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Strongly n-generated semigroup rings

RYŪKI MATSUDA*

Let $A$ be a ring (commutative with identity). If, for each 2-generated ideal $I$ of $A$ and for each nonzero element $a$ of $I$ there exists an element $a_1$ of $A$ such that $I = (a_1, a)$, then $A$ is said to be strongly 2-generated ([3]).

If, for each $(n+1)$-generated ideal $I$ of $A$ and for each nonzero element $a$ of $I$ there exist elements $a_1, \ldots, a_n$ of $A$ such that $I = (a_1, \ldots, a_n, a)$, then we say that $A$ is strongly $(n+1)$-generated.

If, for each $(n+1)$-generated ideal $I$ of $A$ and for each regular element $a$ of $A$ with $a \not\in I$ there exist $a_1, \ldots, a_n$ of $A$ such that $I = (a_1, \ldots, a_n, a)$, then we say that $A$ is strongly $r$-$(n+1)$-generated.

Let $S$ be a torsion-free cancellative abelian semigroup $\neq 0$ (nonzero and written additively). The aim of this paper is to determine a necessary and sufficient condition for the semigroup ring $A[X;S]$ of $S$ over a ring $A$ to be strongly $(n+1)$-generated or strongly $r$-$(n+1)$-generated.

**Lemma 1** ([7, (4.1)]). Let $A$ be a 0-dimensional ring. Let $p$ be a prime number, $n \in N$ and $n_i \in N$ $(1 \leq i \leq n+2)$ such that $p1_A$ is a nilpotent of $A$ and $n_{i+1} > n + n_i$ for each $i$. Assume in $A[X;Q]$ that

$$(p^nY^{n_1}, p^{n-1}Y^{n_2}, \ldots, pY^{n_n}, Y^{n_{n+1}}) = (f_1, \ldots, f_n, Y^{n_{n+2}})$$

with $Y = 1 - X^1$. Then we have $p^n1_A = 0$.

**Lemma 2** ([7, (4.2)]). Assume that $A[X;Q]$ has $r$-n-generator property. Then $\dim(A) = 0$.

Let $G$ be a torsion-free abelian group (nonzero and written additively).

**Lemma 3** ([7, (2.2)]). If $A[X;G]$ has $r$-n(1/2)-generator property, then $A[X;G^*]$ has $r$-n(1/2)-generator property, where $G^*$ is the divisible hull of $G$.

**Lemma 4** ([1, Proposition 4]). Let $M = Aa_1 + \cdots + Aa_n$ be a finitely generated module over a 0-dimensional ring $A$, and assume that $M$ is generated by $m$ elements. Then there exists an orthogonal set $\{e_1, \ldots, e_h\}$ of idempotents in $A$ with $e_1 + \cdots + e_h = 1$ such that, for each $j$, $e_jM$ is generated, as an $Ae_j$-module, by less than $m+1$ elements chosen from the set $\{e_ja_1, \ldots, e_ja_n\}$.

**Lemma 5** ([1, Lemma 7]). Let $S$ be a subsemigroup of $Q_0$ and $0 < s \in S$. Then each idempotent of $A[X;S]/(X^s)$ is the residue class of an idempotent of $A$.

**Lemma 6** ([1, Corollary 12]). $A[X;S]$ has n-generator property if and only if one of the following conditions hold:

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(1) $S$ is isomorphic with a subgroup of $Q$; $\dim(A) = 0$ and for each $\{a_1, \cdots, a_n\} \subset N(A)$ there exists a decomposition $A = A e_1 \oplus \cdots \oplus A e_h$ such that for each $j$, $a_1 e_j = 0$ or $a_i e_j \in (a_1, \cdots, a_{i-1}) e_j$ for some $i \geq 2$.

(2) $S$ is isomorphic with a subgroup of $Q$; $\dim(A) = 0$ and for each finitely generated nil ideal $I$ there exists a decomposition $A = A e_1 \oplus \cdots \oplus A e_h$ such that for each $j$, $\nu(I e_j) < n$.

(3) $S$ is isomorphic with a subsemigroup of $Q_0$ with $o(S) < \infty$; $\dim(A) = 0$ and for each $\{a_1, \cdots, a_m\} \subset N(A)$ with $(m+1) o(S) > n$ there exists a decomposition $A = A e_1 \oplus \cdots \oplus A e_h$ such that for each $j$, $a_1 e_j = 0$ or $a_i e_j \in (a_1, \cdots, a_{i-1}) e_j$ for some $i \geq 2$.

(4) $S$ is isomorphic with a subsemigroup of $Q_0$ with $o(S) < \infty$; $\dim(A) = 0$ and for each finitely generated nil ideal $I$ there exists a decomposition $A = A e_1 \oplus \cdots \oplus A e_h$ such that for each $j$, $(\nu(I e_j) + 1) o(S) \leq n$.

**Lemma 7** ([7,(6.1)]). Assume that $A[X;Q]$ has $r$-$(n+1/2)$-generator property. Let $p$ be a prime number such that $p1_A$ is a nilpotent of $A$. Then we have $p^n 1_A = 0$.

