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1. INTRODUCTION

Throughout this paper, $R$ will be a commutative integral domain with identity and $K$ will denote its quotient field.

Let $F(R)$ be the set of nonzero fractional ideals of $R$, that is, $I \in F(R)$ in case $I$ is a nonzero $R$-submodule of $K$ such that there exists some $0 \neq d \in R$ such that $dI \subset R$.

Each ideal of $R$ is a fractional ideal of $R$ and is called an integral ideal of $R$. We shall denote the set of nonzero integral ideals of $R$ by $I(R)$.

A fractional ideal of $R$ is called a finitely generated fractional ideal in case it is finitely generated as an $R$-module of $K$. We shall denote the set of nonzero finitely generated fractional ideals of $R$ and the set of nonzero finitely generated integral ideals of $R$ by $F_f(R)$ and $I_f(R)$ respectively.

A star operation on $R$ is a mapping $I \mapsto I^*$ of $F(R)$ into itself that satisfies the following three properties, for all $0 \neq a \in K$ and $I, J \in F(R)$:

1. $(a)^* = (a)$ and $(aI)^* = aI^*$;
2. $I \subset I^*$, and if $I \subset J$, then $I^* \subset J^*$; and
3. $(I^*)^* = I^*$.

If we set $I_d = I$ for every $I \in F(R)$, then the mapping $I \mapsto I_d$ is a star operation on $R$ and is called the $d$-operation. For any $I \in F(R)$, let

$$I^{-1} = R : I = \{x \in K \mid xI \subset R\}$$

and denote $(I^{-1})^{-1}$ by $I_v$. Then the mapping $I \mapsto I_v$ is a star operation on $R$ and is called the $v$-operation.

An element $u \in K$ is almost integral over $R$ in case there exists an element $0 \neq r \in R$ such that $ru^n \in R$ for all integers $n \geq 1$, or equivalently, there is a nonzero ideal $I$ of $R$ for which $u \in I : I$.

Similarly an element $u \in K$ is integral over $R$ in case $u$ satisfies an equation

$$u^n + c_1u^{n-1} + \cdots + c_n = 0,$$

where $c_i \in R$, or equivalently, there is a nonzero finitely generated ideal $I$ of $R$ for which $u \in I : I$.

We denote the set of elements of $K$ which are almost integral over $R$ and the set of elements of $K$ which are integral over $R$ by $R'$ and $\bar{R}$ respectively.
In a recent paper [AHZ], D.F. Anderson, E.G. Houston and M. Zafrullah have introduced the notion of pseudo-integrality. An element $u$ of $K$ is pseudo-integral over $R$ if $u \subseteq I_v : I_v$ for some nonzero finitely generated ideal $I_v$ of $R$ and the set of elements of $K$ which are pseudo-integral over $R$ is denoted by $\hat{R}$.

It is well known that $\hat{R}$, the integral closure of $R$, and $R'$, the complete integral closure of $R$, are overrings of $R$, i.e., rings between $R$ and $K$. In [AHZ, Proposition 1.1], they proved that $\hat{R}$, the pseudo-integral closure of $R$, is an overring of $R$.

It is also known that $R'$ is integrally closed, as shown in [W, p.76] but is not necessarily completely integrally closed, that is, $(R')' \neq R'$, as shown in [GH, Example 1]. In [AHZ, Theorem 1.2], they showed that $\hat{R}$ is integrally closed.

In Section 2, we define $R^*$, the *-integral closure of $R$ for any star operation $*$ on $R$. In Theorem 2.8 we show that $R^*$ is an integrally closed overring of $R$. In Theorem 2.11 we give a generalization of Proposition 1.3 in [AHZ] and in Theorem 2.14 we also give a generalization of Corollary 1.6 in [AHZ].

In Section 3, we provide some examples concerning pseudo-integrality. In particular we give some examples of non-Mori pseudo-integrally closed domains which are not completely integrally closed.

