<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>The existence of time-dependent global solutions, asymptotic behavior and stability of the solutions for semi-linear reaction-diffusion equations</td>
</tr>
<tr>
<td>Author(s)</td>
<td>OU ZHUO LANG; HORIUCHI, Toshio; HE MENG XING</td>
</tr>
<tr>
<td>Citation</td>
<td>Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 25: 23-33</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1993</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10109/3029">http://hdl.handle.net/10109/3029</a></td>
</tr>
</tbody>
</table>

このリポジトリに収録されているコンテンツの著作権は、それぞれの著作権者に帰属します。引用、転載、複製等される場合は、著作権法を遵守してください。
The existence of time-dependent global solutions, asymptotic behavior and stability of the solutions for semi-linear reaction-diffusion equations

OU ZHUO LING, TOSHIO HORIUCHI AND HE MENG XING*

Introduction

C.V. Pao used in [2] the so-called upper and lower solution methods to present a qualitative analysis for a coupled system of two reaction-diffusion equations. In this paper, our aim is to generalize the qualitative results to a system of three reaction-diffusion equations. Through suitable constructions of upper and lower solutions, the existence of time-dependent solutions, asymptotic behavior of the solutions and stability of nontrivial steady-state solutions are obtained. In section 1, we establish the existence-comparison theorem for the time-dependent system of three reaction-diffusion equations. The proof of this theorem involves the construction of two sequences which converge monotonically to a unique solution of the corresponding system. The asymptotic behavior of the time-dependent solutions is given in Section 2. We obtain the stability of steady-state solutions in Section 3.

1. The Existence-Comparison Theorem

Let Ω be a bounded domain of $\mathbb{R}^n$ ($n \geq 1$), where the boundary $\partial \Omega$ is smooth. Let $k$ be a nonnegative integer and let $\alpha$ satisfy $0 < \alpha < 1$. By $H^{k+\alpha}(\Omega)$ we denote the set of all functions on $\Omega$ whose $k$-th order partial derivatives are uniformly Hölder continuous with exponent $\alpha$. We consider the following basic reaction-diffusion equations (1.1) together with the boundary and initial conditions (1.2) and (1.3):

\[
\begin{align*}
(u_1)_t - L_1 u_1 &= -u_1 f_1(u_2) \\
(u_2)_t - L_2 u_2 &= u_1 f_1(u_2) - u_2 f_2(u_1) \\
(u_3)_t - L_3 u_3 &= u_2 f_2(u_1) - u_3 f_3(u_1, u_2)
\end{align*}
\]

(1.1)

where $(u_i)_t = \frac{\partial}{\partial t} u_i(t, x)$ and $L_i$ are uniformly elliptic operators of the form

\[
L_i \equiv \sum_{j,k} a_{jk}^i(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^n a_{j}^i(x) \frac{\partial}{\partial x_j}, \quad (i = 1, 2, 3),
\]

Received March 10, 1993.

1991 Mathematics Subject Classification. 35B05.

* Department of Scientific and Research, Wuhan University of Technology, Wuhan, 430070, China; Ibaraki University, Japan and Wuhan University of Technology, China.
where \( a_{i,j,k} \) and \( a_i^j \in H^\alpha(\Omega) \).

The boundary and the initial conditions are defined by

\[
(1.2) \quad B_i[u_i] \equiv \alpha_i(x) \partial u_i / \partial \nu + \beta_i(x) u_i = h_i, \quad (t \in (0,T], x \in \partial \Omega, i = 1, 2, 3),
\]

\[
(1.3) \quad u_i(0, x) = u_{i,0}(x), \quad (x \in \Omega, \quad i = 1, 2, 3),
\]

where \( h_i \in H^{2+\alpha}(\partial \Omega \times [0,T]) \) \( u_{i,0} \in H^{2+\alpha}(\Omega) \), \( \alpha_i \) and \( \beta_i \in H^{2+\alpha}(\Omega) \) and \( \alpha_i \geq 0, \beta_i \geq 0 \) with \( \alpha_i + \beta_i > 0 \) on \( \partial \Omega \). Here \( \nu \) is the outward normal on \( \partial \Omega \), \( f_1 \) and \( f_2 \) are Lipschitz continuous functions on \( R^1 \) and \( f_3 \) is Lipschitz continuous function on \( R^2 \) as well.

For convenience, we set \( D_T = (0,T] \times \Omega, \overline{D_T} = [0,T] \times \Omega \), and by \( C(\overline{D_T}) \) we denote the set of all continuous functions on \( \overline{D_T} \), where \( T < +\infty \).

**DEFINITION 1.1.** \( P = (p_1, p_2, p_3) \) and \( Q = (q_1, q_2, q_3) \) are said to be an ordered pair if \( Q \subset P \) in \( D_T \) and they satisfy the following inequalities:

\[
(1.4) \quad B_i[p_i] - h_i \geq 0 \geq B_i[q_i] - h_i, \quad (t \in (0,T], x \in \partial \Omega, \quad i = 1, 2, 3),
\]

\[
(1.5) \quad p_i(0, x) \geq u_{i,0}(x) \geq q_i(0, x), \quad (x \in \Omega, \quad i = 1, 2, 3).
\]

In this paper, by a smooth function \( U(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x)) \) we mean that \( u_1, u_2 \) and \( u_3 \) are continuously differentiable in \( t \), twice continuously differentiable in \( x \).

