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Propagation Estimate for $N$–Body Quantum Systems

HIDEO TAMURA*

In the present work we study the propagation estimate for $N$–body quantum systems with short–range or long–range interactions, which shows that the relative motion of clusters is asymptotically concentrated on classical trajectories.

For notational brevity, we here consider a simple system of $N$ particles with identical masses $m_j = 1$, $1 \leq j \leq N$, moving in the space $R^3$ through a real–valued pair potential $V_{jk}$, $1 \leq j < k \leq N$. We denote by $r_j \in R^3$ the position vector of the $j$–particle. Then, for such a system, the configuration space $X$ in the center of mass frame is given by

$$X = \{ r = (r_1, \ldots, r_N) \in R^{3N} : \sum_{j=1}^{N} r_j = 0 \}$$

and the energy operator $H$ (Schrödinger operator) takes the form

$$H = -\frac{1}{2} \Delta + \sum_{1 \leq j < k \leq N} V_{jk}(r_j - r_k),$$

where $\Delta$ denotes the Laplacian over $X$. We make the following assumptions on the real-valued pair potentials $V_{jk}(y)$, $y \in R^3$:

**Assumption (V).**

(V.1) $V_{jk}$ is decomposed into $V_{jk} = V^s_{jk} + V^l_{jk}$.

(V.2) $|V^s_{jk}(y)| \leq C(1 + |y|)^{-(1+\rho)}$, $\rho > 0$.

(V.3) $|\partial^\alpha_y V^l_{jk}(y)| \leq C(1 + |y|)^{-(|\alpha|+\rho)}$, $0 \leq |\alpha| \leq 1$.

Without loss of generality, we may assume that $0 < \rho < 1$. Throughout the paper, the constant $\rho$ is used with the meaning ascribed above and assumption (V) is always assumed to be satisfied. By assumption, the operator $H$ defined formally above admits a unique self-adjoint realization in $L_2(X)$. We denote by the same notation $H$ this self-adjoint realization.

The propagation estimate for $N$–body systems has been already studied by Derezinski [2] and Sigal–Soffer [6]. Roughly speaking, in these works, the pair potential $V_{jk}(y)$ has been assumed, in addition to (V), to satisfy the following additional assumptions:

(V.4) $(1 + |y|)^{1+\rho} |\nabla_y V_{jk}(y)| < \infty$. 

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The aim of the present work is to extend the propagation estimate to a slightly wider class of pair potentials without such restrictions. The basic idea of the proof is exactly the same as in [2]. However, in the remarkable work [6], the propagation estimate has played an important role in proving the asymptotic completeness of $N$-body scattering systems. In fact, we can prove the asymptotic completeness of $N$-body Schrödinger operators with short-range interactions under only the assumption that $|V_{jk}(y)| \leq C(1 + |y|)^{-1+\rho}$ (Tamura [8]). This extends slightly the result obtained by [6]. Thus it seems to be important that we here give the detailed proof of the propagation estimate for the class of pair potentials with assumption $(V)$, even if such an extension can be easily done by following carefully the arguments used in the works [2] and [6].

The precise formulation of the obtained results is given in section 1. We require a lot of basic notations in many-body scattering theory.

(1) Cluster decompositions. We use the letter $a$ or $b$ to denote a partition of $\{1, \ldots, N\}$ into non-empty disjoint subsets. Such a decomposition is called a cluster decomposition. For given cluster decomposition $a$, we denote by $\#(a)$ the number of clusters in $a$. Only a cluster decomposition $a$ with $2 \leq \#(a) \leq N$ is considered throughout the entire discussion. We also use the relation $jak$ if $j$ and $k$ are in the same cluster in $a$ and $\sim jak$ if they are in different clusters in $a$. Furthermore, if $b$ is a refinement of $a$, then we denote by $b \subseteq a$ this relation.

(2) Configuration spaces. For given cluster decomposition $a$, we define the subspaces $X^a$ and $X_a$ of $X$ as follows:

$$X^a = \{ r \in X : \sum_{j \in C} r_j = 0 \quad \text{for all clusters } C \text{ in } a \},$$

$$X_a = \{ r \in X : r_j = r_k \quad \text{if} \quad jak \}. $$

The subspace $X^a$ is the configuration space of the internal motion within clusters in $a$ and $X_a$ is the configuration space of the relative motion of clusters in $a$. We introduce the scalar product in $X$ by

$$r \cdot \tilde{r} = \sum_{j=1}^{N} r_j \cdot \tilde{r}_j$$

and denote by $|\cdot|$ the metric (norm) on $X$ induced by the scalar product above. Then the two subspaces $X^a$ and $X_a$ are mutually orthogonal with respect to this scalar product and span $X$; $X = X^a \oplus X_a$, and hence the space $L^2(X)$ can be regarded as the tensor product of spaces $L^2(X^a)$ and $L^2(X_a)$:

$$L^2(X) = L^2(X^a) \otimes L^2(X_a).$$

We write $x$ for the variables in $X$ and denote by $x^a$ and $x_a$ its projection on $X^a$ and $X_a$, respectively.
(3) Hamiltonians. We define the cluster Hamiltonian $H_a$ acting on $L^2(X)$ by
\[
H_a = -\frac{1}{2}\Delta + \sum_{jak} V_{jk}(r_j - r_k).
\]

If $(a) = N$, then we write
\[
H_a = H_0 = -\frac{1}{2}\Delta.
\]

According to (0.2), the Hamiltonian $H_a$ has the following decomposition:
\[
H_a = H^a \otimes Id + Id \otimes T_a
\]
on $L^2(X) = L^2(X^a) \otimes L^2(X_a)$, $Id$ being the identity operator, where the subsystem Hamiltonian $H^a$ associated with cluster decomposition $a$ is defined by
\[
H^a = -\frac{1}{2}\Delta + \sum_{jak} V_{jk}(r_j - r_k) \quad \text{on} \quad L^2(X^a),
\]
and $T_a = -\frac{1}{2}\Delta$ acting on $L^2(X_a)$ denotes the kinetic energy operator of the center mass motion of clusters in $a$. If $(a) = N$, then $H^a$ is defined as the zero operator acting on the scalar field $C$.

(4) Threshold sets. For given self-adjoint operator $A$, we denote by $\sigma_p(A)$ the set of all bound state energies (point eigenvalues) of $A$. Let $H^a$ be defined by (0.3). We define $\Lambda(H)$ by
\[
\Lambda(H) = \bigcup_{2 \leq (a) \leq N} \sigma_p(H^a).
\]
The element of $\Lambda(H)$ is called a threshold energy of $H$. Similarly we define
\[
\Lambda(H^a) = \bigcup_{b \subset a, b \neq a} \sigma_p(H^b).
\]
If $(a) = N$, we define $\Lambda(H^a)$ as $\Lambda(H^a) = \{0\}$. The threshold set $\Lambda(H_a)$ of the truncated Hamiltonian $H_a$ is given by
\[
\Lambda(H_a) = \Lambda(H^a) \cup \sigma_p(H^a).
\]
We know (for example, see [3]) that the threshold sets $\Lambda(H)$, $\Lambda(H^a)$ and $\Lambda(H_a)$ are all closed and countable under assumption (V).

(5) Momentum spaces. We denote by $X'$ and $X'_a$ the spaces dual to $X$ and $X_a$, respectively. We introduce the scalar product in $X'$ in the same way as in (0.1) and denote by $| \cdot |$ the metric (norm) on $X'$ induced by this scalar product.
For generic point \( p \in X' \), we denote by \( p_a \) its projection on \( X_a' \) and write \( \chi(D_a) \) for the pseudodifferential operator with symbol \( \chi(p_a) \).