**Lemma 8** ([7,(6.2)]). Assume that $A[X;Q]$ has $r$-$(n+1/2)$-generator property. Then $A[X;Q]$ has $n$-generator property.

**Lemma 9.** Assume that $A[X;Q]$ is strongly $r$-$(n+1)$-generated. Then $\dim(A) = 0$.

**Proof.** The proof is similar with the proof of Lemma 2.

**Lemma 10.** If $A[X;G]$ is strongly $r$-$(n+1)$-generated, then $A[X;G^*]$ is strongly $r$-$(n+1)$-generated.

**Proof.** The proof is similar with the proof of Lemma 3.

**Lemma 11.** If $A[X;G]$ is strongly $r$-$(n+1)$-generated, then $t.f.r.(G) = 1$.

**Proof.** Suppose that $t.f.r.(G) \geq 2$. By Lemma 10, $(A'[X;Q])[X;Q]$ is strongly $r$-$(n+1)$-generated for some ring $A'$. This contradicts with Lemma 9.

**Lemma 12.** If $A[X;S]$ is strongly $r$-$(n+1)$-generated, then $\dim(A) = 0$.

**Proof.** By Lemma 11, $t.f.r.(G) = 1$. By Lemma 10, $A[X;Q]$ is strongly $r$-$(n+1)$-generated. By Lemma 9, $\dim(A) = 0$.

**Lemma 13.** Let $S$ be a subsemigroup of $Q_0$. If $A[X;S]$ is strongly $r$-$(n+1)$-generated, then $o(S) < \infty$.

**Proof.** Suppose that $o(S) = \infty$. Then $S$ has a finitely generated subsemigroup $T_0$ such that each finitely generated subsemigroup $T'$ of $S$ containing $T_0$ has order $\leq n + 2$. Set $\min\{o(T') : T' \text{ is a finitely generated subsemigroup of } S \text{ containing } T_0\} = m$. We have $m \geq n + 2$. Let $T$ be a finitely generated subsemigroup of $S$ containing $T_0$ such that $o(T) = m$. We may assume that $q(T) = Z$. We have $h + Z_0 \subset T$ for some $h \in Z_0$. Let $l$ be a large natural number. We have

$$(X^h, X^{h+1}, \cdots, X^{h+n}) = (g_1, \cdots, g_n, X^l)$$
in \( A[X;S] \). Let \( M \) be a maximal ideal of \( A \). Set \( A/M = k \). In \( k[X;S] \) we have

\[
(X^h, X^{h+1}, \ldots, X^{h+n}) = (f_1, \ldots, f_n, X^h)
\]

for some \( f_i \in k[X;S] \). Set \( k[X;S]/(X^h) = B \). In \( B \) we have

\[
(X^h, X^{h+1}, \ldots, X^{h+n}) = (f_1, \ldots, f_n).
\]

Apply Lemma 4 and Lemma 5. Then this ideal is generated in \( B \) by \( n \) elements from \( \{X^h, \ldots, X^{h+n}\} \). Considering the order of \( S \), this is impossible. Therefore \( o(S) < \infty \).

**Lemma 14.** Let \( S \) be a subsemigroup of \( Q_0 \). If \( A[X;S] \) is strongly \( r-(n+1) \)-generated, then \( A[X;S] \) has \( n \)-generator property.

**Proof.** By Lemma 12, we have \( \dim(A) = 0 \). By Lemma 13 we have \( o(S) = k \). Set \( o(S) = k \). Let \( \{a_1, \ldots, a_m\} \subset N(A) \) with \( (m+1)k > n \). Set \( S \cap Z_0 = S_1 \). We may assume that \( o(S_1) = k \) and \( q(S_1) = Z \). We have \( g+Z_0 \subset S_1 \) for some \( g \in N \). We set

\[
\beta = \{a_jX^{g+jk-i} \mid 1 \leq j \leq m+1, 1 \leq i \leq k\},
\]

where \( a_{m+1} = 1 \). If \( km < n+1 \), then we set

\[
\beta' = \{f \in \beta \mid g \leq \deg(f) \leq g+n\}.
\]

If \( km \geq n+1 \), then we set

\[
\beta' = \{f \in \beta \mid g \leq \deg(f) \leq g+n-1\} \cup \{X^{g+mk}\}.
\]

Then note that \( X^{g+mk} \in \beta' \). Let \( l \) be a large natural number. We have

\[
\beta' A[X;S] = (f_1, \ldots, f_n, X^{l(g+mk)})
\]

for some \( f_i \in A[X;S] \). Set \( \tilde{A} = A[X;S]/(X^{l(g+mk)}) \). Apply Lemma 4 and Lemma 5. In \( \tilde{A} \), \( \beta' \tilde{A} \) is \( n \)-generated. Hence there exists a decomposition \( \tilde{A} = \tilde{A}e_1 \oplus \cdots \oplus \tilde{A}e_k \) such that \( \beta' \tilde{A}e_\alpha \) is generated by \( n \) elements from the set \( \beta' \tilde{A}e_\alpha \) for each \( \alpha \). We may assume that each \( e_\alpha \in A \). There exists \( a_jX^{g+jk-i} \in \beta' \) such that

\[
a_jX^{g+jk-i}e_\alpha \in (\beta' - \{a_jX^{g+jk-i}\}, X^{l(g+mk)})A[X;S]e_\alpha
\]

for some \( i, j \). It follows that \( a_je_\alpha = 0 \) if \( j = 1 \) and \( a_je_\alpha \in (a_1, \ldots, a_{j-1})e_\alpha \) if \( j > 1 \). By Lemma 6, \( A[X;S] \) has \( n \)-generator property.

