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<tr>
<td>Author(s)</td>
<td>TAKANO, Katsuo</td>
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<tr>
<td>Citation</td>
<td>Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 23: 39-46</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10109/3020">http://hdl.handle.net/10109/3020</a></td>
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On positive and integrable solution of first order delay
differentiable equation with measurable coefficients

KATSUO TAKANO*

1. Introduction
A probability density of which distribution is called an infinitely divisible
distribution satisfies the integral equation

\[ tx(t) = \int_0^t x(t - u)dK(u) \quad (t > 0), \]

where \( K(u) \) is non-decreasing and

\[ \int_1^\infty u^{-1}dK(u) < \infty, \]

(cf. [7. (4.2.16)]). A particular probability density of a distribution in the class
\( L \), which is a subclass of the infinitely divisible distributions, satisfies the first
order delay differential equation

\[ tx'(t) = (\lambda - 1)x(t) - \Sigma_{i=1}^n \lambda_i x(t - p_i) \]

(cf. [8], [11]). It is very interesting for us as we can directly obtain a probability
distribution from the integral equation or the differential equation instead of
obtaining it from Lévy’s inversion formula (cf. [9]). From this point of views, we
will study conditions for the differential equation

\[ tx'(t) = a(t)x(t) - \Sigma_{i=1}^n b_i(t)x(t - p_i) \quad (1) \]

with Lebesgue measurable coefficients to have a positive and integrable solution
on \( (0, \infty) \), though many authors are studying the oscillatory conditions of the
delay differential equations with continuous coefficients.

2. The absolutely continuous solution
Suppose the following conditions:

- \( a(t) \) is Lebesgue measurable and locally bounded on \( [0, \infty) \),

\begin{align*}
(a1) \quad x(t) &= 0 \text{ for } t < 0, \quad x(p_1) = c > 0, \\
&\quad 0 < p_1 < p_2 < \ldots < p_n < \infty,

(a2) \quad a(t) &= \text{Lebesgue measurable and locally bounded on } [0, \infty),
\end{align*}
$a(0) > -1,$

$a(t)$ is differentiable from the right side at $t = 0$.

(a3) $b_i(t)$ are Lebesgue measurable and locally bounded on $[0, \infty)$,

$b_i(t) = 0, \ (0 \leq t \leq p_i); b_i(t) \geq 0, \ a.e. \ (p_i \leq t) \ (i = 1, 2, ..., n)$.

The abbreviation a.e. means almost everywhere with respect to the Lebes-gue measure. We will show that the equation (1) has an absolutely continuous solution under these conditions. From (a1), we have

$$tx'(t) = a(t)x(t) \ a.e. \ (0 < t < p_1).$$

By (a2), the equality

$$a(t) = a(0) + a'_+(0)t + t \cdot o(t) \ (o(t) \to 0 \ as \ t \to +0)$$

holds, where $a'_+(0)$ denotes the right derivative at $t = 0$. Take a positive number $\delta$ such as $0 < \delta < p_1$ and set $g(t) = \int_{t}^{\delta} \frac{a(s)}{s} ds$ for $0 < t \leq p_1$. Then for any positive number $\delta_1$ such as $0 < \delta_1 < \delta$, we obtain an absolutely continuous solution, $x(t) = C \exp(g(t)) \ (\delta_1 \leq t \leq p_1), C = c \exp(-g(p_1))$. In fact, we have

$$g(t) = -\int_{t}^{\delta} \left( \frac{a(0)}{s} + a'_+(0) + o(s) \right) ds$$

$$= -a(0) \log \frac{t}{\delta} - \int_{t}^{\delta} (a'_+(0) + o(s)) ds$$

and

$$x(t) = C \left( \frac{t}{\delta} \right)^{a(0)} \exp \left( -\int_{t}^{\delta} (a'_+(0) + o(s)) ds \right)$$

for $0 < t \leq \delta$. This shows that $x(t)$ is absolutely continuous on $[\delta_1, p_1]$ (cf. [6]).