(6) Function classes. Throughout the entire discussion, we use the following classes of functions:

\[
\begin{align*}
C^\infty_0(R^1) &= \{ f \in C^\infty(R^1) : f \text{ has compact support} \}; \\
C^\infty_1(R^1) &= \{ f \in C^\infty(R^1) : f' \in C^\infty_0(R^1) \}; \\
C^\infty_2(R^1) &= \{ f \in C^\infty(R^1) : f'' \in C^\infty_0(R^1) \}; \\
B^\infty(X) &= \{ f \in C^\infty(X) : \partial^\alpha_x f \text{ is bounded for all } \alpha \}; \\
S^m(X) &= \{ f \in C^\infty(X) : |\partial^\alpha_x f| \leq C_\alpha(x)^{m-|\alpha|} \}
\end{align*}
\]

with

\[
\langle x \rangle = (1 + |x|^2)^{1/2}.
\]

1. In this section we formulate precisely the propagation estimate for the propagator \( \exp(-itH) \). We continue to introduce several new notations. We denote by \( \| \cdot \|_X \) the \( L^2 \) norm in \( L^2(X) \). Let \( \Gamma \) be a Borel set in \( R^1 \) and let \( E_\Gamma(H) \) be the spectral resolution of \( H \) onto \( \Gamma \). We say that \( B : L^2(X) \rightarrow L^2(X) \) is \( H \)-smooth on \( \Gamma \), if there exists a constant \( C_\Gamma > 0 \) such that for any \( \psi \in L^2(X) \),

\[
\int \| B \exp(-itH) E_\Gamma(H) \psi \|_X^2 dt \leq C_\Gamma \| \psi \|_X^2,
\]

where the integration with no domain attached is taken over the whole space.

1.1. We denote by \( X_\nu \) the multiplication operator ; \( X_\nu : \varphi(x) \rightarrow \langle x \rangle^\nu \varphi(x) \). The important class of \( H \)-smooth operators is obtained from the principle of limiting absorption.

Proposition 1.1. Let \( \Gamma \) be a compact interval in \( R^1 \) avoiding all threshold and bound state energies of \( H \). Then the operator \( X_{-\nu} \) with \( \nu > 1/2 \) is \( H \)-smooth on \( \Gamma \).

The proposition above is a consequence of the resolvent estimate

\[
(1.1) \quad \| X_{-\nu}(H - \lambda \mp i\kappa)^{-1} X_{-\nu} \| < \infty,
\]

uniformly in \( \lambda \in \Gamma \) and \( \kappa, 0 < \kappa \leq 1 \), where \( \| \cdot \| \) denotes the operator norm when considered as an operator from \( L^2(X) \) into itself. This estimate was first proved by Mourre [4] in the three-body case and was extended by Perry–Sigal–Simon [5] to the \( N \)-body case under the additional assumptions (V.4) and (V.5). Recently, the result has been slightly improved by Amrein–Berthier–Georgescu [1] and Tamura [7], so that (1.1) remains true under assumption (V) only. As stated above, this makes it possible for us to extend the propagation estimate to a slightly wider class of pair potentials satisfying (V) only.
1.2. We formulate a series of main theorems. The formulation of these theorems further requires new notations.

Let $S_X$ denote the unit sphere in $X$. For given cluster decomposition $a$ with $2 \leq \#(a) \leq N$, we define the subset $S_a$ of $S_X$ by

$$S_a = \{ \omega = (\omega_1, \ldots, \omega_N) \in S_X \cap X_a : \omega_j \neq \omega_k \text{ if } j \neq k \}. \tag{1.2}$$

As is easily seen from the definition, $S_a \cap S_b = \emptyset$ if $a \neq b$ and $S_X$ is decomposed into the disjoint union of $S_a$;

$$S_X = \bigcup_{2 \leq \#(a) \leq N} \Theta S_a.$$

Furthermore, if $\omega \in S_a$, then a point in its small neighborhood belongs to $S_b$ with $b \subset a$. For $\omega \in S_X$ and $\epsilon > 0$, we use the notation

$$\text{con}(\omega, \epsilon) = \{ \theta \in S_X : |\theta - \omega| < \epsilon \}.$$

Recall the notations $\Lambda(H)$ and $\Lambda(H_a)$. For $E \in \mathbb{R}^1$, we define

$$\Sigma(E) = \{ \pm \sqrt{2(E - \lambda)} : E - \lambda \geq 0, \lambda \in \Lambda(H) \},$$

$$\Sigma_a(E) = \{ \pm \sqrt{2(E - \lambda)} : E - \lambda \geq 0, \lambda \in \Lambda(H_a) \}.$$

Since $\Lambda(H)$ and $\Lambda(H_a)$ are closed and countable, it follows that the above sets $\Sigma(E)$ and $\Sigma_a(E)$ are also closed and countable.

Finally we define the operator $\gamma$ by

$$\gamma = \frac{1}{2i} \{ (x/(x)) \cdot \nabla_x + \nabla_x \cdot (x/(x)) \}. \tag{1.3}$$

We know ([6], Theorem 3.2) that $\gamma$ has a self-adjoint realization in $L^2(X)$ with its natural domain. The operator $\gamma$ was first introduced by [6] to decouple channels.

We now formulate the propagation estimate as Theorems 1, 2 and 3 below. In the statements of all the theorems, $E$ is fixed and is assumed to be neither a threshold energy nor a bound state energy of $H$.

**Theorem 1.** Let $f \in C_0^\infty(\mathbb{R}^1)$. Assume that $f$ is supported away from $\Sigma(E)$. Then one can take an interval $\Gamma$ around $E$ so small that the operator $X_{-1/2}f(\gamma)$ is $H$-smooth on $\Gamma$.

**Theorem 2.** Fix $\omega \in S_a$ for some cluster decomposition $a$. Let $Q(x; \omega) \in S^0(X)$ have conical support in $\text{con}(\omega, \epsilon)$ for $\epsilon > 0$ small enough. Assume that $f \in C_0^\infty(\mathbb{R}^1)$ is supported away from $\Sigma_a(E)$. Then one can take $\epsilon$ and an interval $\Gamma$ around $E$ so small that the operator $X_{-1/2}Qf(\gamma)$ is $H$-smooth on $\Gamma$.

**Theorem 3.** Let $\omega \in S_a$ and $Q(x; \omega) \in S^0(X)$ be as above. Fix $\sigma \in \Sigma_a(E)$. Suppose that $f \in C_0^\infty(\mathbb{R}^1)$ is supported in a small interval $\Pi$ around $\sigma$. For given $\beta > 0$, let $\chi \in S^0(X'_a)$ be a symbol with support in $\{ p_a \in X'_a : |p_a - \sigma \omega| > \beta \}$,
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ω ∈ Xa being identified with an element of Xa. Then one can take ε, Π and Γ so small that the operator X−1/2Q_X(Da)f(γ) is H-smooth on Γ.

Roughly speaking, the statements of the main theorems show that the relative motion of clusters is asymptotically concentrated on classical trajectories. Theorems 1, 2 and 3 are proved in sections 5, 6 and 7, respectively. The first half sections are of preliminary character. In section 2, we make a brief review of the commutator method due to Mourre [4]. In sections 3 and 4, we develop the commutator calculus and derive several important formulas. These are used as a basic tool in proving the main theorems above.

2. As stated above, the proof of the main theorems is based on the commutator method initiated by Mourre [4] and developed by Froese–Herbst [3]. We here make a brief review of this remarkable method. For details, see [3].