**DEFINITION 1.2.** Let \( P = (p_1, p_2, p_3) \), and \( Q = (q_1, q_2, q_3) \) be an ordered pair of smooth functions in \( D_T \) satisfying inequalities (1.4), and (1.5). Then \( P \) and \( Q \) are called upper and lower solutions of (1.1)-(1.3), respectively, if

\[
\begin{align*}
(p_1)_t - L_1 p_1 + p_1 f_1(q_2) & \geq 0 \geq (q_1)_t - L_1 q_1 + q_1 f_1(p_2), \\
(p_2)_t - L_2 p_2 - p_2 f_2(q_1) + p_2 f_2(q_1) & \geq 0 \geq (q_2)_t - L_2 q_2 - q_2 f_1(q_2) + q_2 f_2(p_1), \\
(p_3)_t - L_3 p_3 - p_3 f_2(p_1) + p_3 f_3(q_1, q_2) & \geq 0 \geq (q_3)_t - L_3 q_3 - q_3 f_2(p_1) + q_3 f_3(p_1, p_2),
\end{align*}
\]

Suppose that \( f_1, f_2 \) and \( f_3 \) are non-negative monotone increasing, and let \( M_1, M_2 \) and \( M_3 \) be positive numbers and satisfy the inequalities \( M_1 \geq f_1(p_2), \ M_2 \geq f_2(p_1), \ M_3 \geq f_3(p_1, p_2) \). In order to establish the existence-comparison theorem in terms of upper and lower solutions we consider the sequence \( \{U^k\} = \{(u_1^k, u_2^k, u_3^k)\} \) which solves the following linear system

\[
\begin{align*}
(u_1^k)_t - L_1 u_1^k + u_1^k M_1 u_1^{k-1} = M_1 u_1^{k-1} - u_1^{k-1} f_1(u_2^{k-1}), \\
(u_2^k)_t - L_2 u_2^k + M_2 u_2^{k-1} = M_2 u_2^{k-1} + u_2^{k-1} f_1(u_2^{k-1}) - u_2^{k-1} f_2(u_1^{k-1}), \\
(u_3^k)_t - L_3 u_3^k + M_3 u_3^{k-1} = M_3 u_3^{k-1} + u_3^{k-1} f_2(u_1^{k-1}) - u_3^{k-1} f_3(u_1^{k-1}, u_2^{k-1}).
\end{align*}
\]
under the boundary and initial conditions:

\begin{align}
B_i[u_i^k] &= h_i(x), \quad (t \in (0,T), x \in \partial \Omega, \quad i = 1, 2, 3, \cdots), \\
u_i^k(0, x) &= u_{i,0}(x), \quad (x \in \Omega, \quad i = 1, 2, 3),
\end{align}

where \(k = 1, 2, 3, \cdots\). For each \(k\), this system consists of three linear uncoupled initial boundary value problems, and therefore the existence of \(\{U^k\} = \{(u_1^k, u_2^k, u_3^k)\}\) follows from the standard existence theorem. To ensure that \(\{U^k\}\) is a monotone sequence and converges to the unique solution of (1.1)-(1.3) it is necessary to choose a proper initial iteration. We choose \((p_1, p_2, p_3)\) and \((q_1, q_2, q_3)\) as two distinct initial iterations and denote the corresponding sequences from (1.7)-(1.9) by \(\{V^k\} = \{(v_1^k, v_2^k, v_3^k)\}\) and \(\{N^k\} = \{(n_1^k, n_2^k, n_3^k)\}\), respectively. We use the initial iteration \((v_0^1, v_0^2, v_0^3) = (p_1, p_2, p_3)\) to construct the sequence \(\{V^k\} = \{(v_1^k, v_2^k, v_3^k)\}\) from the equations

\begin{align}
\begin{cases}
(v_1^k)_t - L_1 v_1^k + M_1 v_1^k &= M_1 v_1^{k-1} - v_1^{k-1} f_1(n_2^{k-1}), \\
(v_2^k)_t - L_2 v_2^k + M_2 v_2^k &= M_2 v_2^{k-1} + v_2^{k-1} f_1(v_1^{k-1}) - v_2^{k-1} f_2(n_1^{k-1}), \\
(v_3^k)_t - L_3 v_3^k + M_3 v_3^k &= M_3 v_3^{k-1} + v_3^{k-1} f_1(v_1^{k-1}) - v_3^{k-1} f_3(n_1^{k-1}, n_2^{k-1}).
\end{cases}
\end{align}

While the sequence \(\{N^k\} = \{(n_1^k, n_2^k, n_3^k)\}\) with \((n_0^1, n_0^2, n_0^3) = (q_1, q_2, q_3)\) is determined from the equations

\begin{align}
\begin{cases}
(n_1^k)_t - L_1 n_1^k + M_1 n_1^k &= M_1 n_1^{k-1} - n_1^{k-1} f_1(v_2^{k-1}), \\
(n_2^k)_t - L_2 n_2^k + M_2 n_2^k &= M_2 n_2^{k-1} + n_2^{k-1} f_1(n_2^{k-1}) - n_2^{k-1} f_2(v_1^{k-1}), \\
(n_3^k)_t - L_3 n_3^k + M_3 n_3^k &= M_3 n_3^{k-1} + n_2^{k-1} f_2(n_1^{k-1}) - n_3^{k-1} f_3(v_1^{k-1}, v_2^{k-1}).
\end{cases}
\end{align}

In these systems, the boundary and initial conditions are given by (1.8) and (1.9). With this construction we can establish our existence-comparison theorem in relation to upper and lower solutions.

