I_0 = \bigcup_{\mathcal{F} \in \mathcal{F}} I \). If not so then \( \bigcup_{\mathcal{F} \in \mathcal{F}} \not\subset I_0 \). Putting \( \mathcal{F}_1 = \mathcal{F} \), the family \( \mathcal{F}_1 \) covers any interval of \( \mathcal{F} \), which contradicts (4). Let \( I \) be any subinterval of \( I_0 \) such that \( \overline{I} \subset I_0 \). Then, since \( \overline{I} \subset \bigcup_{\mathcal{F} \in \mathcal{F}} I \), there exists a finite set of \( I_k \in \mathcal{F} \) such that \( \overline{I} \subset \cup I_k \). By (2), the interval \( I \) is represented as the finite sum of consecutive intervals \( J_k \in \mathcal{F} \), and hence \( I \in \mathcal{F} \) by (1). Therefore, by (3), \( I_0 \in \mathcal{F} \).

PROOF OF THEOREM 12.1. It is sufficient to show that \( F \) is nondecreasing on \( (a, b) \) since \( F \) is \( uCM \) on \( I \). Let \( \mathcal{F} \) be the family of all open intervals on each of which \( F \) is nondecreasing. It is clear that \( \mathcal{F} \) satisfies (1), (2) and (3) in Lemma. To show that \( \mathcal{F} \) also satisfies (4), let \( \mathcal{F}_1 \) be a subfamily of \( \mathcal{F} \) such that \( \mathcal{F}_1 \) does not cover \((a, b)\), and let \( E \) be the set of all points which are not covered by \( \mathcal{F}_1 \). Then \( E \) is a closed set. Since \( F \in ACab(I) \), there exists a sequence of closed sets \( \{E_k\} \) such that \( I = \bigcup E_k \) and \( F \in AC(E_k) \) for each \( k \). It follows from Baire's categorization theorem that there is an interval \((l, m)\) and an integer \( n \) such that \((l, m) \cap E \subset E_n\). Hence \( f \in ACab(H) \), where \( H = [l, m] \cap E \) by (AC1).

It is shown by the method of repeated bisection that \( F \) is nondecreasing on each complementary interval of \( H \) with respect to \((l, m)\). Since \( F \in ACab(H) \) and \( D_{ab}F(x) \geq 0 \) a.e. on \( H \), \( F \) is nondecreasing on \((l, m)\) by (AC5) and hence \((l, m) \in \mathcal{F} \). But \( \mathcal{F}_1 \) does not cover \((l, m)\), for it contains points of \( E \). Thus condition (4) of Lemma is satisfied, which complete the proof.

For an upper semi-linear space \( uL \) contained in \( uCM \), that is, \( uL \) is closed under linear combinations with nonnegative coefficients, we put \( L = \{F : F, -F \in uL\} \). Then \( L \) is a linear space. We fix a linear space \( L \) in \( uCM \).

DEFINITION 12.1. A measurable function \( f : I \rightarrow R \) is \( LD_{ab} \)-integrable on \( I = [a, b] \) if there exists a function \( F \in L \cap AC_{ab} \) such that \( D_{ab}F(x) = f(x) \) a.e. on \( I \), and the \( LD_{ab} \)-integral of \( f \) on \( I \) is \( F(b) - F(a) \).

If we put \( D_{ab} = AD \) and \( AC_{ab} = AC \) in the above then the corresponding integral is the \( LDG \)-integral defined by C.M.Lee.

THEOREM 12.2. Let \( \{f_n\} \) be a sequence of \( LD_{ab} \)-integrable functions on \( I \) and let \( \lim_{n \rightarrow \infty} f_n = f \) a.e.. If there are \( LD_{ab} \)-integrable functions \( g \) and \( h \) such that \( g \leq f_n \leq h \) a.e. for all \( n \) then \( f \) is \( LD_{ab} \)-integrable on \( I \) and

\[
\lim_{n \rightarrow \infty} (LD_{ab}) \int_I f_n = (LD_{ab}) \int_I f.
\]

We have been unable to get any more effective convergence theorem such as P.Y. Lee's convergence theorem.