Let A be the generator of the dilation unitary group:

\[ A = \frac{1}{2i} \{ x \cdot \nabla + \nabla \cdot x \}. \]

We know that A has a self-adjoint realization in L^2(X). Recall the notation \( \Lambda(H) \). We define

\[ s(\lambda) = \sup \{ \sigma \in \Lambda(H) : \sigma \leq \lambda \} \]

for \( \lambda \geq \inf \Lambda(H) \) and

\[ s(\lambda) = -M_0 \]

for \( \lambda < \inf \Lambda(H) \), where \( M_0 \) is chosen so large that \( -M_0 \) lies below all eigenvalues of \( H \).

**Proposition 2.1.** For any \( \delta > 0 \) and \( \lambda \in \mathbb{R}^1 \), there exist an interval \( \Gamma \) around \( \lambda \) and a compact operator \( K : L^2(X) \rightarrow L^2(X) \) such that

\[ E_\Gamma(H) i[H, A] E_\Gamma(H) \leq \{ 2(\lambda - s(\lambda)) - \delta \} E_\Gamma(H) + K \]

in the form sense.

We apply the proposition above to the subsystem Hamiltonian \( H^a \) acting on \( L^2(X^a) \). Let

\[ A^a = \frac{1}{2i} \{ x^a \cdot \nabla^a + \nabla^a \cdot x^a \}, \]

where \( \nabla^a \) denotes the gradient operator in \( X^a \). In a similar way as above, we define

\[ s^a(\lambda) = \sup \{ \sigma \in \Lambda(H^a) : \sigma \leq \lambda \} \]

for \( \lambda \geq \inf \Lambda(H^a) \) and

\[ s^a(\lambda) = -M_0 \]
for \( \lambda < \inf \Lambda(H^a) \), where \( M_0 \) is the same as in (2.2). If \( \lambda \) is not a bound state energy of \( H^a \), then we can take an interval \( \Gamma \) around \( \lambda \) so small that

\[
E_{\Gamma}(H^a)i[H^a, A^a]E_{\Gamma}(H^a) \geq \{2(\lambda - s^a(\lambda)) - \delta\} E_{\Gamma}(H^a)
\]

On the other hand, if \( \lambda \) is a bound state energy of \( H^a \), then we can prove by use of the virial theorem that

\[
E_{\Gamma}(H^a)i[H^a, A^a]E_{\Gamma}(H^a) \geq -\delta E_{\Gamma}(H^a)
\]

for a small interval \( \Gamma \) around \( \lambda \). Taking account of the relation \( \Lambda(H_a) = \Lambda(H^a) \cup \sigma_p(H^a) \), we now define

\[
\theta_a(\lambda) = 2\lambda - 2\sup\{\sigma \in \Lambda(H_a) : \sigma \leq \lambda\}
\]

for \( \lambda \geq \inf \Lambda(H_a) \) and

\[
\theta_a(\lambda) = 2\lambda + 2M_0
\]

for \( \lambda < \inf \Lambda(H_a) \). Then we have, combining (2.3) and (2.4), that for any \( \delta > 0 \), there exists an interval \( \Gamma \) around \( \lambda \) such that

\[
E_{\Gamma}(H^a)i[H^a, A^a]E_{\Gamma}(H^a) \geq (\theta_a(\lambda) - \delta) E_{\Gamma}(H^a).
\]

We further define

\[
\theta^\kappa_a(\lambda) = \inf\{\theta_a(\mu) : |\mu - \lambda| < \kappa\}
\]

for \( \kappa > 0 \) small enough. By definition, we can easily obtain that \( \theta^\kappa_a(\lambda) \geq 0 \) for \( \lambda \geq \inf \Lambda(H_a) \).

**Proposition 2.2.** Let \( J \) be a compact interval in \( R^1 \). Let \( \Gamma_{\lambda, \epsilon} = (\lambda - \epsilon, \lambda + \epsilon) \) for \( \lambda \in J \). Then, for any \( \delta > 0 \) and \( \kappa > 0 \) small enough, there exists \( \epsilon = \epsilon(\delta, \kappa) > 0 \) independent of \( \lambda \) such that

\[
E_{\Gamma_{\lambda, \epsilon}}(H^a)i[H^a, A^a]E_{\Gamma_{\lambda, \epsilon}}(H^a) \geq (\theta^\kappa_a(\lambda) - \delta) E_{\Gamma_{\lambda, \epsilon}}(H^a).
\]

**Proof.** Let

\[
\epsilon(\lambda, \kappa, \delta) = \sup\{\epsilon : (2.6) \text{ holds with } \kappa \text{ and } \delta\}.
\]

By (2.5), there exists \( \nu = \nu(\lambda, \delta) > 0 \) such that

\[
E_{\Gamma_{\lambda, \nu}}(H^a)i[H^a, A^a]E_{\Gamma_{\lambda, \nu}}(H^a) \geq (\theta_a(\lambda) - \delta) E_{\Gamma_{\lambda, \nu}}(H^a).
\]

Take \( \mu \) so that \( |\mu| < \min\{\kappa/2, \nu/4\} \) and define

\[
\zeta = \nu - 2|\mu| > \nu/2 > 0.
\]

Then we have \( \Gamma_{\lambda + \mu, \zeta} \subseteq \Gamma_{\lambda, \nu} \) and also \( \theta_a(\lambda) \geq \theta^\kappa_a(\lambda + \mu) \). Hence, by definition, it follows that

\[
\epsilon(\lambda + \mu, \kappa, \delta) \geq \zeta \geq \nu/2 > 0
\]

for \( \mu \) as above. Thus, for any \( \lambda \in J \), there exists an open interval \( O_\lambda \) around \( \lambda \) such that \( \epsilon(\mu, \kappa, \delta) > \epsilon_\lambda > 0 \) for \( \mu \in O_\lambda \). This proves the proposition.

We now apply the commutator method to the cluster Hamiltonian \( H_a \) acting on \( L^2(X) \). In the \( p_a \)-representation, the commutator \( i[H_a, A] \) is expressed as \( |p_a|^2 + i[H^a, A^a] \). Define \( \theta^\kappa_a(p_a; \lambda) \) by

\[
\theta^\kappa_a(p_a; \lambda) = |p_a|^2 + \theta^\kappa_a(\lambda - |p_a|^2/2).
\]

Then, by Proposition 2.2, we obtain the following
**Proposition 2.3.** For any \( \delta > 0 \) and \( \kappa > 0 \) small enough and for \( \lambda \in R^1 \), there exists an open small interval \( \Gamma \) around \( \lambda \) such that

\[
E_\Gamma(H_a)i[H_a,A]E_\Gamma(H_a) \geq (\theta^\kappa(D_a;\lambda) - \delta)E_\Gamma(H_a)
\]

in the form sense.

3. In the present section, we collect several properties of commutator operators which are used without further references as a basic tool to prove the main theorems. These properties can be easily verified or can be verified in almost the same way as in [2] and [6].

3.1. Recall the notations \( C^\infty_1(R^1) \) and \( C^\infty_2(R^1) \). For \( f \in C^\infty_1(R^1) \) or \( C^\infty_2(R^1) \), we denote its (distributional) Fourier transformation by the formula

\[
\hat{f}(\tau) = (2\pi)^{-1/2} \int \exp(-i\tau s)f(s)\,ds.
\]

It follows immediately that \( \hat{f}(\tau) \in S(R^1) \) (the Schwartz space) for \( f \in C^\infty_1(R^1) \) and \( \tau^2 \hat{f}(\tau) \in S(R^1) \) for \( f \in C^\infty_2(R^1) \).