By (2), we define $x(0)$ such as $x(0) = x(+0)$ if $x(+0)$ is finite and $x(0) = +\infty$ if $x(+0) = \infty$. Then if $a(0) > 0, x(t)$ becomes continuous on $(-\infty, p_1)$, and if $0 \geq a(0) > -1, x(t)$ is continuous on $(-\infty, p_1)$ except $t = 0$. Let

$$x_0(t) = 0 \ (t < 0),$$

$$= x(t) \ (0 \leq t \leq p_1).$$

Let $\sigma = \min\{p_1, p_2 - p_1, ..., p_n - p_{n-1}\}$. To simplify the explanation of the step by step integrals, suppose that $p_1 < p_1 + \sigma < p_2 < p_1 + 2\sigma$. Then we solve the equation

$$x'(t) - \frac{a(t)}{t} x(t) = -\frac{b_1(t)}{t} x_0(t - p_1) \ a.e. \ (p_1 < t < p_1 + \sigma)$$

and we obtain an absolutely continuous solution,

$$x(t) = \left[ -\int_{p_1}^{t} b_1(s) x_0(s - p_1) \exp \left( -\int_{p_1}^{s} \frac{a(u)}{u} du \right) ds + C_0 \right] \exp \left( \int_{p_1}^{t} \frac{a(s)}{s} ds \right)$$

$(p_1 \leq t \leq p_1 + \sigma),$$
because the integral including $x_0(t)$ exists from (2) and the local boundedness of $a(t)$ and $b_1(t)$. To make $x(t)$ continuous on $(0, \infty)$, we let $x(p_1) = C_0 = x_0(p_1)$.

Next, we define $x_1(t)$ by

$$x_1(t) = x_0(t) \quad (t \leq p_1),$$

$$= x(t) \quad (p_1 \leq t \leq p_1 + \sigma),$$

and solve

$$x'(t) - \frac{a(t)}{t} x(t) = -\frac{b_1(t)}{t} x_1(t - p_1) \quad \text{a.e.} \quad (p_1 + \sigma < t < p_2),$$

$$x'(t) - \frac{a(t)}{t} x(t) = -\frac{b_1(t)}{t} x_1(t - p_1) - \frac{b_2(t)}{t} x_1(t - p_2) \quad \text{a.e.} \quad (p_2 < t < p_1 + 2\sigma),$$

and we obtain an absolutely continuous solution,

$$x(t) = \left[ -\int_{p_1 + \sigma}^{t} \frac{1}{s} \left( b_1(s)x_1(s - p_1) + b_2(s)x_1(s - p_2) \right) \exp \left( -\int_{p_1 + \sigma}^{s} \frac{a(u)}{u} du \right) ds + C_1 \right] \exp \left( \int_{p_1 + \sigma}^{t} \frac{a(s)}{s} ds \right) \quad (p_1 + \sigma \leq t \leq p_1 + 2\sigma),$$

because the integral including $x_1(s - p_2)$ exists from (2) and the local boundedness of $a(t)$ and $b_i(t)$. To make $x(t)$ continuous, we let $x(p_1 + \sigma) = C_1 = x_1(p_1 + \sigma)$. Repeating this procedure, we obtain an absolutely continuous solution,

$$x(t) = \left[ -\int_{p_1 + (m-1)\sigma}^{t} \frac{1}{s} \left( \sum_{i=1}^{n} b_i(s)x_{m-1}(s - p_i) \right) \exp \left( -\int_{p_1 + (m-1)\sigma}^{s} \frac{a(u)}{u} du \right) ds \right. $$

$$+ x_{m-1}(p_1 + (m - 1)\sigma) \exp \left( \int_{p_1 + (m-1)\sigma}^{t} \frac{a(s)}{s} ds \right) \quad (p_1 + (m - 1)\sigma \leq t \leq p_1 + m\sigma),$$

of the equation

$$x'(t) - \frac{a(t)}{t} x(t) = -\frac{1}{t} \sum_{i=1}^{n} b_i(t)x_{m-1}(t - p_i) \quad \text{a.e.} \quad (p_1 + (m - 1)\sigma < t < p_1 + m\sigma).$$

3. Main result

Denote $\max\{t, s\}$ by $t \vee s$. 

THEOREM. The solution $x(t)$ of the equation (1) under the conditions (a1), (a2), (a3) is positive on $(0, \infty)$ and integrable over $(0, \infty)$ if the following conditions are satisfied.

(b1) $\sum_{i=1}^{n} \int_{(t-p_i)v_0}^{t} b_i(u + p_i)du > 0 \quad (t \geq p_1)$.

(b2) $a(t) + 1 = \sum_{i=1}^{n} b_i(t + p_i)$ a.e. $(t \geq 0)$.

