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<td>Author(s)</td>
<td>SHIMOMURA, Katsunori</td>
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<tr>
<td>Citation</td>
<td>Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 24: 31-37</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1992</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10109/3025">http://hdl.handle.net/10109/3025</a></td>
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A characterization of the inner NTA domain by the quasi-hyperbolic metric

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Dedicated to Professor M. Kishi on the occasion of his 60th birthday

1. Introduction

Let $D$ be a proper subdomain of the $n$-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$). For each $x \in D$, we denote by $\delta_D(x)$ the distance between $x$ and $D^c$, the complement of $D$. According to Gehring and Osgood [2], we define the quasi-hyperbolic metric $k_D$ in $D$ by

$$k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\delta_D(z)}$$

where the infimum is taken over all rectifiable arcs $\gamma$ joining $x$ to $y$ in $D$ and $ds$ is the Euclidean arc length-element on $\gamma$. We define another metric $j_D$ on $D$ by

$$j_D(x, y) = \log \left(1 + \frac{|x - y|}{\min\{\delta_D(x), \delta_D(y)\}}\right).$$

It is known ([3]) that the inequalities

$$\left| \log \frac{\delta_D(x)}{\delta_D(y)} \right| \leq k_D(x, y),$$

$$j_D(x, y) \leq k_D(x, y)$$

hold for all $x, y \in D$.

In [6], we defined an inner NTA domain and estimated the boundary growth of positive harmonic functions on an inner NTA domain in terms of $\delta_D(x)$. Herron and Vuorinen obtained a similar result in [4].

The purpose of this paper is to give a characterization of inner NTA domains using the quasi-hyperbolic metric.

In this paper, we shall prove the following

Received May 26, 1992.
1991 Mathematics Subject Classification. Primary 30C99 Secondary 31B25, 35B05.
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THEOREM. A bounded domain \( D \) is an inner NTA domain if and only if there exists a point \( x_0 \) in \( D \) such that the inequality

\[ k_D(x_0, x) \leq c_jD(x_0, x) + C \]

holds for all \( x \in D \) with some positive constants \( c \) and \( C \).

It is easily seen that if \( D \) is bounded, the above condition is equivalent to

\[ \delta_D(x_0, x) \leq c_1 \log \frac{\delta_D(x_0)}{\delta_D(x)} + c_2 \]

for some positive constants \( c_1 \) and \( c_2 \). The condition (2) are said to be a quasi-hyperbolic boundary condition ([1]), and domains with (2) are called H"older domains ([7]).

The author would like to thank Professor M. Ito for his helpful comments.

2. Preliminaries

For \( x \in \mathbb{R}^n \) and \( r > 0 \), \( B(x, r) \) denotes the closed ball with center \( x \) and radius \( r \), and for a closed ball \( B \), \( r(B) \) denotes its radius.

Let \( M \) be a positive constant and \( N \) a positive integer. An \( M \)-Harnack chain of the length \( N \) in \( D \) is a finite sequence of closed balls \( \{B_j\}_{j=1}^N \) in \( D \) such that \( B_j \cap B_{j+1} \neq \emptyset \) \( j = 1, \ldots, N-1 \), and \( r(B_j) = M d(B_j, D^c) \) \( j = 1, \ldots, N \), where \( d(B_j, D^c) \) denotes the distance between \( B_j \) and \( D^c \). For \( x, y \in D \), we say that \( y \) can be connected with \( x \) by an \( M \)-Harnack chain \( \{B_j\}_{j=1}^N \) of length \( N \) in \( D \) if \( x \) is the center of \( B_1 \) and \( y \in B_N \). \( H_{MN}(x) \) denotes the set of points which can be connected with \( x \) by an \( M \)-Harnack chain of the length \( N \) in \( D \).

Let \( M \) be a positive constant, \( N \) a positive integer and \( 0 < \nu < 1 \) a constant. A \((M,N,\nu)\)-sequence in \( D \) is a possibly finite sequence \( z = \{z_j\} \) of points in \( D \) such that

(a) \( z_{j+1} \in H_{MN}(z_j) \quad j = 1, 2, \ldots \)

and

(b) \( \sup_j \nu^{-j} \delta_D(z_j) < +\infty. \)

For an \((M,N,\nu)\)-sequence \( z \) in \( D \), we put \( H(z) = \bigcup_j H_{MN}(z_j) \).