**Proposition 3.1 ([6], Lemma A.1).** Let \( B \) and \( C \) be self-adjoint operators on \( L^2(X) \). Let \( f \in C^\infty_1(R^1) \). Assume that \( B \) and \([C,B]\) are bounded. Then \([B,f(C)]\) is also bounded and

\[
[B,f(C)] = (2\pi)^{-1/2} \int \hat{f}(s)\,ds \int_0^\infty \exp(i(s - \tau)C)i[B,C]\exp(i\tau C)\,d\tau.
\]

**Proposition 3.2 ([6], Lemma A.7).** Let \( B \) and \( C \) be as in Proposition 3.1. Let \( f \in C^\infty_2(R^1) \). Assume that \( B, [C,B] \) and \([C,[C,B]]\) are all bounded. Then \([B,f(C)]\) is also bounded and

\[
[B,f(C)] = f'(C)[B,C] + R
\]

with

\[
R = -(2\pi)^{-1/2} \int \hat{f}(s)\,ds \int_0^\infty (s - \tau)\exp(i(s - \tau)C)[C,[C,B]]\exp(i\tau C)\,d\tau.
\]

**Proposition 3.3 ([2], Lemma 5.4).** Let \( \gamma \) be defined by (1.3). Then, for any \( k \in R^1 \), one has

\[
\|X_k\exp(it\gamma)X_{-k}\| \leq C_k(1 + |t|)^{|k|}.
\]

3.2. We here introduce new notations. For an operator \( B \) (not necessarily bounded) acting on \( L^2(X) \), we write \( B \in O((x)^m) \), if \( X_{-m+k}BX_{-k} \) extends to a bounded operator on \( L^2(X) \) for any \( k \in R^1 \). We also use the notation \( B_1 = B_2 + O((x)^m) \), if the difference \( B_1 - B_2 \) is of class \( O((x)^m) \).

We now take \( c \) so large that \((H + c)^{-1}: L^2(X) \rightarrow L^2(X) \) is bounded. Then the following three lemmas are easy to prove.
LEMMA 3.1. If $|\alpha| + |\beta| \leq 2$, then $\partial_x^\alpha (H + c)^{-1} \partial_x^\beta \in O((x)^0)$ and

$$[Q, \partial_x^\alpha (H + c)^{-1} \partial_x^\beta] \in O((x)^{m-1})$$

for $Q \in S^m(X)$.

LEMMA 3.2. Let $\gamma$ be defined by (1.3). Then one has the following statements:

1. $[\gamma, \partial_x^\alpha (H + c)^{-1} \partial_x^\beta] \in O((x)^{-1})$ for $|\alpha| + |\beta| \leq 1$.
2. $[Q, [\gamma, (H + c)^{-1}]] \in O((x)^{m-2})$ for $Q \in S^m(X)$.
3. $[\gamma, [\gamma, (H + c)^{-1}]] \in O((x)^{1-\rho})$.

LEMMA 3.3. Let $A$ be defined by (2.1). Then one has the following statements:

1. $[A, \partial_x^\alpha (H + c)^{-1} \partial_x^\beta] \in O((x)^0)$ for $|\alpha| + |\beta| \leq 1$.
2. $[Q, [A, (H + c)^{-1}]] \in O((x)^{m-1})$ for $Q \in S^m(X)$.
3. $[\gamma, [A, (H + c)^{-1}]] \in O((x)^{-\rho})$.

LEMMA 3.4. Let $f \in C_0^\infty(R^1)$. Then $f(\gamma) \in O((x)^0)$ and

$$[Q, f(\gamma)] \in O((x)^{m-1})$$

for $Q \in S^m(X)$.

PROOF. The first assertion follows from Proposition 3.3. If $Q \in S^m(X)$ with $m \leq 0$, then the second assertion follows from Proposition 3.1 with $B = Q$ and $C = \gamma$. If $m > 0$, then we write $Q = X_m Q_0$ with $Q_0 \in S^0(X)$ and calculate

$$[Q, f(\gamma)] = X_m [Q_0, f(\gamma)] + [X_m, f(\gamma)] Q_0.$$ 

Since

$$[X_m, f(\gamma)] = X_m [f(\gamma), X_{-m}] X_m \in O((x)^{m-1}),$$

this proves the lemma.

LEMMA 3.5. Let $g \in C_0^\infty(R^1)$. Then $g(H) \in O((x)^0)$ and

$$[Q, g(H)] \in O((x)^{m-1})$$

for $Q \in S^m(X)$.

PROOF. We write $g(H) = g_1((H + c)^{-1})$ with $g_1 \in C_0^\infty(R^1)$. The operator $\exp(it(H + c)^{-1})$ has the same property as $\gamma$ in Proposition 3.3. Hence, the same argument as in the proof of Lemma 3.4 applies.

LEMMA 3.6. Let $\chi \in B_0^\infty(X'_\alpha)$. If $\chi(D_a)$ is considered as an operator acting on $L^2(X)$, then $\chi(D_a) \in O((x)^0)$ and

$$[Q, \chi(D_a)] \in O((x)^{m-1})$$
for $Q \in S^m(X)$.

**Proof.** The lemma is proved by use of the standard pseudodifferential calculus.

**Lemma 3.7.** Let $f \in C_0^\infty(R^1)$ and $g \in C_0^\infty(R^1)$. Then

$$[f(\gamma), g(H)] \in O(\langle x \rangle^{-1}).$$

**Proof.** By Lemma 3.2, $[\gamma, (H + c)^{-1}] \in O(\langle x \rangle^{-1})$. Hence we have by Proposition 3.1 with $B = (H + c)^{-1}$ and $C = \gamma$ that $[f(\gamma), (H + c)^{-1}] \in O(\langle x \rangle^{-1})$. We again use Proposition 3.1 with $B = f(\gamma)$ and $C = (H + c)^{-1}$ to obtain the conclusion.

**Lemma 3.8.** Let $f \in C_0^\infty(R^1)$. Then one has the following statements:

1. If $h \in C_1^\infty(R^1)$, then

$$[h(\omega \cdot D_a), f(\gamma)] \in O(\langle x \rangle^{-1})$$

for $\omega \in S_a$.

2. If $\chi \in S_0(X'_a)$, then

$$[\chi(D_a), f(\gamma)] \in O(\langle x \rangle^{-1}).$$

**Proof.** We prove (1) only. A similar argument applies to the proof of (2). The operator $\gamma$ is related to $A$ through the relation

$$\gamma = AX^{-1} + Q$$

with $Q \in S^{-1}(X)$. By use of this relation, we have

$$[\gamma, h(\omega \cdot D_a)] = \gamma B_1 + B_2$$

with $B_1$ and $B_2 \in O(\langle x \rangle^{-1})$ ([2], Lemma 5.6). Hence it follows that

$$[(\gamma + i)^{-1}, h(\omega \cdot D_a)] \in O(\langle x \rangle^{-1}),$$

because $(\gamma + i)^{-1} \in O(\langle x \rangle^0)$. By the formula in Proposition 3.1 with $B = h(\omega \cdot D_a)$ and $C = \gamma$, we obtain that

$$(\gamma + i)^{-1}[f(\gamma), h(\omega \cdot D_a)] \in O(\langle x \rangle^{-1}).$$

Thus, if we write

$$f(\gamma) = (\gamma + i)^{-1} f_1(\gamma) (\gamma + i)^{-1}$$

with $f_1 \in C_0^\infty(R^1)$, then (1) is verified by a simple calculation.