(b3) For some $M$ and for a sufficiently large $T$

$$\int_{T}^{t} \frac{1}{s} \left( a(s) - \sum_{i=1}^{n} b_i(s) \exp(- \int_{s-p_i}^{a(u)} \frac{a(u)}{u}du) \right) ds \leq M \quad (t > T)$$

holds.

(b4) $\int_{T}^{\infty} \frac{1}{t} \left( \int_{t-p_j}^{t} \frac{1}{u} b_j(u + p_j)(\int_{u-p_i}^{u} b_i(v + p_i)dv)du \right)dt$

$(i,j = 1, 2, \ldots, n)$ are convergent for a sufficiently large $T$.

PROOF. At first, let us show that the solution $x(t)$ is positive on $(0, \infty)$. The solution $x(t)$ is positive on $(0, p_1]$. Suppose that $\alpha$ is the smallest value such as $x(\alpha) = 0$. Then, integrating from 0 to $\alpha$, we obtain

$$\int_{0}^{\alpha} (tx'(t) + x(t))dt = \int_{0}^{\alpha} \left( (a(t) + 1)x(t) - \sum_{i=1}^{n} b_i(t)x(t - p_i) \right) dt$$

and

$$0 = \sum_{i=1}^{n} \int_{(\alpha-p_i)v_0}^{\alpha} b_i(t + p_i)x(t)dt. \quad (3)$$

From (b1), it holds that for some $\ell$

$$\int_{(\alpha-p_\ell)v_0}^{\alpha} b_\ell(u + p_\ell)du > 0.$$

For positive integers $\sigma, \rho$, put $A_{\sigma, \rho} = \{ t : (\alpha - p_\ell) \vee 0 + \frac{1}{\sigma} < t < \alpha - \frac{1}{\rho} \}$. For sufficiently large $\sigma, \rho$, by the monotone convergence theorem it holds that

$$\int_{A_{\sigma, \rho}} b_\ell(u + p_\ell)du > 0.$$

For a sufficiently small positive $\epsilon$, we have $\{ t : (\alpha - p_\ell) \vee 0 < t < \alpha, x(t) > \epsilon \} \subset A_{\sigma, \rho}$ and hence,

$$\sum_{i=1}^{n} \int_{(\alpha-p_i)v_0}^{\alpha} b_i(u + p_i)du \geq \epsilon \int_{A_{\sigma, \rho}} b_\ell(u + p_\ell)du > 0.$$
This is a contradiction and hence the solution $x(t)$ of (1) is positive on $(0, \infty)$. From this fact and the equation (1), we obtain
\[
\frac{x'(t)}{x(t)} \leq \frac{a(t)}{t} \quad \text{a.e.} \quad (t > p_n)
\]
and, integrating from $t - \tau$ to $t$,
\[
\frac{x(t - \tau)}{x(t)} \geq \exp \left( - \int_{t-\tau}^{t} \frac{a(u)}{u} \, du \right) \quad (t > p_n + \tau, \tau > 0).
\]
By this inequality and (1), we see that
\[
x'(t) \leq \frac{1}{t} \left[ a(t) - \sum_{i=1}^{n} b_i(t) \exp \left( - \int_{t-p_i}^{t} \frac{a(u)}{u} \, du \right) \right] x(t) \quad \text{a.e.} \quad (t > T)
\]
for a sufficiently large $T$. Hence it holds that for a sufficiently large $T$
\[
x(t) \leq x(T) \exp \left\{ \int_{T}^{t} \frac{1}{s} \left( a(s) - \sum_{i=1}^{n} b_i(s) \exp \left( - \int_{s-p_i}^{s} \frac{a(u)}{u} \, du \right) \right) ds \right\}
\]
and by (b3), $x(t)$ is upper bounded for $t > T$. Next let us show that $x(t)$ is integrable over $(0, \infty)$. By the equality
\[
xt(t) = \sum_{i=1}^{\infty} \int_{t-p_i}^{t} b_i(u + p_i) x(u) \, du
\]
we have
\[
0 < x(t) \leq \frac{B}{t} \sum_{i=1}^{\infty} \int_{t-p_i}^{t} b_i(u + p_i) \, du,
\]
because the inequality $0 < x(t) \leq B \quad (t \geq T)$ holds for a sufficiently large $T$. Substituting this into (5), we have
\[
0 < x(t) \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{B}{t} \int_{t-p_i}^{t} \frac{b_j(u + p_j)}{u} \left( \int_{u-p_i}^{u} b_i(v + p_i) \, dv \right) \, du.
\]
We see that $x(t)$ is integrable over $(0, \infty)$ as (b3) holds. q.e.d.