To establish a formula of integration by parts for the integral, we set two more axioms about abstract derivative and abstract \( AC \) function:
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(D3) \( D_{ab}[f(x)g(x)] = D_{ab}f(x) \cdot g(x) + f(x) \cdot D_{ab}g(x) \).

(AC6) If \( F \in AC_{ab}(E) \) and \( G \in AC(E) \) then \( FG \in AC_{ab}(E) \).

**Theorem 12.3.** Let \( f \) be \( LD_{ab} \)-integrable on \([a, b]\) and \( F \) be its primitive. Let \( g \in VB[a, b] \) and \( G(x) = \int_a^x g \). If \( F \) is \( D \)-integrable on \([a, b]\) then \( fG \) is \( LD_{ab} \)-integrable there and

\[
(LD_{ab}) \int_a^b fG = [FG]_a^b - (D) \int_a^b Fg.
\]

§13. Addendum

We add two more integrals of Lusin type defined by Gordon and Fu respectively.

(I) The \( AK_N \)-integral [Gordon; 17].

**Definition 13.1.** A function \( F : [a, b] \to R \) is \( BVG_N \) on \([a, b]\) if \( F \in C_{af} \cap BVG \) on \([a, b]\) and \( F \) satisfies condition \((N)\) on \([a, b]\).

**Definition 13.2.** A function \( f : [a, b] \to R \) is \( AK_N \)-integrable on \([a, b]\) if there is a \( BVG_N \) function \( F \) on \([a, b]\) such that \( ADF(x) = f(x) \) a.e. on \([a, b]\).

To show the uniqueness of the integral, it is sufficient to prove the following theorem:

**Theorem 13.1.** If \( F \) is \( BVG_N \) on \([a, b]\) and \( ADF(x) \geq 0 \) a.e. on \([a, b]\) then \( F \) is nondecreasing there.

Gordon proves this theorem simply by using Bruckner’s monotonicity theorem (Theorem 8.7).

**Definition 13.3.** A function \( f : [a, b] \to R \) is \( AD_0 \)-integrable on \([a, b]\) if there exists a function \( F : [a, b] \to R \) such that \( F \in C_{af} \cap AC_G \) and \( ADF(x) = f(x) \) a.e. on \([a, b]\).

The corresponding monotonicity theorem of this integral is obtained as a corollary of Theorem 13.1.

Thus we have

\[
AD \subset AD_0 \subset AK_N.
\]

**Theorem 13.2.** If \( f : [a, b] \to R \) is approximately continuous Perron integrable on \([a, b]\) in the sense of Gordon [16; p.259] then it is also \( AK_N \)-integrable on \([a, b]\).

The proof follows directly from Theorems 17.14, 17.15, 16.18 and Exercise 16.8 in [16].

It follows that the \( AK_N \)-integral includes Burkill’s approximately continuous Perron integral [1].

(II) The \( \lambda \)-power integral on Cantor type sets [Fu; 15].

For any function defined on any Null set, its Lebesgue integral is always zero. Fu succeeded in defining a new integral which gives non-zero value for trivial functions defined on an interval including Cantor type sets.
Let $[a, b]$ be a compact interval on $R$. For a partition of $[a, b]$:

$$a = a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n = b,$$

we set $U_1 = \bigcup_{j=1}^n U^j_1$, where $U^j_1 = [a^j_1, b^j_1]$ ($j = 1, 2, \ldots, n$).

Given a partition of $U^j_1$ ($j = 1, 2, \ldots, n$):

$$a_1^j = a^j_1 + (j-1)n+1 < b_1^j < \cdots < a_n^j < b_n^j = b_1^j,$$

we set $U_2 = \bigcup_{j=1}^{n^2} U^j_2$, where $U^j_2 = [a^j_2, b^j_2]$ ($j = 1, 2, \ldots, n^2$).