**Lemma 3.9.** Let $f \in C_1^\infty(R^1)$. Then one has the following statements:

1. $[f(\gamma), [\gamma, (H + c)^{-1}]] \in O(\langle x \rangle^{-1-\rho}).$

2. $[f(\gamma), [A, (H + c)^{-1}]] \in O(\langle x \rangle^{-\rho}).$

**Proof.** We prove (1) only. By Lemma 3.2, $[\gamma, [\gamma, (H + c)^{-1}]] \in O(\langle x \rangle^{-1-\rho})$. Hence the statement immediately follows from Proposition 3.1 with $B = \gamma, (H + c)^{-1}$ and $C = \gamma$.

3.3. Throughout this subsection, $\omega \in S_a$ is fixed with some cluster decomposition $a$ and $Q \in S_0(X)$ is assumed to have support in a conical neighborhood of $\omega$: If $x$ with $|x| \geq 1$ is in the support of $Q$, then $x/|x| \in con(\omega, \epsilon)$ for $\epsilon > 0$ small enough.
**Lemma 3.10.** One has the following statements:

1. \( Q(H + c)^{-1} = Q(H_a + c)^{-1} + O((x)^{-\rho}). \)
2. \( Qg(H) = Qg(H_a) + O((x)^{-\rho}), \quad g \in C^\infty_0(R^1). \)

**Proof.** Let \( I_a \) be the intercluster potential for \( a \) which is defined by

\[
I_a = H - H_a = \sum_{j<k} V_{jk}(r_j - r_k).
\]

This potential has the decaying property that \( I_a = O(|x|^{-\rho}), |x| \to \infty \), on the support of \( Q \). Therefore, (1) can be easily proved and also (2) follows from (1).

**Lemma 3.11.**

\[
Q[A,(H + c)^{-1}]Q = Q[A,(H_a + c)^{-1}]Q + O((x)^{-\rho}).
\]

**Proof.** We calculate \( Q[A,(H_a + c)^{-1} - (H + c)^{-1}]Q \). Then this is decomposed into the following three terms:

\[
T_1 = Q[A,(H_a + c)^{-1}]I_a(H + c)^{-1}Q,
T_2 = Q(H_a + c)^{-1}[A,I_a](H + c)^{-1}Q,
T_3 = Q(H_a + c)^{-1}I_a(A,(H + c)^{-1})Q.
\]

The operators \( I_a(H + c)^{-1}Q \) and \( Q(H_a + c)^{-1}I_a \) are both of class \( O((x)^{-\rho}) \). Hence the operators \( T_1 \) and \( T_2 \) are of class \( O((x)^{-\rho}) \). The commutator \([A, I_a]\) takes the form

\[
[A, I_a] = \sum_{|\alpha| + |\beta| \leq 1} \partial_\alpha^\alpha O((x)^0) \partial_\beta^\beta
\]

and also we have

\[
Q[A, I_a]Q = \sum_{|\alpha| + |\beta| \leq 1} \partial_\alpha^\alpha O((x)^{-\rho}) \partial_\beta^\beta.
\]

Hence the operator \( T_2 \) is also of class \( O((x)^{-\rho}) \). This proves the lemma.

**Lemma 3.12.** Let \( \chi \in B^\infty(X'_a) \). Then one has

\[
Q[\chi(D_a), I_a] \in O((x)^{-1-\rho}).
\]

**Proof.** We take \( Q_1 \in S^0(X) \) so that \( Q_1 Q = Q \). We may assume that \( Q_1 \) has the same property as \( Q \). By the pseudodifferential calculus,

\[
Q\chi(D_a)(1 - Q_1) \in O((x)^{-L})
\]

for any \( L \) large enough. By assumption (V), we can decompose the intercluster potential \( I_a \) into \( I_a = I_a^a + I_a^s \) in such a way that:

\[
|\partial_\alpha^\alpha I_a^a| \leq C(x)^{-1-\rho} \quad \text{and} \quad |\partial_\alpha^\alpha I_a^s| \leq C(x)^{-1-\rho}, \quad |\alpha| \geq 1,
\]

and therefore

\[
Q\chi(D_a)(1 - Q_1) \in O((x)^{-L})
\]
on the support of $Q_1$. Thus we can easily prove the lemma.

**Lemma 3.13.** Let $\chi \in B^\infty(X'_a)$. Then one has

$$Q[\chi(D_a), (H + c)^{-1}] \in O(\langle x \rangle^{-1-\rho})$$

**Proof.** Since $[\chi(D_a), H_a] = 0$, we can calculate the commutator under consideration as

$$Q(H + c)^{-1}[I_a, \chi(D_a)](H + c)^{-1}.$$ This, together with Lemma 3.12, proves the lemma.

**Lemma 3.14.** Let $f \in C^\infty_0(R^1)$. Then one has

$$Q(f(\omega) - f(\omega \cdot D_a)) = \varepsilon Q B_0(D) + B_1(D)$$

with $B_0 \in O(\langle x \rangle^{\rho})$ and $B_1 \in O(\langle x \rangle^{-1})$, where $B_0$ satisfies $\|X_{-k}B_0X_k\| = O(1)$ uniformly in $\varepsilon > 0$ small enough.

**Proof.** The operator $Q(\gamma - \omega \cdot D_a)$ takes a form similar to that in the lemma. Hence the lemma follows from this relation.

4. In this section, we derive, by use of the commutator calculus developed in section 3, several important formulas which play a basic role in proving the main theorems.

**Proposition 4.1.** One has

$$i[\gamma, (H + c)^{-1}]$$

$$= X_{-1/2}[i[A, (H + c)^{-1}] - (H + c)^{-1}\gamma^2(H + c)^{-1}]X_{-1/2} + O(\langle x \rangle^{-2}).$$

**Proof.** The proposition is easy to prove. By relation (3.1), we have

$$[\gamma, (H + c)^{-1}] = [A, (H + c)^{-1}]X_{-1} + A[X_{-1}, (H + c)^{-1}] + O(\langle x \rangle^{-2})$$

and also, by a direct calculation,

$$i[H_0, X_{-1}] = -\gamma X_{-2} + O(\langle x \rangle^{-3}).$$

Therefore the proposition follows at once.

We use Proposition 3.2 with $B = (H + c)^{-1}$ and $C = \gamma$. The following two propositions can be easily proved by making use of the formula in Proposition 4.1 and of the commutator calculus in section 3.
Proposition 4.2. Let $F \in C^\infty_c(R^1)$. Assume that $F'$ is of the form $F' = f^2$ with $f \in C^\infty_1(R^1)$. Then one has

$$i[F(\gamma)(H + c)^{-1}] = X_{-1/2}f(\gamma)i[A,(H + c)^{-1}]f(\gamma)X_{-1/2}$$

$$- (H + c)^{-1}X_{-1/2}f(\gamma)^2X_{-1/2}(H + c)^{-1} + O(\langle x \rangle^{-1 - \rho}).$$

Proposition 4.3. Let $Q \in S^0(X)$ be supported in a small conical neighborhood of $\omega \in S_\alpha$. Let $F \in C^\infty_1(R^1)$. Assume that $F'$ is of the form $F' = f^2$ with $f \in C^\infty_0(R^1)$. Then one has

$$i[F(\gamma),(H + c)^{-1}]Q^2 = X_{-1/2}Qf(\gamma)i[A,(H + c)^{-1}]f(\gamma)QX_{-1/2}$$

$$- (H + c)^{-1}X_{-1/2}Qf(\gamma)^2f(\gamma)^2QX_{-1/2}(H + c)^{-1} + O(\langle x \rangle^{-1 - \rho}).$$

The next proposition also follows from Propositions 3.2 and 4.1. However, we should note that the proof uses Lemma 3.8 and the relation

$$[[A,(H_a + c)^{-1}], \chi(D_a)] = 0$$

for $\chi \in B^\infty_0(X'_a)$.