**Theorem 1.1.** Let \((p_1, p_2, p_3), (q_1, q_2, q_3)\) be an ordered pair of upper and lower solutions of (1.1)-(1.3), let \(f_i\) be non-negative, Lipschitz continuous and monotone increasing \((i = 1, 2, 3)\). Then the sequences \(\{V^k\} = \{(v_1^k, v_2^k, v_3^k)\}\) and \(\{N^k\} = \{(n_1^k, n_2^k, n_3^k)\}\) obtained from (1.8)-(1.11) converge monotonically from above and below, respectively, to a unique solution \(u = (u_1, u_2, u_3)\) of (1.1)-(1.3) such that

\begin{align}
q_i(x, t) \leq u_i(t, x) \leq p_i(t, x) \quad ((t, x) \in \overline{D_T}, \quad i = 1, 2, 3).
\end{align}
Before the proof we give some remarks. First the Lipschitz continuity of \( f_i \) is needed only to ensure the uniqueness of the solutions for the system (1.1). Hence in order to get the comparison inequalities (1.12) in this theorem, it suffices to assume the Hölder continuity on each \( f_i \). Secondly it is sufficient that each \( f_i \) is defined only on the set \( S_i \) \( (i = 1, 2, 3) \) respectively, where

\[
S_1 = \{ x : \min_{\bar{D}_T} q_2 \leq x \leq \max_{\bar{D}_T} p_2 \}
\]

\[
S_2 = \{ x : \min_{\bar{D}_T} q_1 \leq x \leq \max_{\bar{D}_T} p_1 \}
\]

\[
S_3 = \{ (x, y) : \min_{\bar{D}_T} q_1 \leq x \leq \max_{\bar{D}_T} p_1, \min_{\bar{D}_T} q_2 \leq y \leq \max_{\bar{D}_T} p_2 \}.
\]

Lastly we also note that the assertions in this theorem are able to be generalized in various ways. For example, we can increase the number of chemical substances reacting on each other, namely it is not difficult to deal with \( n \)-reaction-diffusion equations of the same type, and we can also treat the case under some assumptions that each function \( f_i \) \( (i = 1, 2, \ldots, n) \) depends on the variables \((t, x)\). (c.f. [2]).

**Proof.** Since \( f_i \) \( (i = 1, 2, 3) \) satisfy the Lipschitz condition, the existence and the uniqueness of the solution of problem (1.1), (1.2) and (1.3) can be shown by the standard method of successive approximation (cf [2]). Hence our main aim is to show that the sequences defined by (1.10) and (1.11) converge monotonically to the unique solution. Let \( w_i = v_i^0 - v_i^1 = p_i - v_i^1, (i = 1, 2, 3), \) then by (1.4)-(1.6) and (1.10),

\[
(w_1)_t - L_1 w_1 + M_1 w_1 = ((p_1)_t - L_1 p_1 + M_1 p_1)
- (M_1 p_1 - p_1 f_1(q_2)) \geq 0,
\]

\[
(w_2)_t - L_2 w_2 + M_2 w_2 = ((p_2)_t - L_2 p_2 + M_2 p_2)
- (M_2 p_2 + p_1 f_1(p_2) - p_2 f_2(q_1)) \geq 0,
\]

\[
(w_3)_t - L_3 w_3 + M_3 w_3 = ((p_3)_t - L_3 p_3 + M_3 p_3)
- (M_3 p_3 + p_2 f_2(p_1) - p_3 f_3(q_1, q_2)) \geq 0,
\]

\[
B_i[w_i] = B[p_i] - h_i \geq 0,
\]

\[
w_i(0, x) = p_i(0, x) - u_{i,0}(x) \geq 0, \quad i = 1, 2, 3.
\]

By the maximum principle for the weakly coupled system of diffusion equations, the above inequalities imply that \( w_i \geq 0 \) on \( \bar{D}_T \) for each \( i = 1, 2, 3 \). Similarly, using relations (1.4)-(1.6) and (1.11), the functions \( w_i = n_i^1 - n_i^0 = n_i^1 - q_i \) satisfy the inequalities in (1.14)-(1.17) and thus \( n_i^0 \leq n_i^1, i = 1, 2, 3 \). Now let \( w_i = v_i^1 - n_i^1 \).
Then the monotone increasing property of \( f_i(i = 1, 2, 3) \) and the relations in (1.10) and (1.11) imply that

\[
\begin{align*}
(w_1)_t - L_1 w_1 + M_1 w_1 &= (M_1 p_1 - p_1 f_1(q_2)) \\
-(M_1 q_1 - q_1 f_1(p_2)) &\geq 0 \\
(w_2)_t - L_2 w_2 + M_2 w_2 &= (M_2 p_2 + p_1 f_1(p_2) - p_2 f_2(q_1)) \\
-(M_2 q_2 + q_1 f_1(q_2) - q_2 f_2(p_1)) &\geq 0 \\
(w_3)_t - L_3 w_3 + M_3 w_3 &= (M_3 p_3 + p_2 f_2(p_1) - p_3 f_3(q_1, q_2)) \\
-(M_3 q_3 + q_2 f_2(q_1) - q_3 f_3(p_1, p_2)) &\geq 0.
\end{align*}
\]