**Corollary 1.** The solution $x(t)$ of the equation (1) is positive on $(0, \infty)$ and integrable over $(0, \infty)$ and $\lim_{t \to +\infty} x(t) = 0$ under the conditions (a1), (a2), (a3), (b1), (b2), (b3), (b4) and the following (c1).
\[
(c1) \lim_{s \to +\infty} \sup_{s \leq t} \left\{ \frac{1}{t} \sup \{ b_i(u + p_i) : t - p_i \leq u \leq t \} \right\} \text{ are finite.}
\]
**Proof.** From Theorem and (5), we see that $0 < x(t) \leq \sum_{i=1}^{\infty} \frac{1}{t} \sup \{ b_i(u + p_i) : t - p_i \leq u \leq t \} \int_{t-p_i}^{t} x(u) \, du$. By the integrability of $x(t)$ and (c1), we have $\lim_{t \to +\infty} \sup \{ b_i(u + p_i) : t - p_i \leq u \leq t \} \int_{t-p_i}^{t} x(u) \, du = 0$ and hence $\lim_{t \to +\infty} x(t) = 0$. q.e.d.

**Corollary 2.** The solution $x(t)$ of the equation (1) is positive on $(0, \infty)$ and integrable over $(0, \infty)$ and $\lim_{t \to +\infty} x(t) = 0$ under the conditions (a1), (a2), (a3), (b1), (b2), (b3) and the following (d1).
\[
(d1) b_i(t) \text{ are bounded on } [p_i, \infty).
\]
**Proof.** The condition (d1) satisfies (b4) and (c1). q.e.d.
4. Examples
At first we shall consider an equation with discontinuous coefficients.

EXAMPLE 1. Let

\[ 0 < p_1 < p_2, \quad \tau = p_2 - p_1, \quad 1 < m, \quad 0 < \lambda < \frac{m}{m - 1}, \quad 0 < \kappa, \]

\[ p_1 = t_1, \quad p_2 = t_2, \quad t_k = (k - 2)\tau + p_2 \quad (k \geq 3), \]

and

\[ b_1(t) = \begin{cases} 0, & 0 \leq t < t_1, \\ \lambda, & t_1 \leq t < t_2, \\ \frac{\lambda}{m}, & t_2 \leq t < t_{2i+1}, \\ \lambda, & t_{2i+1} \leq t < t_{2i+2}, \quad i = 1, 2, \ldots. \end{cases} \]

and

\[ b_2(t) = \begin{cases} 0, & 0 \leq t < t_2, \\ \kappa, & t_2 \leq t < t_{2i+1}, \\ \frac{m - 1}{m} \lambda + \kappa, & t_{2i+1} \leq t < t_{2i+2}, \quad i = 1, 2, \ldots. \end{cases} \]

Put

\[ a(t) + 1 = b_1(t + p_1) + b_2(t + p_2) = \lambda + \kappa, \quad t \geq 0. \]

Then the equation is

\[ tx'(t) = a(t)x(t) - b(t)x(t - p_1) - b_2(t)x(t - p_2), \quad \text{a.e.} \quad (6) \]

\[ x(t) = 0, \quad t < 0, \]

\[ x(p_1) = c > 0. \]

We see that

\[ a(t) - b_1(t) \exp\left(-\int_{t-p_1}^{t} \frac{a(u)}{u} du\right) - b_2(t) \exp\left(-\int_{t-p_2}^{t} \frac{a(u)}{u} du\right) \]

\[ \leq (\lambda + \kappa - 1) - \left(\frac{\lambda}{m} + \kappa\right) \]

\[ = \frac{m - 1}{m} \lambda - 1 < 0 \quad (t > T) \]

for a sufficiently large \( T \). The solution \( x(t) \) is monotonically decreasing for \( t > T \).

From Corollary (2), the solution \( x(t) \) of (6) is positive on \((0, \infty)\) and integrable over \((0, \infty)\) and \( \lim_{t \to +\infty} x(t) = 0 \).