Proposition 4.4. Let $\omega \in S_\alpha, Q \in S^0(X)$ and $F \in C^\infty_1(R^1)$ be as in Proposition 4.3. Assume either that $\chi \in S^0(X'_a)$ or that $\chi(p_a) = h(\omega \cdot p_a)$ with $h \in C^\infty_0(R^1)$. Then one has

$$i[F(\gamma),(H + c)^{-1}]Q^2 \chi(D_a)^2$$

$$= X_{-1/2}f(\gamma)Q\chi(D_a)[A,(H + c)^{-1}]\chi(D_a)Qf(\gamma)X_{-1/2}$$

$$- (H + c)^{-1}X_{-1/2}Q\chi(D_a)\gamma^2f(\gamma)^2\chi(D_a)QX_{-1/2}(H + c)^{-1}$$

$$+ O(\langle x \rangle^{-1 - \rho}).$$

5. In this section we shall prove Theorem 1. Throughout the proof, we fix $g_0 \in C^\infty_0(R^1)$, so that $g_0 = 1$ on $\Gamma$ for a small interval $\Gamma$ around $E$. Hence we have $g_0(H) E_\Gamma(H) = E_\Gamma(H)$. The support of $g_0$ is taken so small that it avoids all the threshold and bound state energies of $H$. We also use the notation $Re B = (B + B^*)/2$ for a bounded operator $B$ acting on $L^2(X)$.

5.1. We first consider the case in which $f \in C^\infty_1(R^1)$ satisfies $f(s) = 1$ for $|s| \geq M$ and $f(s) = 0$ for $|s| \leq M/2$ with $M > 0$ large enough. Define $F \in C^\infty_2(R^1)$ to satisfy $F' = -f^2$ and the operator $B_0$ as

$$B_0 = f(\gamma)X_{-1/2}E_\Gamma(H).$$

Then, in order to prove the theorem for $f$ as above, it suffices to show that

$$E_\Gamma(H)[H,F(\gamma)] \geq cB_0^* B_0 + E_\Gamma(H)O(\langle x \rangle^{-1 - \rho})E_\Gamma(H).$$

(5.1)
for some $c > 0$. Set $g_1(H) = g_0(H)(H + c)$. Then
\[
g_0(H)i[H, F(\gamma)]g_0(H) = g_1(H)i[F(\gamma), (H + c)^{-1}]g_1(H).
\]
We now apply Proposition 4.2 to the commutator on the right side of (5.1). Since $\gamma^2 f(\gamma)^2 \geq (M/2)^2 f(\gamma)^2$ and
\[
-g_1(H)[A, (H + c)^{-1}]g_1(H) \leq Cg_0(H)^2
\]
in the form sense, we can obtain (5.1) by taking $M$ large enough. This proves that the operator $X_{-1/2} f(\gamma)$ is $H$-smooth on $\Gamma$.

5.2. Next we consider the case in which $f \in C_0^\infty(R^1)$ has compact support and prove (5.1) for such a $f(\gamma)$. To this end, we here introduce a non-negative smooth partition of unity over $X$ with the following properties:
(Q.1) $\sum_j Q_j(x)^2 = 1$ on $X$ ; (Q.2) $Q_j \in S^0(X)$ ; (Q.3) $Q_j$ has conical support in $\text{con}(\omega_j, \epsilon)$ for $\omega_j \in S_{a_j}$, $\epsilon > 0$ being chosen small enough and uniformly in $j$. We fix one of $Q_j$ as $Q$ and define the operator $B_1$ as
\[
B_1 = f(\gamma)QX_{-1/2}E_\Gamma(H).
\]
The function $Q(x)$ is assumed to have conical support in $\text{con}(\omega, \epsilon)$ for some $\omega \in S_a$.

We now define $F \in C_1^\infty(R^1)$ to satisfy that $F = f^2$. Then, to prove (5.1), it suffices by the above partition of unity to show that
\[
\text{Re } E_\Gamma(H)g_1(H)i[F(\gamma), (H + c)^{-1}]Q^2g_1(H)E_\Gamma \geq cB_1^*B_1 + E_\Gamma(H)O((x)^{-\rho})E_\Gamma(H)
\]
for some $c > 0$. We apply Proposition 4.3 to the commutator on the left side of (5.2). Then we first obtain that $-B_1^*\gamma^2 B_1 \geq -\Pi B_1^*B_1$ with
\[
\Pi = \sup\{s^2 : s \in \text{supp } f\}.
\]
We take $f_1 \in C_0^\infty(R^1)$ to satisfy the relation $f_1 f = f$. Then, by Lemmas 3.10 and 3.14, it follows that
\[
g_1(H)Qf(\gamma) = Qf(\gamma)(f_1(\omega \cdot D_a)g_1(H_a) + \epsilon O((x)^0)) + O((x)^{-\rho}).
\]
Similarly we have
\[
f(\gamma)Qg_1(H) = (g_1(H_a)f_1(\omega \cdot D_a) + \epsilon O((x)^0))Qf(\gamma) + O((x)^{-\rho}).
\]
Furthermore, it follows from Proposition 2.3 that for any $\delta > 0$ and $\kappa > 0$ small enough, we can take the support of $g_0$ so small that
\[
f_1(\omega \cdot D_a)g_0(H_a)i[H_a, A]g_0(H_a)f_1(\omega \cdot D_a)
\geq (\Theta - \delta)(g_0(H_a)f_1(\omega \cdot D_a))^*(g_0(H_a)f_1(\omega \cdot D_a))
\]
Propagation estimate

\[ \Theta = \inf\{ \theta_\alpha^*(p_a; E) : \omega \cdot p_a \in \text{supp } f_1 \} . \]

Thus we have that the term on the left side of (5.2) is less than or equal to

\[ (\Theta - \Pi - \delta - O(\varepsilon)) B_1^2 B_1 + E_\Gamma(H) O((x)^{-\frac{1}{2}} - \rho) E_\Gamma(H) \]

in the form sense.

Let

\[ \sigma_0 = \inf\{|\sigma| : \sigma \in \Sigma(E)\} \quad \text{and} \quad \sigma_1 = \sup\{|\sigma| : \sigma \in \Sigma(E)\} . \]

By assumption, \( E \) is neither a threshold energy nor a bound state energy of \( H \) and hence \( \sigma_0 > 0 \) strictly. The proof is divided into the following three cases according as the support of \( f \) : (1) \( \text{supp } f \subset (d, M) \) or \( \text{supp } f \subset (-M, -d) \) for some \( d, \sigma_1 < d < M \); (2) \( \text{supp } f \subset (d_0, k_0) \) or \( \text{supp } f \subset (-k_0, -d_0) \) for some \( d_0 \) and \( k_0, 0 < d_0 < k_0, \) in \( \Sigma(E) \); (3) \( \text{supp } f \subset (-d, d) \) for some \( d, 0 < d < \sigma_0 \). We may assume that \( f_1 \) has the same property as \( f \). We now evaluate the value \( \Pi \) from above and \( \Theta \) from below in each case.

We first deal with case (1). We consider the + case only. We can easily obtain that \( \Pi \leq M^2 \). If \( \omega \cdot p_a \in \text{supp } f_1 \), then \( |p_a|^2 > d^2 \). Hence, if \( \kappa \) is chosen so small that \( 0 < \kappa < (d^2 - \sigma_1^2)/2 \), then it follows that

\[ E' - |p_a|^2/2 < E - \sigma_1^2/2 = \inf\Lambda(H) \leq \inf\Lambda(H_a) \]

for \( E' \) such that \( |E' - E| < \kappa \). Thus we have

\[ \Theta \geq 2(E + M_0 - \kappa) , \]

\( M_0 \) being as in (2.2). Since \( M_0 \) can be taken large enough, this proves (5.2) in case (1).