(1.18)

Since \( B_i[w_i] = 0 \) and \( w_i(0, x) = 0 \), then \( w_i \geq 0 \) and

\[ n_i^0 \leq n_i^1 \leq v_i^0, \quad (i = 1, 2, 3). \]

Assume inductively that

\[ n_i^{k-1} \leq n_i^k \leq v_i^k \leq v_i^{k-1}, \quad (i = 1, 2, 3, \quad k = 1, 2, \cdots, m). \]

(1.19)

Then the functions \( w_i = v_i^m - v_i^{m+1} \) satisfy \( B_i[w_i] = 0, \ w_i(0, x) = 0 \) and the relations

\[
\begin{align*}
(w_1)_t - L_1 w_1 + M_1 w_1 &= (M_1 v_1^{m-1} - v_1^{m-1} f_1(n_2^{m-1})) \\
-(M_1 v_1^m - v_1^m f_1(v_2^m)) &\geq 0 \\
(w_2)_t - L_2 w_2 + M_2 w_2 &= (M_2 v_2^{m-1} + v_1^{m-1} f_1(v_2^{m-1})) \\
-v_2^{m-1} f_2(n_1^{m-1}) &\geq 0 \\
(w_3)_t - L_3 w_3 + M_3 w_3 &= (M_3 v_3^{m-1} + v_2^{m-1} f_2(v_1^{m-1})) \\
-v_3 f_3(n_1^{m-1}, n_2^{m-1}) &\geq 0 \\
(w_3)_t - L_3 w_3 + M_3 w_3 &= (M_3 v_3^m + v_2^m f_2(n_1^m) - v_3^m f_3(n_1^m, n_2^m)) \geq 0,
\end{align*}
\]

(1.20)

which ensure that \( v_i^m \geq v_i^{m+1}, i = 1, 2, 3 \). The same reasoning leads to the inequalities \( n_i^m \leq n_i^{m+1} \) and \( n_i^{m+1} \leq v_i^{m+1} \). This proves the monotone relation (1.19) for every \( k \). It follows from this monotone property that the pointwise limits

\[
\begin{align*}
\lim_{k \to \infty} v_i^k(t, x) &= v_i(t, x), \\
\lim_{k \to \infty} n_i^k(t, x) &= n_i(t, x), \quad i = 1, 2, 3.
\end{align*}
\]

(1.21)

exist and \( n_i \leq v_i \) on \( \mathcal{D}_T \). A standard regularity argument shows that the set of functions \((v_1, v_2, v_3, n_1, n_2, n_3)\) and \((n_1, n_2, n_3, v_1, v_2, v_3)\) are the solutions of the
system

\[
\begin{align*}
(w_1)_t - L_1 w_1 &= -w_1 f_1(w_5) \\
(w_2)_t - L_2 w_2 &= w_1 f_1(w_2) - w_2 f_2(w_4) \\
(w_3)_t - L_3 w_3 &= w_2 f_2(w_1) - w_3 f_3(w_5, w_6) \\
(w_4)_t - L_4 w_4 &= -w_4 f_1(w_2) \\
(w_5)_t - L_2 w_5 &= w_4 f_1(w_5) - w_2 f_2(w_1) \\
(w_6)_t - L_3 w_6 &= w_5 f_2(w_4) - w_6 f_3(w_1, w_2) \\
B_i[w_i] &= h^*_i, \quad u_i(0, x) = u^*_{i,0}(x), \quad (i = 1, 2, \ldots, 6),
\end{align*}
\]

where \( h^*_1 = h^*_2 = h^*_3 \), \( h^*_2 = h^*_5 = h^*_6 \), \( h^*_3 = h^*_6 = h^*_3 \), \( u^*_{1,0} = u^*_{4,0} = u^*_1, u^*_{2,0} = u^*_{5,0} = u^*_{2,0}, u^*_{3,0} = u^*_{5,0} = u^*_3 \). As we mentioned just after the statement of this theorem, the modified system (1.22) as well as the system (1.1) has a unique solution under the boundary and the initial conditions as above. In fact, one can show the existence and the uniqueness of the solution of (1.22) by the method of successive approximation. Therefore, we have \( v_1 = n_1, v_2 = n_2, v_3 = n_3 \) by uniqueness, and this proves that \((v_1, v_2, v_3)\) is the unique solution of (1.1) with (1.2) and (1.3). Thus the proof of the theorem is completed.