EXAMPLE 2. Suppose that \( b_i(t) = 0 \quad (0 < t < p_i) \) and each \( b_i(t) \) is positive and continuous on \([p_i, \infty)\) and has the right derivative at \( p_i \). Suppose that \( 0 < \alpha_i < \frac{1}{2} < \beta_i < 1 \) and for sufficiently large \( t \), \( b_i(t) / t^{\alpha_i} = \kappa_i + O(1 / t^{\beta_i}) \)
Positive and integrable solution

Then the solution $x(t)$ of the equation (1) is positive on $(0, \infty)$ and integrable over $(0, \infty)$ and $\lim_{t \to \infty} x(t) = 0$ as the conditions $a(1), a(2), a(3), (b1), (b2), (b3), (b4)$ hold. Let us show that

$$
\int_0^\infty \frac{b_i(s + p_i - b_i(s) \exp(-\int_{s-p_i}^s \frac{a(u)}{u} du))}{s} ds
$$

are convergent. Set $\alpha = \max\{\alpha_1, ..., \alpha_n\}$. We have

$$
\frac{1}{s} \left( b_i(s + p_i) - b_i(s) \exp\left(-\int_{s-p_i}^s \frac{a(u)}{u} du\right) \right)
= \frac{1}{s} \kappa_i \left( (s + p_i)^{\alpha_i} - s^{\alpha_i} \exp\left(-\int_{s-p_i}^s \frac{a(u)}{u} du\right) \right) + O\left(\frac{1}{s^{1+\beta_i}}\right)
= \kappa_i \alpha_i p_i s^{\alpha_i - 2} + \kappa_i \sum_{j=1}^n \kappa_j p_i s^{\alpha_i + \alpha_j - 2} + \kappa_i p_i s^{\alpha_i - 2} + O\left(\frac{1}{s^{1+\beta_i}}\right) + O\left(\frac{1}{s^{3-\alpha_i - 2\alpha}}\right)
$$

for sufficiently large $s$. This shows that the integrals of (7) converge and by (4), $\lim_{t \to \infty} x(t) = 0$.

Let us show that the integrals

$$
\int_T^\infty \frac{1}{t} \left( \int_{t-p_i}^t \frac{1}{u} b_j(u + p_j) \left( \int_{u-p_i}^u b_i(v + p_i) dv \right) du \right) dt
$$

are finite. From the following estimates,

$$
\int_{u-p_i}^u b_i(v + p_i) dv \approx \kappa_i p_i u^{\alpha_i},
$$

for sufficiently large $t$, it holds that for a sufficiently large $T$

$$
\int_T^\infty \frac{1}{t} \left( \int_{t-p_j}^t \frac{1}{u} b_j(u + p_j) \left( \int_{u-p_i}^u b_i(v + p_i) dv \right) du \right) dt
\approx \kappa_j \kappa_i p_j p_i \int_T^\infty \frac{1}{t^{2-\alpha_i - \alpha_j}} dt < \infty.
$$

Hence (b4) holds.

**Example 3.** Suppose that each $b_i(t)$ is positive and continuous on $[p_i, \infty)$ and has the right derivative at $p_i$ and $b_i(t) = 0 : 0 < t < p_i$. Suppose $\lim_{t \to \infty} b_i(t) / t = \kappa_i \geq 0$ (finite limit), $\sum_{i=1}^n \kappa_i p_i = D > 1$, $b_i(t + p_i) \geq \kappa_i p_i / D$ ($0 < t < \infty$), $b_i(t + p_i) / (t + p_i) \geq \kappa_i / D$ ($0 < t < \infty$). Put $\sum_{i=1}^n b_i(t + p_i) = a(t) + 1$. Then the solution $x(t)$ of (1) is positive on $(0, \infty)$ but not integrable over $(0, \infty)$. In fact, we have

$$
x(t) - x(+0) = \sum_{i=1}^n \int_0^{t-p_i} \frac{b_i(s + p_i)}{s + p_i} \left( \frac{p_i}{D} \right) x(s) ds
\quad + \sum_{i=1}^n \int_{t-p_i}^t \left( b_i(s + p_i) - \frac{\kappa_i p_i}{D} \right) x(s) ds > \epsilon > 0
$$

for a sufficiently small $\epsilon$ and for sufficiently large $t$.
References