**2.*-INTEGRAL CLOSURES** The set of all star operations on $R$ will be denoted by $S(R)$.

For any $* \in S(R)$, a fractional ideal $I \in F(R)$ is called a $*$-ideal if $I^* = I$. We will denote the set of all $*$-ideals of $R$ by $F_*(R)$.

We may define a partial order $\leq$ on $S(R)$ by $*_1 \leq *_2$ if and only if $I^*_1 \subseteq I^*_2$ for every $I \in F(R)$.

**PROPOSITION 2.1 (cf. [AA]).** For $*_1, *_2 \in S(R)$, the following conditions are equivalent:

1. $*_1 \leq *_2$,
2. $(I^*_2)^*_1 = I^*_2$ for every $I \in F(R)$,
3. $(I^*_1)^*_2 = I^*_2$ for every $I \in F(R)$,
4. $F_{*_2}(R) \subseteq F_{*_1}(R)$.

**PROOF.** The proof is easy and is omitted.

Recall that for any $I \in F(R)$, we have $I_v = \cap \{Rx \mid I \subseteq Rx, x \in K\}$. Hence we have $I \subseteq I^* \subseteq I_v$ for all star operations $*$ on $R$ and all $I \in F(R)$, and so the $d$-operation is a smallest element in $S(R)$ and the $v$-operation is a greatest element in $S(R)$.

**DEFINITION 2.2.** For any star operation $*$ on $R$, we set

$$R^* = \cup \{I^* : I \in F_*(R)\} = \cup \{I^* : I \in I_v(R)\}$$

and we shall call $R^*$ the $*$-integral closure of $R$. 
THEOREM 2.3. For any star operation $*$ on $R$, $R^*$ is an overring of $R$.

PROOF. First note that for any $I, J \in F_f(R)$, $I^* : I^* \subseteq (IJ)^* : (IJ)^*$. In fact, if $uI^* \subseteq I^*$, then $uI^*J^* \subseteq I^*J^*$ and so

$$u(IJ)^* = (uIJ)^* = (u(I^*J^*))^* \subseteq (I^*J^*)^* = (IJ)^*.$$ 

Thus

$$(I^* : I^*) \cup (J^* : J^*) \subseteq (IJ)^* : (IJ)^*$$

for all $I, J \in F(R)$, and therefore $R^* = \bigcup \{I^* : I^* | I \in F_f(R)\}$ is a directed union. Since each $I^* : I^*$ is a ring, $R^*$ is a ring, as required.

REMARK 2.4. For any star operation $*$ on $R$, it is easy to see that

$$(I^* : I^*) \cup (J^* : J^*) \subseteq (IJ)^* : (IJ)^*$$

for all $I, J \in F(R)$.

PROPOSITION 2.5. If $*_1 \leq *_2$ in $S(R)$, then we have $R^{*_1} \subseteq R^{*_2}$.

PROOF. Let $x \in I^{*_1} : I^{*_1}$ with $I \in F_f(R)$. Then, it follows from Proposition 2.1 that

$$xI^{*_2} = x(I^{*_1})^{*_2} = (xI^{*_1})^{*_2} \subseteq (I^{*_1})^{*_2} = I^{*_2}.$$ 

Thus $I^{*_1} : I^{*_1} \subseteq I^{*_2} : I^{*_2}$ for all $I \in F_f(R)$ and hence $R^{*_1} \subseteq R^{*_2}$, as wanted.

COROLLARY 2.6. For any star operation $*$ on $R$, we have $\tilde{R} \subseteq R^* \subseteq \check{R}$.

DEFINITION 2.7. An element $u \in K$ is said to be $*$-integral over $R$ if $u \in I^* : I^*$ for some $I \in F_f(R)$. An overring $T$ of $R$ is said to be $*$-integral over $R$ if each element of $T$ is $*$-integral over $R$.

We now generalize Theorem 1.2 in [AHZ] to the case of an arbitrary star operation $*$ on $R$.

THEOREM 2.8. Let $T$ be an overring of $R$ and let $x$ be an element of $K$. Suppose that $T$ is $*$-integral over $R$ and that $x$ is integral over $T$. Then $x$ is $*$-integral over $R$. In particular, $R^*$ is integrally closed.