2. The asymptotic behavior of the solution for a system

In this section, we consider a system as follows:

\[
\begin{align*}
(u_1)_t - \nabla(A_1 \nabla u_1) &= -c_1 G_1(u_2)u_1 \\
(u_2)_t - \nabla(A_2 \nabla u_2) &= c_1 G_1(u_2)u_1 - c_2 G_2(u_1)u_2 \\
(u_3)_t - \nabla(A_3 \nabla u_3) &= c_2 G_2(u_1)u_2 - c_3 G_1(u_2)u_3.
\end{align*}
\]

together with the boundary and initial conditions (1.2) and (1.3), where \( u_i \equiv u_i(t, x), A_i \equiv A_i(x) \) and \( c_i \) are positive constants \((i = 1, 2, 3)\). The functional \( G_i(u) \) is given by

\[
(G_i(u))(t, x) \equiv \int_{\Omega} m_i(x, x') u(t, x') dx',
\]

where \( m_i \) is a given positive continuous function in \( \Omega \times \Omega \) and \( A_i \) \((i=1,2,3)\) are assumed positive and smooth. It is clear that (2.1) is the special case of (1.1). In the present system, upper and lower solutions are required to satisfy the relations

\[
\begin{align*}
(p_1)_t - \nabla(A_1 \nabla p_1) + c_1 G_1(q_2)p_1 &\geq 0 \\
(q_1)_t - \nabla(A_1 \nabla q_1) + c_1 G_1(p_2)q_1, \\
(p_2)_t - \nabla(A_2 \nabla p_2) - c_1 G_1(p_2)p_1 + c_2 G_2(q_1)p_2 &\geq 0 \\
(q_2)_t - \nabla(A_2 \nabla q_2) - c_1 G_1(q_2)q_1 + c_2 G_2(p_1)q_2, \\
(p_3)_t - \nabla(A_3 \nabla p_3) - c_2 G_2(p_1)p_2 + c_3 G_1(q_2)p_3 &\geq 0 \\
(q_3)_t - \nabla(A_3 \nabla q_3) - c_2 G_2(q_1)q_2 + c_3 G_1(p_2)q_3,
\end{align*}
\]
The main goal of this section is to construct suitable functions \((p_1, p_2, p_3)\) and \((q_1, q_2, q_3)\) so that the asymptotic behavior of the solution can be determined. Our construction of upper and lower solutions often makes use of the smallest eigenvalue \(\lambda_i\) and the corresponding eigenfunction \(\phi_i\) of the eigenvalue problem

\[
\begin{cases}
\nabla(A_i \nabla \phi_i) + \lambda_i \phi_i = 0, & (x \in \Omega), \\
B_i[\phi_i] = 0, & (x \in \partial \Omega),
\end{cases}
\]

\(i = 1, 2, 3.\)

It is clear that \(\lambda_i\) and \(\phi_i\) are positive in \(\Omega\). We normalize each \(\phi_i\) so that \(\max_{x \in \Omega} \phi_i(x) = 1\). In the following theorem we establish the existence and asymptotic property of the solution for the problem of (2.1), (1.2) and (1.3).

**Theorem 2.1.** Let \(h_i \geq 0, i=1,2,3,\) and let \(u_{i,0} \geq 0, i=1,2,3.\) Then the system (2.1), with (1.2) and (1.3) has the unique non-negative global solution \((u_1, u_2, u_3)\). Furthermore, if \(h_i = 0\) and \(\lambda_i > 0\) then there exist positive constants \(\rho_1, \rho_2, T_1\) and \(T_2\) such that for any \(b_1 < \lambda_2, b_2 < \lambda_3,\) the solution \((u_1, u_2, u_3)\) satisfies

\[
\begin{cases}
0 \leq u_1(t, x) \leq \rho_1 e^{\lambda_1 t}, & (t \geq 0, x \in \overline{\Omega}) \\
0 \leq u_2(t, x) \leq \rho_2 e^{b_1(t-T_1)}, & (t \geq T_1, x \in \overline{\Omega}) \\
0 \leq u_3(t, x) \leq \rho_3 e^{b_2(t-T_2)}, & (t \geq T_2, x \in \overline{\Omega}).
\end{cases}
\]

**Proof.** Let \(u_{1}^*, u_{2}^*\) and \(u_{3}^*\) be the solutions of the linear uncoupled systems, respectively,

\[
\begin{cases}
(u_1)_t - \nabla(A_1 \nabla u_1) = 0, \\
B_1[u_1] = h_1, \quad u_1(0, x) = u_{1,0}(x),
\end{cases}
\]

\[
\begin{cases}
(u_2)_t - \nabla(A_2 \nabla u_2) = c_1 u_1^* G_1(u_2), \\
B_2[u_2] = h_2, \quad u_2(0, x) = u_{2,0}(x),
\end{cases}
\]

\[
\begin{cases}
(u_3)_t - \nabla(A_3 \nabla u_3) = c_2 u_2^* G_2(u_3^*), \\
B_3[u_3] = h_3, \quad u_3(0, x) = u_{3,0}(x),
\end{cases}
\]

where \(c_1\) and \(c_2\) are positive constants. By the non-negative property of \(h_i\) and \(u_{1,0}\), the non-negative solution \(u_1^*\) to (2.5) exists. Since \(u_1^*\) is known and \(c_1 G_1(u_2)\) is a linear functional of \(u_2\), the existence of a solution \(u_2^*\) to (2.6) follows from the standard method of successive approximations. To show that \(u_2^*\) is non-negative on \([0, T] \times \overline{\Omega}\) for every \(T < +\infty\), we make the transformation \(w_1 = e^{-\sigma t} u_2^*\) for a sufficiently large \(\sigma\). Then the system (2.6) is transformed into the form