DEFINITION. Let \( M \) be a positive constant, \( N \) a positive integer and \( 0 < \nu < 1 \) a constant. A bounded domain \( D \) in \( \mathbb{R}^n \) is called an \((M,N,\nu)\)-inner NTA domain if there exist positive constants \( r_0 \), \( h_0 \), \( M_0 \), and a positive integer \( N_0 \) satisfying the following condition: If \( z \in D \) and \( \delta_D(z) \leq r_0 \), we can find an \((M,N,\nu)\)-sequence \( z = \{z_j\} \) with \( \delta_D(z_1) \geq r_0 \) and \( \delta_D(z_j) \leq h_0 \nu^j \quad j = 1, 2, \ldots \) such that \( H_{M_0N_0}(x) \cap H(z) \neq \emptyset. \)
A bounded domain $D$ in $\mathbb{R}^n$ is simply called an inner NTA domain if there exist $M$, $N$ and $\nu$ such that $D$ is an $(M, N, \nu)$-inner NTA domain.

**Remark.** The above definition is a generalization of the original definition of inner NTA domains given in [6], and all results in [6] hold for a present inner NTA domain, which can be verified by a similar way.

For every $x, y \in D$, there exists a quasi-hyperbolic geodesic $\gamma$ joining $x$ to $y$ ([2]). Thus, if $x_1, y_1 \in \gamma$ and $\gamma(x_1, y_1)$ denotes the portion of $\gamma$ joining $x_1$ and $y_1$, then

$$k_D(x_1, y_1) = \int_{\gamma(x_1, y_1)} \frac{ds}{\delta_D(z)}$$

holds.

Let $0 < a < 1$. Then we have

$$k_D(x, y) \geq \log(1 + a)$$

for $|x - y| \geq a\delta_D(x)$. In other words,

$$|x - y| \leq a\delta_D(x)$$

holds if $k_D(x, y) \leq \log(1 + a)$.

On the other hand, we obtain

$$k_D(x, y) \leq \log \frac{1}{1 - a}$$

for $|x - y| \leq a\delta_D(x)$ considering the line segment from $x$ to $y$.

As is easily seen, the two metrics $k_D$ and $j_D$ give $D$ the same topology as the relative Euclidean topology.

3. Main Result

We start with two lemmas.

**Lemma 1.** $k_D(x, y) \leq (2N - 1)\log(1 + M)$ holds for all $y \in H_{MN}(x)$. Conversely if $k_D(x, y) \leq (2N - 1)\log(1 + \frac{M}{1 + M})$, then $y \in H_{MN}(x)$.

**Proof.** If $y \in H_{MN}(x)$, then $y$ can be connected with $x$ by an $M$-Harnack chain $\{B_j\}_{j=1}^N$. Let $x_1 = x$, $x_2 = y$, $x_{2j-1} = \text{the center of } B_j$ and choose $x_{2j} \in B_j \cap B_{j+1}$, $j = 1, 2, \ldots, N - 1$. By the condition of $M$-Harnack chain, we have

$$\delta_D(x_{2j-1}) = r(B_j) + d(B_j, D^c) = \frac{1 + M}{M} r(B_j),$$

for $j = 1, \ldots, N$. This and $x_{2j} \in B_j \cap B_{j+1}$ show

$$|x_{2j-1} - x_{2j}| \leq r(B_j) \leq \frac{M}{1 + M} \delta_D(x_{2j-1}),$$

$$|x_{2j+1} - x_{2j}| \leq r(B_{j+1}) \leq \frac{M}{1 + M} \delta_D(x_{2j+1}).$$
Since $0 < \frac{M}{1 + M} < 1$, the inequality (4) implies
\[ k_D(x_{2j-1}, x_{2j}) \leq \log(1 + M), \]
and then
\[ k_D(x, y) \leq \sum_{i=1}^{2N-1} k_D(x_i, x_{i+1}) \leq (2N - 1) \log(1 + M). \]

Conversely we assume that $k_D(x, y) \leq (2N - 1) \log\left(1 + \frac{M}{1 + M}\right)$. Let $\gamma$ be a quasi-hyperbolic geodesic joining $x$ to $y$. Choose the points $\{z_j\}_{j=1}^{2N}$ on $\gamma$ satisfying $x_1 = x$, $x_{2N} = y$ and $k_D(x_j, x_{j+1}) \leq \log\left(1 + \frac{M}{1 + M}\right)$. Then by the inequality (3'), $|x_{2j-1} - x_{2j}| \leq \frac{M}{1 + M} \delta_D(x_{2j-1})$.