**Lemma 15.** Assume \( A[X;Q] \) is strongly \( r-(n+1) \)-generated. Let \( p \) be a prime number such that \( p1_A \) is a nilpotent of \( A \). Then \( p^n1_A = 0 \).

**Proof.** By Lemma 9, \( \dim(A) = 0 \). Lemma 1 implies that \( p^n1_A = 0 \).
LEMMA 16. Assume that $A[X; Q]$ is strongly $r$-$(n+1)$-generated and that $A$ has the direct sum decomposition $A = B \oplus C$. Let $p$ be a prime number such that $p^1_B$ is a nilpotent of $B$. Then $p^n1_B = 0$.

PROOF. $B[X; Q]$ is strongly $r$-$(n+1)$-generated. By Lemma 15, we have $p^n1_B = 0$.

LEMMA 17. Assume that $A[X; Q]$ is strongly $r$-$(n+1)$-generated. Then $A[X; Q]$ has $n$-generator property.

PROOF. By Lemma 9 we have $\dim(A) = 0$. Then, using Lemma 16, the proof is similar with the proof of Lemma 8.

THEOREM 18. $A[X; S]$ is strongly $r$-$(n+1)$-generated if and only if $A[X; S]$ has $n$-generator property.

PROOF. The necessity: Set $q(S) = G$. By Lemma 11 we may assume that $S$ is isomorphic either with a subsemigroup of $Q_0$ or with a subgroup of $Q$. The first case: By Lemma 14 we see that $A[X; S]$ has $n$-generator property. The second case: By Lemma 10, $A[X; Q]$ is strongly $r$-$(n+1)$-generated. By Lemma 17, $A[X; Q]$ has $n$-generator property. By Lemma 6, $A[X; S]$ has $n$-generator property.

COROLLARY 19. $A[X; S]$ is strongly $(n+1)$-generated if and only if $A[X; S]$ has $n$-generator property.

APPENDIX [10] and [8] consider minimal overrings of a Noetherian domain.

PROPOSITION 20. Let $A$ be a ring (not necessarily neither noetherian nor integral). Then there exists an extension ring $B$ of $A$ such that:

1. $B \supsetneq A$.

2. If $C \supset A$ is a subring of $B$, then $C = B$ or $C = A$.

PROOF. Let $M$ be a maximal ideal of $A$. Let $I$ be an ideal of $A[X]$ generated by the set $\{X^2, mX \mid m \in M\}$. There exists a natural injection of $A$ into $A[X]/I$. Therefore there exists an extension ring $B$ of $A$ such that $B \supsetneq A$, $B = A[x]$, $x^2 = 0$ and $mx = 0$ for each $m \in M$. $B/M$ is a 2-dimensional vector space over $A/M$. Let $C$ be an intermediate ring of $B \supset A$. Then $C/M$ is a subspace of $B/M$. If $C/M$ is 2-dimensional, then $C = B$. If $C/M$ is 1-dimensional, then $C = A$.

Let $\{p_i\}_{i=1}^{\infty}$ be the set of positive prime numbers, and $x, y$ are indeterminates over the rational integers $Z$. Let $\Pi(D)$ denote the set of prime ideals of $D$ which are minimal over some ideal $(a) :_D (b)$ for $a, b \in D$. In [2] we proved that the ring $R = \mathbb{Z}\{x/p_i, y/p_i\}_{i=1}^{\infty}$ resolves all the following questions/conjectures:

(1) Conjecture [9]. If $D$ is an almost Krull domain and each height one prime ideal of $D$ is divisorial, then $D$ is a Krull domain.

(2) Question [5]. If $D_P$ is a valuation ring for each $P \in \Pi(D)$, is $D[X]_U$ a Prüfer ring?
(3) Conjecture[4]. There exists an essential ring which is not a Prüfer v-multiplication ring.

(4) Question[6]. Is every almost Krull domain a Prüfer v-multiplication ring?

In a letter to the author, Professor Houston wrote, "In [2], are the terms with "y" necessary? That is, would \( R = \mathbb{Z}[x/p_i] \) not work also?"

**Proposition 21.** Let \( x_1, \ldots, x_n \) be indeterminates over \( \mathbb{Z} \), where \( n \) is any integer \( \geq 1 \). Then \( \mathbb{Z}[x_1/p_i, x_2/p_i, \ldots, x_n/p_i, p_i^\infty] \) resolves all the above four questions/conjectures.

The proof is similar with that of [2].

**References**