\[
\begin{cases}
(w_1)_t - \nabla(A_2 \nabla w_1) + \sigma w_1 = c_1 u_1^* G_1(w_1), \\
B_2[w_1] = e^{-\sigma t} h_2, \quad w_1(0, x) = u_{2,0}(x).
\end{cases}
\]
Suppose, by contradiction, \( w_1 \) attains a negative minimum at some point \((t_1, x_1) \in [0, T] \times \Omega \). Then by the boundary and initial conditions in (2.8), \( x_1 \not\in \partial \Omega, t_1 \neq 0 \). This implies that \((t_1, x_1) \in (0, T] \times \Omega \) and thus we have \( w_1(t_1, x_1) \leq 0 \) and \( \nabla(A_2(x_1) \nabla w_1(t_1, x_1)) \geq 0 \). In view of the relation (2.6) and the non-negative property of \( u_1^* \), we get

\[
\sigma w_1(t_1, x_1) \geq c_1 u_1^*(t_1, x_1) (G_1(w_1)(t_1, x_1)) \geq w_1(t_1, x_1) [c_1 u_1^*(t_1, x_1) \int \Omega m_1(x, x') dx'].
\]

This relation is impossible since \( \sigma \) can be arbitrarily large. So now we get \( w_1(t_1, x_1) \geq 0 \), and this proves the existence and non-negative property of \( u_2^* \). Now we make the transformation \( w_2 = e^{-\sigma t} u_2^* \), for a sufficiently large \( \sigma \). Then the system (2.7) is transformed into the form

\[
\begin{cases}
(w_2)_t - \nabla(A_3 \nabla w_2) + \sigma w_2 = c_2 u_2^* G_2(u_1^*), \\
B_3[w_2] = e^{-\sigma t} h_3, \quad w_2(0, x) = u_{3,0}(x),
\end{cases}
\]

(2.9)

so in a similar way, it is easy to show the existence and non-negative property of \( u_3^* \). Putting \((p_1, p_2, p_3) = (u_1^*, u_2^*, u_3^*), (q_1, q_2, q_3) = (0, 0, 0)\), it is immediately shown that the inequalities in (2.2), (1.2) and (1.3) are satisfied. Then it follows from theorem 1.1 that the system (2.1) with (1.2) and (1.3) has the unique solution \((u_1, u_2, u_3)\) such that

\[
\begin{cases}
0 \leq u_1(t, x) \leq u_1^*(t, x) \\
0 \leq u_2(t, x) \leq u_2^*(t, x) \\
0 \leq u_3(t, x) \leq u_3^*(t, x).
\end{cases}
\]

(2.10)

If the boundary condition (1.3) is replaced by the Neumann type, the same argument shows that the problem has a unique non-negative solution \((u_1, u_2, u_3)\) such that (2.10) holds. In this situation, \( u_1^*, u_2^* \) and \( u_3^* \) are the non-negative solutions of (2.5), (2.6) and (2.7) respectively with the boundary conditions:

\[
\partial u_1^*/\partial \nu = \partial u_2^*/\partial \nu = \partial u_3^*/\partial \nu = 0.
\]

To show that relation (2.4) when \( h_i = 0 \), we note that the solution \( u_1^* \) of (2.5) satisfies the relation \( 0 \leq u_1^* \leq \rho_1 e^{-\lambda_1} \) \((\text{cf} [1])\). Hence the first relation in (2.4) follows from (2.10). For the second relation we apply the comparison theorem, in terms of upper and lower solutions, for the scalar system (2.6). In fact, since the function \( c_1 u_1^* G_1(u_2) \) is monotone increasing in \( u_2 \), we may consider (2.6) as a special case of (1.1)-(1.3). In view of Theorem 1.1 it suffices to find a suitable pair of upper and lower solutions. For this purpose, we choose \( T_1 > 0 \) and a corresponding \( \rho_2 > 0 \) such that

\[
\begin{cases}
u_1^*(T_1, x) \leq \rho_2 \phi_2(x) \\
u_1^*(t, x) \leq m_2 \phi_2 \left( c_2 \int \Omega m_2(x, x') \phi_2(x') dx' \right)^{-1}, \quad (t \geq T_1),
\end{cases}
\]

(2.11)
The existence, asymptotic behavior and stability of solutions

where $m = \lambda_2 - b_1 > 0$. This is possible since $u^*_1 \to 0$ as $t \to \infty$. By considering $u^*_1(T_1, x)$ as the initial function, the function $p_2 = \rho_2 e^{-b_1(t-T_1)}$ is an upper solution of the system (2.6) in the domain $[T_1, +\infty) \times \Omega$ provided that

$$
(2.12) \quad (-b_1 \phi_2 - \nabla(A_2 \nabla \phi_2)) \rho_2 e^{-b_1(t-T_1)} \geq c_2 u^*_1 \rho_2 e^{-b_1(t-T_1)} \int_{\Omega} m_2(x, x') \phi_2(x') dx', \quad (t > T_1, x \in \Omega).
$$