Now let $B_j (j = 1, \ldots, N)$ be the closed ball centered at $x_{2j-1}$ with radius $\frac{M}{1 + M} \delta_D(x_{2j-1})$. Then $\tau(B_j) = M d(B_j, D^c)$ and $x_{2j-2}, x_{2j} \in B_j$ i.e. $B_j \cap B_{j+1} = \emptyset$.

Thus $\{B_j\}_{j=1}^{N}$ is an $M$-Harnack chain connecting $y$ with $x$, and it shows $y \in H_{M,N}(x)$.

**Lemma 2.** Let $z = \{z_j\}$ be an $(M,N,\nu)$-sequence with $\sup_j \nu^{-j} \delta_D(z_j) \leq h$. Then there exists a positive constant $C$ depending on $M$, $N$, $\nu$, and $\delta_D(z_1)$ such that the inequality
\[ k_D(y, z_1) \leq \frac{(2N - 1) \log(1 + M)}{-\log \nu} j_D(y, z_1) + C \]
holds for all $y \in H(z)$.

**Proof.** Lemma 1 shows that the condition (a) of $(M,N,\nu)$-sequences is
\[ k_D(z_j, z_{j+1}) \leq (2N - 1) \log(1 + M) \quad j = 1, 2, \ldots. \]
If $y \in H(z)$, $y \in H_{M,N}(z_l)$ for some $l$, then
\[ k_D(y, z_l) \leq (2N - 1) \log(1 + M), \]
\[ j_D(y, z_l) \leq C_1 = C_1(M, N) \]
because $\delta_D(y)$ is comparable to $\delta_D(z_l)$. Together with (5), we have
\[ k_D(y, z_1) \leq l(2N - 1) \log(1 + M). \]

On the other hand, $\delta_D(z_j) \leq h \nu^j$ implies that there exists an integer $j_0$ depending only on $h$ and $\delta_D(z_1)$ such that
\[ j_D(z_j, z_1) \geq \left| \log \frac{\delta_D(z_1)}{2 \delta_D(z_j)} \right| \geq \log \frac{\delta_D(z_1)}{2h \nu^j} \]
holds for $j \geq j_0$. Thus we obtain

$$k_D(y, z_1) \leq \frac{(2N-1)\log(1+M)}{-\log \nu} j_D(x_1, z_1) + C_2$$

$$\leq \frac{(2N-1)\log(1+M)}{-\log \nu} j_D(y, z_1) + C$$

with $C$ depending on $M$, $N$, $\nu$, $h$ and $\delta_D(z_1)$.

Now we shall state and prove our main results.

**Theorem 1.** Let $D$ be an $(M, N, \nu)$-inner NTA domain. Then for any $x_0 \in D$, there exists a positive constant $C$ such that the inequality

$$k_D(x_0, x) \leq \frac{(2N-1)\log(1+M)}{-\log \nu} j_D(x_0, x) + C$$

holds for all $x \in D$.

**Proof.** Since $E_0 = \{x \in D : \delta_D(x) \geq r_0\}$ is compact, $E_0 \subset E_1 = \{x \in D : k_D(x_0, x) \leq C_1\}$ for some $C_1 > 0$.

For every $x \in D$ with $\delta_D(x) \leq r_0$, there exists an $(M, N, \nu)$-sequence $z = \{z_j\}$ with $\delta_D(z_1) \geq r_0$ and $\delta_D(z_j) \leq h_0 \nu^j$ such that $H_{M_0 N_0}(x) \cap H(z) \neq \emptyset$. Let $y \in H_{M_0 N_0}(x) \cap H(z)$. Then

$$k_D(x_0, x) \leq k_D(x_0, z_1) + k_D(z_1, y) + k_D(y, x).$$

Since $z_1 \in E_0$, $k_D(x_0, z_1) \leq C_1$. Put $c = \frac{(2N-1)\log(1+M)}{-\log \nu}$. Then Lemmas 1 and 2 imply that

$$k_D(y, x) \leq C_2 := (2N_0 - 1)\log(1 + M_0),$$

$$k_D(y, z_1) \leq c j_D(z_1, y) + C_3.$$

On the other hand, $j_D(x_0, z_1) \leq C_1$ and $j_D(y, x) \leq C_2$ since $j_D \leq k_D$. Putting $C = C_1 + (1 + c)(C_2 + C_3)$, we obtain the desired inequality.

**Remark.** Theorem 1 and $j_D \leq k_D$ shows that $M, N, \nu$ must satisfy $(1 + M)^{1-2N} \leq \nu$.

**Theorem 2.** Let $D$ be a bounded domain. Assume that there exist a point $x_0 \in D$ and constants $c \geq 1$ and $C > 0$ such that

$$(6)\quad k_D(x_0, x) \leq cj_D(x_0, x) + C$$

for all $x \in D$. Then $D$ is an inner NTA domain with $M = (e^\theta - 1)/(2 - e^\theta)$, $N = 1$, $\nu = e^{-\theta/c}$ for each $0 < \theta < \log 2$.