Next we consider the case (2). We again deal with the + case only. Assume that

\[ \text{supp } f_1 \subset (d_1, k_1) \subset (d_0, k_0) . \]

Then we have \( \Pi \leq k_1^2 \). We may further assume that there are no points of \( \Sigma(E) \) in the interval \((d_0, k_0)\). We can write \( E \) as \( E = d_0^2/2 + c_0 \) or \( E = k_0^2/2 + c_1 \) with \( c_0, c_1 < d_0 \) and \( c_1 < c_0 \), in \( \Lambda(H) \). If \( \omega \cdot p_a \in \text{supp } f_1 \), then \( |p_a|^2 \geq d_1^2 \). Hence, if \( \kappa \) is chosen so small that \( 0 < \kappa < (d_1^2 - d_0^2)/2 \), then we have that

\[ E' - |p_a|^2 < c_0 \]

for \( E' \) such that \( |E' - E| < \kappa \). This implies that

\[ \theta_\alpha^*(p_a; E) \geq k_0^2 - 2\kappa \]

for \( p_a \) as above. Thus, (5.2) can be verified in case (2) also.
The case (3) can be dealt with in the same way as in case (2). In fact, we have \( \Theta \geq \sigma_0^2 - 2 \kappa \), if we take \( \kappa \) as \( 0 < \kappa < (\sigma_0^2 - d^2)/2 \). Thus, in any case, we can prove (5.2) and hence the operator \( X^{-1/2} f(\gamma) \) is shown to be \( H \)-smooth on \( \Gamma \), if \( f \in C_0^\infty(\mathbb{R}^1) \) is supported away from \( \Sigma(E) \). Combining this with the result of subsection 5.1 now completes the proof of Theorem 1.

6. The present section is devoted to proving Theorem 2. We begin by fixing several notations. For given \( \omega \in S_\alpha \), we take \( \nu, 0 < \nu < 1 \), so close to 1 that if \( x \cdot \omega / \langle x \rangle > \nu \) with \( |x| \) large enough, then \( x/|x| \) lies in some \( S_b \) with \( b \subset a \). Let

\[
1 > \nu^\pm > \nu_1^\pm > \nu.
\]

We introduce non-negative functions \( \tilde{q}_\pm \in C_0^\infty(\mathbb{R}^1) \) with support in \( (\nu_1^\pm, \nu^\pm) \) and define

\[
q_\pm(x) = \tilde{q}_\pm(x \cdot \omega / \langle x \rangle)
\]

and

\[
Q_\pm(x) = \tilde{Q}_\pm(x \cdot \omega / \langle x \rangle)
\]

with

\[
\tilde{Q}_\pm(t) = \int_{-\infty}^{t} \tilde{q}_\pm(s)^2 ds.
\]

By definition, the functions \( q_\pm \) and \( Q_\pm \) belong to \( S_0(X) \) and also \( Q_\pm \) is supported in \( \{x \in X : x \cdot \omega > \nu_1^\pm \} \). We can choose \( \tilde{q}_\pm \in C_0^\infty(\mathbb{R}^1) \) so that \( \sqrt{Q_\pm} \in S^0(X) \). Theorem 2 is obtained as an immediate consequence of the following proposition.

PROPOSITION 6.1. Let \( Q_\pm \) be as above. Let \( \sigma \notin \Sigma_a(E) \). Suppose that \( f \in C_0^\infty(\mathbb{R}^1) \) is supported in a small interval around \( \sigma \) and is supported away from \( \Sigma_a(E) \). Fix \( \lambda^- \) and \( \lambda^+ \) as \( \lambda^- < \sigma \nu^- \) and \( \lambda^+ > \sigma \nu^+ \) and take \( h_- \) and \( h_+ \in C_1^\infty(\mathbb{R}^1) \) to satisfy that \( h_- \) and \( h_+ \) have support in \( (-\infty, \lambda^-) \) and \( (\lambda^+, \infty) \), respectively. Then one can take a conical support of \( Q_\pm \) and an interval \( \Gamma \) around \( E \) so small that the operator

\[
X_{-1/2} h_\pm (\omega \cdot D_\alpha) \sqrt{Q_\pm} f(\gamma)
\]

is \( H \)-smooth on \( \Gamma \).

6.1. We shall prove the proposition for the \( - \) case only. We write \( Q \) for \( Q_- \) throughout the proof. As in the proof of Theorem 1, the proof is separated into the three cases according as the support of \( f \). We here consider only the case in which

\[
\text{supp } f \subset (d_2, k_2) \subset (d_0, k_0)
\]

for some \( d_0 \) and \( k_0 \), \( 0 < d_0 < k_0 \), in \( \Sigma_a(E) \). The remaining two cases can be dealt with in a similar way. Since \( \Sigma_a(E) \) is closed and countable, we can take \( d_1 \).
and \(d_2, d_0 < d_1 < d_2\), so that there are no points of \(\Sigma(E)\) in the interval \([d_1, d_2]\). Let \(f_1 \in C^{\infty}_0(R^1)\) be such that \(\text{supp } f_1 \subset [d_1, d_2]\) and
\[
\int f_1(s)^2 ds = \int f(s)^2 ds.
\]
Define \(F_1\) and \(F \in C^{\infty}_0(R^1)\) so that \(F'_1 = f_2^2\) and \(F' = f^2\), respectively. By construction, \(F_1 \geq F\) and \(F_1 - F \in C^{\infty}_0(R^1)\). We can also take \(f\) and \(f_1\) so that \(\sqrt{F_1 - F} \in C^{\infty}_0(R^1)\).

Next we choose \(\lambda^1\) such that
\[
\lambda^- < \lambda^1 < d_1 \nu^* < \sigma \nu^-.
\]
This is possible because \(d_1\) and \(\nu^*\) can be taken close enough to \(\sigma\) and \(\nu^*\), respectively. We introduce \(h \in C^{\infty}_0(R^1)\) such that \(h\) has support in \((-\infty, \lambda^-)\) and \(h = 1\) on \((-\infty, \lambda^-]\) and define \(\chi(D_0)\) as \(\chi(D_a) = h(\omega \cdot D_a)\).

With the notations above, we now define
\[
(6.1) \quad \Phi = F(\gamma)Q\chi(D_a)^2 + F_1(\gamma)(1 - Q\chi(D_a)^2)
\]
and
\[
(6.2) \quad B_2 = \chi(D_a)f(\gamma)\sqrt{Q}X_{-1/2}E_{\Gamma}(H).
\]
Set \(g_1(H) = g_0(H)(H + c)\) again. Then we assert that there exists \(c > 0\) such that
\[
(6.4) \quad \text{Re } E_{\Gamma}(H)g_1(H)i[\Phi_1(H + c)^{-1}]g_1(H)E_{\Gamma}(H) \geq cB_2^*B_2 + E_{\Gamma}(H)(B^*B + O((x)^{-1-\rho}))E_{\Gamma}(H)
\]
with some \(H\)-smooth operator \(B\) on \(\Gamma\). If the assertion above is proved, then the proposition follows at once.