This inequality is equivalent to

$$
(2.13) \quad (\lambda_2 - b_1) \phi_2 \geq c_2 u^*_1 \int_{\Omega} m_2(x, x') \phi_2(x') dx', \quad (t > T_1, x \in \Omega),
$$

which is clearly satisfied by the relation (2.11). Since $q_2 = 0$ is obviously a lower solution of (2.6) in the domain $[T_1, +\infty) \times \Omega$, the comparison theorem for scalar system ensures that

$$
(2.14) \quad 0 \leq u_2 \leq \rho_2 e^{-b_1(t-T_1)} \phi_2, \quad (t \in [T_1, +\infty), x \in \Omega),
$$

thus we have

$$
(2.15) \quad 0 \leq u_2 \leq \rho_2 e^{-b_1(t-T_1)} \phi_2.
$$

The same reasoning leads to

$$
(2.16) \quad 0 \leq u_3 \leq \rho_3 e^{-b_3(t-T_3)} \phi_3, \quad (t \in [T_2, +\infty), x \in \Omega).
$$

This completes the proof of the theorem.

3. The asymptotic stability of a given non-negative steady-state solution

Now we investigate the stability problem of the inhomogeneous system (2.1) with (1.2) and (1.3) when all $h_i$ are not identically zero. Here the definitions of stability and asymptotic stability are in the usual sense of Liapunov. Here a steady-state solution is a solution $(u_1^s, u_2^s, u_3^s)$ of the boundary value problem

$$
(3.1) \quad \begin{cases}
-A_1 \Delta u_1 = -c_1 G_1(u_2) u_1 \\
-A_2 \Delta u_2 = c_1 G_1(u_2) u_1 - c_2 G_2(u_1) u_2 \\
-A_3 \Delta u_3 = c_2 G_2(u_1) u_2 - c_3 G_1(u_2) u_3
\end{cases} \quad (x \in \Omega),
$$

$$
(3.2) \quad B[u_i] = \alpha \partial u_i / \partial \nu + \beta u_i = h_i, \quad (x \in \partial \Omega, \quad i = 1, 2, 3),
$$

where $G_i(u)(i = 1, 2, 3)$ is defined in the Section 2. The determination of the stability of $(u_1^s, u_2^s, u_3^s)$ can be achieved by a suitable construction of upper and lower solutions. We shall construct such functions by means of the smallest eigenvalue $\lambda_0$ and the corresponding eigenfunction $\phi_0$ of the eigenvalue problem:

$$
(3.3) \quad \begin{cases}
\nabla^2 \phi_0 + \lambda_0 \phi_0 = 0, \quad (x \in \Omega), \\
B[\phi_0] = 0, \quad (x \in \partial \Omega).
\end{cases}
$$

For convenience, we set $G_0 = \max_i \sup_{x \in \Omega} |G_i(\phi_0)|$. In the following theorem we establish a sufficient condition for the asymptotic stability of a given non-negative steady-state solution.
THEOREM 3.1. Let $A_i$ be positive constants and let $(u_{1s}, u_{2s}, u_{3s})$ be a non-negative steady-state solution of (3.1) and (3.2). If there exist positive constants $\gamma_1, \gamma_2$ and $\varepsilon$ such that

\[
\begin{align*}
\lambda_0 A_1 + c_1 G_1(u_{2s}) - \gamma_1 c_1 G_1(\phi_0)u_{1s}/\phi_0 &\geq \varepsilon \\
\gamma_1 \lambda_0 A_2 + \gamma_2 G_2(u_{1s}) - \gamma_1 c_1 G_1(\phi_0)u_{1s}/\phi_0 &\geq \varepsilon \\
- c_1 G_1(u_{2s}) - G_2(\phi_0)u_{2s}/\phi_0 &\geq \gamma_1 \varepsilon \\
\gamma_2 \lambda_0 A_3 + \gamma_2 c_3 G_1(u_{2s}) - \gamma_1 c_2 G_2(\phi_0)u_{1s} &\geq \gamma_2 \varepsilon \\
- \gamma_1 c_3 G_1(\phi_0)u_{3s}/\phi_0 - c_2 G_2(\phi_0)u_{2s}/\phi_0 &\geq \gamma_2 \varepsilon ,
\end{align*}
\]

then the solution $(u_1, u_2, u_3)$ of (2.1) with (1.2) and (1.3) satisfies the relation

\[
(3.5) \\
\begin{align*}
(u_{1s}(x) - g(t))\phi_0(x) &\leq u_1(t, x) \leq u_{1s}(x) + g(t)\phi_0(x) \\
u_{2s}(x) - \gamma_1 g(t)\phi_0(x) &\leq u_2(t, x) \leq u_{2s}(x) + \gamma_1 g(t)\phi_0(x) \\
u_{3s}(x) - \gamma_2 g(t)\phi_0(x) &\leq u_3(t, x) \leq u_{3s}(x) + \gamma_2 g(t)\phi_0(x),
\end{align*}
\]

where $g(t)$ is defined by

\[
(3.6) \\
g(t) = \left[\eta^{-1} + (g(0)^{-1} - \eta^{-1})e^{\eta t}\right]^{-1},
\]

with $g(0) < \eta < (e/\eta)^{\min\{1, c_1 + c_2\}^{-1}, [\gamma_1 (c_2 + c_3)]^{-1}}$.