This procedure continues to get a sequence of intervals $U_m = \{U^m_j\}_{j=1}^{n^m}$.

Define $C = \cap_{m=1}^\infty U_m$. Then $C$ is a perfect set.

**Definition 13.4.** A perfect set $C$ is said to be Cantor type if it is of zero measure, $H \cdot \dim(C) = \lambda$, $0 < H \cdot \dim(C) < \infty$ and

$$H \cdot \dim \left( C \cap (a_m^{j(m)}, b_m^{j(m)}) \right) = (b_m^{j(m)} - a_m^{j(m)})^\lambda,$$

where $H \cdot \dim$ and $H \cdot \dim$ mean the Hausdorff dimension and the $\lambda$-Hausdorff measure respectively.

Let $C$ be the Cantor type set with $H - \dim(C) = \lambda$ and let $D$ be the set of all end points of intervals $U^j_m$ ($m = 1, 2, \ldots; j = 1, 2, \ldots, n^m$).

**Definition 13.5.** Let $x \in C \setminus D$ and $x = \cap_{m=1}^\infty (a_m^{j(m)}, b_m^{j(m)})$. A function $F : [a, b] \to R$ is $\lambda$-power derivable at $x$ if

$$\lim_{m \to \infty} \frac{F(b_m^{j(m)}) - F(a_m^{j(m)})}{(b_m^{j(m)} - a_m^{j(m)})^\lambda}$$

exists. The limit is called the $\lambda$-power derivative of $F$ at $x$ and is denoted by $F'_{\lambda P}(x)$.

**Definition 13.6.** Let $f : [a, b] \to R$ be a trivial function with $f(x) = 0$ ($x \in [a, b] \setminus C$), where $H - \dim(C) = \lambda$. If there exists a continuous function $F : [a, b] \to R$ such that $F'(x) = f(x)$ on $[a, b] \setminus C$ and $F'_{\lambda P}(x) = f(x)$ n.e. on $C \setminus D$ then $f$ is $\lambda$-power integrable on $[a, b]$ and $(\lambda P) \int_a^b f = F(b) - F(a)$.

The following theorem guarantees uniqueness of the $\lambda P$-integral.

**Theorem 13.3.** If $F : [a, b] \to R$ is continuous, $F'(x) \geq 0$ on $[a, b] \setminus C$ and $F'_{\lambda P}(x) \geq 0$ n.e. on $C \setminus D$ then $F$ is nondecreasing on $[a, b]$.

**Example.** Let $C$ be the Cantor set on $[0, 1]$ and $F$ be the Cantor ternary function.

1. It is known that $\lambda = H - \dim(C) = \log 2 / \log 3$.
2. If $x \in C \setminus D$ and $x = \cap_{m=1}^\infty (a_m^{j(m)}, b_m^{j(m)})$ then

$$\frac{F(b_m^{j(m)}) - F(a_m^{j(m)})}{(b_m^{j(m)} - a_m^{j(m)})^\lambda} = 1,$$

that is, $F'_{\lambda P}(x) = 1$ on $C \setminus D$. Also $F'(x) = 0$ on $[0, 1] \setminus C$. 


(3) Let $f : [0, 1] \to \mathbb{R}$ be such that

$$f(x) = 0 \ (x \in [0, 1] \setminus C), = 1 \ (x \in C).$$

Since $F$ is continuous on $[0, 1]$, $F'(x) = 0 = f(x) \ (x \in [0, 1] \setminus C)$, and $F'_\lambda(x) = 1 = f(x) \ (x \in C \setminus D)$, the function $f$ is $\lambda P$-integrable on $[0, 1]$ and

$$(\lambda P) \int_0^1 f = F(1) - F(0) = 1.$$

References


[37] P. Romanovskii. Integrale de Denjoy dans les espaces abstraits, Mat. Sb. 9 (1941), 67–120.