PROOF. Since $x$ is integral over $T$, there exist elements $a_1, \ldots, a_k \in T$ such that $x$ is integral over $S = R[a_1, \ldots, a_k]$. Then, since each $a_i$ is $*$-integral over $R$, there exists an element $J_i$ in $F_f(R)$ such that $a_i \in J_i^* : J_i^*$.

If we set $J = J_1J_2 \cdots J_k \in F_f(R)$, then it follows that $a_iJ^* \subseteq J^*$ for all $i = 1, 2, \ldots, k$. Thus we have $S \subseteq J^* : J^*$. Now, since $x$ is integral over $S$, there is a nonzero finitely generated ideal $I = Sb_1 + \cdots + Sb_m$ of $S$ such that $xI \subseteq I$.

If we set $H = Jb_1 + \cdots + Jb_m$, then $H$ is a finitely generated fractional ideal of $R$. In fact,

$$H = Jb_1 + \cdots + Jb_m \subseteq JS \subseteq J^*S \subseteq J^*,$$

and so if $dJ^* \subseteq R$ with some $0 \neq d \in R$, then, clearly $dH \subseteq R$. 


Next, since $JSb_i \subset J^*Sb_i \subset J^*b_i$ for each $i = 1, \ldots, m$, it follows that

$$JI = JSb_1 + \cdots + JSb_m \subset J^*b_1 + \cdots + J^*b_m = (Jb_1)^* + \cdots + (Jb_m)^* \subset (Jb_1 + \cdots + Jb_m)^* = H^*.$$  

Then, since $xb_i \in xI \subset I$ for each $i = 1, \ldots, m$, we have

$$xH = x(Jb_1 + \cdots + Jb_m) \subset JI \subset H^*.$$  

Hence we have

$$xH^* = (xH)^* \subset (H^*)^* = H^*.$$  

Thus $x \in H^* : H^* \subset R^*$, completing the proof.

**Definition 2.9.** Let $T$ be an overring of $R$ and let $*$ and $*'$ be star operations on $R$ and $T$ respectively. Then the star operation $*'$ is said to be $*$-linked (or $T$ is said to be $*$-linked) in case, for any $I \subseteq I(R)$, such that $I^* = R$, we have $(IT)^* = T$.

**Definition 2.10.** An integral domain $R$ is called a $*$-UMT ring if each nonzero prime $P$ upper to zero in $R[X]$ contains an element $f$ such that $(c_R(f))^* = R$.

A star operation $*$ on $R$ is said to have finite character or is said to be of finite type if for each $I \subseteq F(R)$, $I^* = \bigcup\{J^* \mid J \in F_f(R), J \subseteq I\}$. Given a star operation $*$ on $R$, we can always define a new star operation $*_*$ on $R$ as follows:

$$I^*_* = \bigcup\{J^* \mid J \in F_f(R) \text{with} J \subseteq I\}.$$  

Then $*_*$ is a star operation of finite type on $R$. It is easily seen that a star operation $*$ has finite character if and only if $I^* = I^*_*$ for all $I \subseteq F(R)$.

If $*$ is a star operation on $R$, then we call $R$ a Prüfer $*$-multiplication ring if for each nonzero finitely generated ideal $I$ of $R[X]$ there is a finitely generated fractional ideal $J$ of $R$ such that $(IJ)^* = R$.

It is shown in [HMM, Theorem 1.1] that if $*$ is a star operation of finite type on $R$, then $R$ is a Prüfer $*$-multiplication ring if and only if $R$ is integrally closed and $R$ is a $*$-UMT ring.

**Theorem 2.11.** Let $*$ be a star operation on $R$ and let $*'$ be a star operation of finite type on $R^*$ such that $*'$ is $*$-linked. Then, if $R$ is a $*$-UMT ring, then $R^*$ is a Prüfer $*'$-multiplication ring.