PROOF. Let $p_i = u_{is} + g_i\phi_0$, $q_i = u_{is} - g_i\phi_0$, $i = 1, 2, 3$, and $g_i = g_i(t)$ are some positive differentiable functions with $g_1(0) \leq g(0), g_2(0) \geq g_1(0), g_3(0) \geq g_2(0)$ and $g(0) < \eta$. Since by (3.2) and (3.3),

\[
(3.7) \\
B(p_i) = B(q_i) = B(u_{is}) = h_i, \quad i = 1, 2, 3.
\]

the functions $(p_1, p_2, p_3)$ and $(q_1, q_2, q_3)$ are an ordered pair of upper and lower solutions if they satisfy the relation (2.2). Using relations (3.1) for $(u_{1s}, u_{2s}, u_{3s})$, (3.3) for $\phi_0$, and the linearity property of the function $G$, a simple calculation shows that the inequalities in (2.2) hold if $g_1$ and $g_2$ satisfy the relations

\[
\begin{align*}
\begin{cases}
[g'_1 + \lambda_0 A_1 g_1]\phi_0 + c_1 [g_1 G_1(u_{2s}) - (u_{1s} + g_1\phi_0)G_1(g_2\phi_0)] &\geq 0 \\
- [g'_1 + \lambda_0 A_1 g_1]\phi_0 + c_1 [-g_1 G_1(u_{2s}) + (u_{1s} + g_1\phi_0)G_1(g_2\phi_0)] &\leq 0 \\
g'_2 + \lambda_0 A_2 g_2\phi_0 - c_1 [g_1 G_1(u_{2s}) + G_1(g_2\phi_0)u_{1s} + g_1\phi_0 G_1(g_2\phi_0)] + c_2 ([G_2(u_{1s})g_2\phi_0 - G_2(g_1\phi_0)u_{2s}] - G_2(g_1\phi_0)g_2\phi_0) &\geq 0 \\
- [g'_2 + \lambda_0 A_2 g_2]\phi_0 - c_1 [-g_1 G_1(u_{2s}) - G_1(g_2\phi_0)u_{1s} + g_1\phi_0 G_1(g_2\phi_0)] + c_2 ([G_2(u_{1s})g_2\phi_0 - G_2(g_1\phi_0)u_{2s}] - G_2(g_1\phi_0)g_2\phi_0) &\geq 0 \\
g'_3 + \lambda_0 A_3 g_3\phi_0 - c_2 [g_2\phi_0 G_2(g_1\phi_0) + g_2\phi_0 G_2(g_1\phi_0)] + c_3 ([G_1(u_{2s})g_3\phi_0 - G_1(g_2\phi_0)u_{3s}] - G_1(g_2\phi_0)g_3\phi_0) &\geq 0 \\
- [g'_3 + \lambda_0 A_3 g_3]\phi_0 - c_2 [-g_2\phi_0 G_2(g_1\phi_0) - G_2(g_1\phi_0)u_{2s} + g_2\phi_0 G_2(g_1\phi_0)] + [G_1(u_{2s})g_3\phi_0 - G_1(g_2\phi_0)g_3\phi_0] &\leq 0.
\end{cases}
\end{align*}
\]
Now we choose $g_1 = g$, $g_2 = \gamma_1 g$ and $g_3 = \gamma_2 g$. Then it suffices to find $g > 0$ such that

$$
\begin{aligned}
g' + \left[ \lambda_0 A_1 + c_1 G_1(u_{2s}) - \gamma_1 c_1 G_1(\phi_0)u_{1s}/\phi_0 \right] g \\
\geq \gamma_1 c_1 G_1(\phi_0)g^2 \\
\gamma_1 g' + \left[ \gamma_1 \lambda_0 A_2 + c_2 \gamma_1 G_2(u_{1s}) - \gamma_1 c_1 G_1(\phi_0)u_{1s}/\phi_0 \\
- c_1 G_1(u_{2s}) - c_2 G_2(\phi_0)u_{2s}/\phi_0 \right] g \\
\geq \gamma_1 (c_1 + c_2) G_2(\phi_0)g^2 \\
\gamma_2 g' + \left[ \gamma_2 \lambda_0 A_3 + c_2 \gamma_2 G_3(u_{2s}) - \gamma_1 c_3 G_1(\phi_0)u_{3s}/\phi_0 \\
- c_1 G_1(u_{1s}) - c_2 G_2(\phi_0)u_{2s}/\phi_0 \right] g \\
\geq \gamma_1 (c_3 \gamma_2 G_1(\phi_0) + c_2 G_2(\phi_0))g^2
\end{aligned}
$$

(3.9)

By the hypothesis of (3.4) these inequalities are valid if

$$
g' + \varepsilon g \geq (\varepsilon/\eta) g^2,
$$

where $\varepsilon/\eta = \max(\gamma_1 c_1 G_0, (c_1 + c_2) G_0, \gamma_1 (c_2 + c_3) G_0)$. Then we see that the functions $(u_{1s} + g \phi_0, u_{2s} + \gamma_1 g \phi_0, u_{3s} + \gamma_2 g \phi_0)$ and $(u_{1s} - g \phi_0, u_{2s} - \gamma_1 g \phi_0, u_{3s} - \gamma_2 g \phi_0)$ are an ordered pair of upper and lower solutions. It follows from Theorem 1.1 that the unique solution $(u_1, u_2, u_3)$ to (2.1) with (1.2) and (1.3) exists and satisfies the relation (3.5).

References