**Proof.** Put $C_0 = C + 1$. Since $E_0 = \{x \in D : k_D(x_0, x) \leq C_0\}$ is compact, there exists $r_0 > 0$ such that $E_0 \subset E_1 = \{x \in D : \delta_D(x) \geq r_0\}$. We may
assume $r_0 < \delta_D(x_0)$. Since $E_1$ is also compact, there exists $C_1 > 0$ such that $E_1 \subset \{x \in D : k_D(x_0, x) \leq C_1\}$.

Let $x \in D$, $\delta_D(x) \leq r_0$ and $\gamma$ be a quasi-hyperbolic geodesic joining $x_0$ to $x$. Fix $0 < \theta < \log 2$ and choose the points $\{z_j\}_{j=1}^l$ on $\gamma$ such that $k_D(x_0, z_1) = C_1$, $z_1 = x$ and $k_D(z_j, z_{j+1}) = \theta$ where $l$ is the smallest integer such that $l\theta \geq k_D(x_0, x) - C_1$. If we put $M = (e^\theta - 1)/(2 - e^\theta)$, then Lemma 1 implies $z_j \in H_{M1}(z_{j+1})$ for $j = 1, \ldots, l - 1$.

On the other hand,

$$\frac{C_1 + (j - 1)\theta}{C} \leq \frac{k_D(x_0, z_j)}{c} \leq \frac{\log \left(1 + \frac{|x_0 - z_j|}{\min\{\delta_D(x_0), \delta_D(z_j)\}}\right)}{\min\{\delta_D(x_0), \delta_D(z_j)\}},$$

and

$$C_2 e^{\frac{\theta \theta}{c}} \leq 1 + \frac{|x_0 - z_j|}{\delta_D(z_j)}.$$

Here $C_2 = e^{(C_1 - C - \theta)/c} > 1$ because $C_1 \geq C_0 > C + \theta$. Putting $\nu = e^{-\theta/c}$ and $K = \text{diameter of } D$, we obtain

$$\frac{\delta_D(z_j)}{C_2 - 1} \leq \nu^j,$$

which shows that $z = \{z_j\}_{j=1}^l$ is an $(M, 1, \nu)$-sequence. This proves the theorem.

4. The growth of the positive $L$-harmonic functions near the boundary

Let $L$ be a uniformly elliptic partial differential operator of second order with Hölder continuous coefficients. In [6] we proved the following theorem. (The 0th order term need not be non-positive. See Hervé [5, p. 560].)

**Theorem.** Let $D$ be an $(M, N, \nu)$-inner NTA domain. Put

$$m = m(M, N, \nu) = \frac{(2N - 1) \log K^{-1}(M + 1)^{n-2}(2M + 1)^{1-n}}{\log \nu},$$

and

$$m' = m'(M, N, \nu) = \frac{(2N - 1) \log K (M + 1)^{n-2}(2M + 1)}{-\log \nu}.$$

Then for any fixed $x_0 \in D$, there exist positive constants $C$ and $C'$ satisfying the following condition. If $u$ is a non-negative solution of $Lu = 0$,

$$Cu(x_0)(\delta_D(x))^m \leq u(x) \leq C'u(x_0)(\delta_D(x))^{-m'}$$

holds for all $x \in D$.

Here $K \geq 1$ is the constant appeared in Harnack's inequality [6, Prop. 1].
Since
\[ \log \frac{\delta_D(x_0)}{\delta_D(x)} \leq k_D(x_0, x), \]
\[ \delta_D(x) \geq \delta_D(x_0)e^{-k_D(x_0,x)}. \]
The above Theorem together with these inequalities gives a boundary growth estimates for $L$-harmonic functions in terms of $k_D(x_0, x)$ in the following form.

\[ C u(x_0)e^{-m k_D(x_0,x)} \leq u(x) \leq C' e^{-m' k_D(x_0,x)} \]

where $C$ and $C'$ are positive constants independent of $x$ and $u$. The inequality of this type is also valid for arbitrary domain and "Harnack functions" ([4]).

Inner NTA domain, or equivalently the domain satisfying
\[ k_D(x_0, x) \leq c j_D(x_0, x) + C \]
is the domain such that the estimate (7) is equivalent to (8).

References