**Proof.** First note that $R^*$ is integrally closed by Theorem 2.8.

By [HMM, Theorem 1.1], we need only to show that $R^*$ is a $*'$-UMT ring. Let $P$ be a nonzero prime upper to zero in $R^*[X]$. Then, since $P \cap R[X]$ is a nonzero prime upper to zero in $R[X]$, it contains an element $f$ such that $(c_R(f))^* = R$. Now, set $I = c_R(f)$. Since $*'$ is $*$-linked, we have $(IR^*)^* = R^*$, and a fortiori, $(c_R(f))^* = R^*$. Thus $R^*$ is a $*'$-UMT ring, as wanted.
REMARK 2.12. It is obvious that we may replace * and *' by *s and (*')s, respectively in Definitions 2.9 and 2.10, and that $R^* = R^{**}$. If $*=v$, then the $v$-operation is usually called the t-operation.

REMARK 2.13. In [HH, Proposition 4.3], it is shown that if * denotes either the v-operation or the t-operation, then $(IR[X])^* = I^*R[X]$ for each $I \in F(R)$. Hence it is easily seen that if we take * = *' = t, then the conditions in Theorem 2.11 are satisfied.

If $*=v$ or $*=t$, then a *-UMT ring is the same as a UMT-domain, and $R^* = \tilde{R}$. Therefore, Theorem 2.11 is a generalization of Proposition 1.3 in [AHZ].

As noted in Remark 2.13, if $*=v$ or $*=t$, then $(IR[X])^* = I^*R[X]$ for all $I \in F(R)$. Now let * be a star operation on $R[X]$. If we define *' on $R$ by $I^* = (IR[X])^* \cap R$, then, by [HMM, Proposition 2.1], *' is a star operation on $R$ and $(I^* R[X])^* = (IR[X])^*$ for each ideal $I$ of $R$.

THEOREM 2.14. Let * be a star operation on $R[X]$ and let *' be the star operation on $R$ induced from * as above. Suppose that $(IR[X])^* = I^*R[X]$ for each nonzero finitely generated ideal $I$ of $R$. Then we have $(R[X])^* = R^*[X]$

Proof. Let $u$ be an element of $R^*$. Then $uI^* \subset I^*$ for some $I \in I_f(R)$, and hence $uI^* R[X] \subset I^* R[X]$. Then, by hypothesis, $u(IR[X])^* \subset (IR[X])^*$, and therefore we have $u \in (R[X])^*$. Thus we get $R^* \subset (R[X])^*$ and so $R^*[X] \subset (R[X])^*$.

Conversely, let $v \in (R[X])^*$. Then $vI^* \subset I^*$ for some $I \in I_f(R[X])$. Note that $(R[X])^* \subset K[X]$. In fact, since $\tilde{R}[X] = (\tilde{R}[X])$ by [AHZ, Corollary 1.6], we have

(email the rest of the theorem)
3. EXAMPLES  It is well known that if $R$ is a Noetherian domain, then $\tilde{R} = R'$, and so $\tilde{R} = \bar{R} = R'$.

Suppose that $R$ is a quasi-coherent domain, i.e., each intersection of finitely many principal ideals of $R$ is finitely generated, or equivalently, $I^{-1} = R : I$ is finitely generated for each nonzero finitely generated ideal $I$ of $R$. Then we have $\tilde{R} = \bar{R}$. Hence, if $R$ is a coherent domain, i.e., each intersection of two finitely generated ideals of $R$ is finitely generated, then $\tilde{R} = \bar{R}$.

If $R$ is a Mori domain, then for each nonzero ideal $I$ of $R$ there is a finitely generated ideal $J$ of $R$ for which $I_v = J_v$. Hence, if $R$ is a Mori domain, then we have $\tilde{R} = R'$.

First we shall show an example of a non-Mori domain which is pseudo-integrally closed, where $R$ is pseudo-integrally closed if and only if $R = \tilde{R}$.

**Example 3.1.** Let $V = \mathbb{Q}[[X]] = \mathbb{Q} + X\mathbb{Q}[[X]]$ where $\mathbb{Q}$ is the set of rational numbers. Then $V$ is a DVR with maximal ideal $M = X \mathbb{Q}[[X]]$. Let $\varphi : V \rightarrow V/M = \mathbb{Q}$ be the canonical surjection and let $R = \varphi^{-1}(\mathbb{Z})$.

Then $R = \mathbb{Z} + X\mathbb{Q}[[X]] = \mathbb{Z} + M$. By [B, Corollary 3.5], $R$ is a non-Mori domain. Next, by [AHZ, Proposition 1.8(i)],

$$\tilde{R} = \tilde{\mathbb{Z}} + M = \mathbb{Z} + M = R,$$

because $\mathbb{Z}$ is Noetherian and so $\tilde{\mathbb{Z}} = \tilde{\mathbb{Z}} = \mathbb{Z}$. Hence $R$ is pseudo-integrally closed.

By [BG, Theorem 2.1 (a)], the complete integral closure of $R = \mathbb{Z} + X\mathbb{Q}[[X]]$ is $V = \mathbb{Q}[[X]]$, and hence $R$ is not completely integrally closed. Thus $R = \tilde{R} \subset \neq R'$.

In this example, each ideal of $R$ containing $M$ is of the form $n\mathbb{Z} + M$ with some $n \in \mathbb{N}$, the set of positive integers. Hence, by [BG, Theorem 4.1],

$$(n\mathbb{Z} + M)_v = (n\mathbb{Z})_v + M = n\mathbb{Z} + M.$$ 

Thus $n\mathbb{Z} + M$ is a divisorial ideal. Hence any ideal of $R$ properly containing $M$ is divisorial.

It is shown in [AHZ, Lemma 1.4] that if $T$ is a flat extension domain of $R$, then $\tilde{R} \subset \tilde{T}$. But, in general, the containment $R \subset T$ does not imply $\tilde{R} \subset \tilde{T}$ (cf. [AHZ, Example 3.2]).

**Remark 3.2.** An example of an integral domain $R$ such that $R \subset \neq \tilde{R} \subset \neq \tilde{R}$, is given in [AHZ, Example 1.9]. Of course, $R$ is neither quasi-coherent nor Mori.

**Example 3.3.** Assume that $R \subset \tilde{R}$ and let $*$ be an e.a.b. star operation on $R$. If $S$ is the Kronecker function ring of $R$ with respect to the star operation $*$, then $R \subset S$, but $\tilde{R} \subset \neq \tilde{S}$. In fact, $S$ is a Bezout ring ([G,(32.7),(b)]). Therefore $\tilde{S} = S$. Since $S \cap K = R$ ([G,(32.7),(a)]), we see that

$R \nsubseteq \tilde{S}$
As shown in [AHZ, Example 2.1], the pseudo-integral closure of a domain need not be pseudo-integrally closed.

Here we give an example of an integral domain \( R \) such that \( \tilde{R} = \hat{R} \).

**Example 3.4.** Let \( V \) be a DVR of the form \( F + M \), where \( F \) is a field and \( M \) is the maximal ideal of \( V \). Let \( D \) be a subring of \( F \) such that the quotient field of \( D \) is properly contained in \( F \). If we set \( R = D + M \), then, by [AHZ, Proposition 1.8 (ii)], we have \( \hat{R} = V \). Since \( V \) is completely integrally closed, \( \tilde{R} = \hat{V} = V' = V = \hat{R} \).

Let us provide an example of a non-Mori domain \( R \) for which \( \tilde{R} = R' \).

**Example 3.5 (cf.[AM]).** Let \( R = \mathbb{Z}[\{X/p_i, Y/p_i\}_{i=1}^\infty] \), where \( \mathbb{Z} \) is the set of integers, \( \{p_i\}_{i=1}^\infty \) is the set of positive primes numbers, and \( X, Y \) are indeterminates over \( \mathbb{Z} \). Then, [AM,(a),p.52] shows that \( R \) is not a Krull ring. Moreover, its proof shows that \( R_M \) is completely integrally closed for each maximal ideal \( M \) of \( R \). Therefore \( R \) is a completely integrally closed ring and is not a Mori ring.

**References**