We calculate the commutator on the left side of (6.4). This is decomposed into the four terms as follows:
\[
i[\Phi_1(H + c)^{-1}] = \sum_{j=1}^4 T_j,
\]
where
\[
T_1 = i[F(\gamma), (H + c)^{-1}]Q\chi(D_a)^2,
T_2 = i[F_1(\gamma), (H + c)^{-1}](1 - Q\chi(D_a)^2),
T_3 = (F(\gamma) - F_1(\gamma))i[Q, (H + c)^{-1}]\chi(D_a)^2,
T_4 = (F(\gamma) - F_1(\gamma))Qi[\chi(D_a)^2, (H + c)^{-1}].
\]
We use Proposition 4.4 to handle the operator \(T_1\). The same argument as in the proof of Theorem 1 applies to obtain that
\[
\text{Re } E_{\Gamma}(H)g_1(H)T_1g_1(H)E_{\Gamma}(H) \geq cB_2^*B_2 + E_{\Gamma}(H)O((x)^{-1-\rho})E_{\Gamma}(H).
\]
The operator \(T_2\) is represented as
\[
T_2 = f_1(\gamma)X_{-1/2}O((x)^0)X_{-1/2}f_1(\gamma) + O((x)^{-1-\rho}).
\]
By assumption, \(f_1\) is supported away from \(\Sigma(E)\) and hence, by Theorem 1, \(X_{-1/2}f_1(\gamma)\) is \(H\)-smooth on \(\Gamma\). By Lemma 3.13, the operator \(T_4\) is of class \(O((x)^{-1-\rho})\). Thus it remains to analyze the operator \(T_3\) only. The next lemma proves (6.4) and hence the proof of the proposition is complete.
LEMMA 6.1. \( \text{Re} T_3 \geq O(\langle x \rangle^{-1-\rho}). \)

6.2. We shall prove Lemma 6.1. For notational brevity, we set \( q = q_- \in S^0(X) \) and \( \varphi = \sqrt{\mathcal{F}_1 - \mathcal{F}} \in C_0^\infty(R^1). \) The function \( q \) has support in a small conical neighborhood of \( \omega \in S_a \) and \( \varphi \) is supported in \((d_1, k_2)\). We further set \( q_\omega = x \cdot \omega / \langle x \rangle \in S^0(X), \) which satisfies \( q_\omega \geq \nu_1^- \) on the support of \( q. \)

By a direct calculation, we have

\[
T_3 = S_1 - S_2 + O(\langle x \rangle^{-2}),
\]

where

\[
S_1 = \varphi(\gamma)^2 (H + c)^{-1} q X^{-1/2} q \varphi(\gamma)^2 X^{-1/2} q (H + c)^{-1} \chi(D_a)^2,
S_2 = \varphi(\gamma)^2 (H + c)^{-1} q X^{-1/2} \omega \cdot D_a X^{-1/2} q (H + c)^{-1} \chi(D_a)^2.
\]

We first study the operator \( S_1. \) By use of the properties of commutator operators in section 3, we have

\[
\text{Re} \, S_1 \sim (H + c)^{-1} q X^{-1/2} q \varphi(\gamma)^2 X^{-1/2} q (H + c)^{-1} \chi(D_a)^2,
\]

where the relation \( \sim \) denotes that the difference of operators on the both sides is of class \( O(\langle x \rangle^{-1-\rho}). \) Set \( \varphi_1(s) = \sqrt{s} \varphi(s) \in C_0^\infty(R^1). \) By lemma 3.10, we further have

\[
\text{Re} \, S_1 \sim X^{-1/2}(H_a + c)^{-1} \chi(D_a) \varphi_1(\gamma) q q \varphi_1(\gamma) \chi(D_a)(H_a + c)^{-1} X^{-1/2}.
\]

Since \( q_\omega q^2 \geq \nu_1^- q^2 \) and \( \varphi_1^2 \geq d_1 \varphi_1^2, \) we obtain that

\[
\text{Re} \, S_1 \geq d_1 \nu_1^- B_3^* B_3 + O(\langle x \rangle^{-1-\rho})
\]

with

\[
B_3 = X^{-1/2}(H_a + c)^{-1} \chi(D_a) \varphi(\gamma) q.
\]

A similar argument applies to the operator \( S_2. \) By Lemma 3.10 again, we have

\[
\text{Re} \, S_2 \sim \varphi(\gamma)^2 (H_a + c)^{-1} q X^{-1/2} \omega \cdot D_a X^{-1/2} q (H_a + c)^{-1} \chi(D_a)^2.
\]

Since \( \omega \cdot p \chi(p)^2 \leq \lambda_1^- \chi(p)^2, \) it follows that

\[
\text{Re} \, S_2 \leq \lambda_1^- B_3^* B_3 + O(\langle x \rangle^{-1-\rho}).
\]

This, together with (6.1), proves the lemma.

7. In this final section we prove Theorem 3. We keep the same notations as in section 6. We again fix \( \omega \in S_a \) for some cluster decomposition \( a. \) Theorem 3 is obtained as an immediate consequence of the following
PROPOSITION 7.1. Let $Q_\pm \in S^0(X)$ be as in section 6. Let $\sigma (\neq 0) \in \Sigma_\omega (E)$. Assume that $f \in C^\infty_0 (R^1)$ is supported in a small neighborhood $\Pi$ around $\sigma$. For $\sigma$ as above, define $h_\pm \in C^\infty_1 (R^1)$ in the same way as in Proposition 6.1. For given $\beta > 0$, define $\chi \in S^0(X'_a)$ with support in $\{ p_a : |p_a - \sigma \omega | > \beta \}$ and define $\chi_\pm \in B^\infty_1 (X'_a)$ as $\chi_\pm (p_a) = h_\pm (\omega \cdot p_a) \chi (p_a)$. Then one can take a conical support of $Q_\pm$, an interval $\Pi$ around $\sigma$ and $\Gamma$ around $E$ so small that the operator

$$X^{-1/2} \chi_\pm (D_a) \sqrt{Q_\pm f (\gamma)}$$

is $H$-smooth on $\Gamma$.

The proposition above can be verified in almost the same way as in the proof of Proposition 6.1, so we give only a sketch for the proof. For brevity, we again assume that $\sigma > 0$ and consider the $-$ case only. We also write $Q$ for $Q_-$.

We start with the following simple lemma.

LEMMA 7.1. For given $\beta > 0$, there exists $\nu_0 > 0$ such that if $\omega \cdot p_a > \sigma - \nu_0$ and $|p_a - \sigma \omega | > \beta$, then $|p_a| > \sigma + \nu_0$.

PROOF. The lemma is easy to prove.

Let $\nu_0$ be as above. Then we may assume that

$$\text{supp } f \subset (d_2, k_2) \subset (\sigma - \nu_0, \sigma + \nu_0).$$

We again take $d_1$ and $d_2$, $\sigma - \nu_0 < d_1 < d_2$, to satisfy that there are no points of $\Sigma (E)$ in the interval $[d_1, d_2]$. We further introduce $f_1 \in C^\infty_0 (R^1)$, $F_1$, $F$ and $h \in C^\infty_1 (R^1)$ in the same way as in the proof of Proposition 6.1. Let $\chi \in S^0(X'_a)$ be as in the proposition and set $\chi_0 (p_a) = h (\omega \cdot p_a) \chi (p_a)$.

With the notations above, we now define $\Phi$ and $B_2$ by (6.2) and (6.3) with $\chi = \chi_0$, respectively, and calculate the commutator $i [\Phi, (H + c)^{-1}]$. Set

$$T_1 = i [F (\gamma), (H + c)^{-1}] Q \chi_0 (D_a)^2.$$ 

Then we assert that

$$\text{Re } E_\Gamma (H) g_1 (H) T_1 g_1 (H) E_\Gamma (H) \geq c B^*_2 B_2 + E_\Gamma (H) O((x)^{-1}) E_\Gamma (H)$$

for some $c > 0$. If this is proved, then the same argument as in the proof of Proposition 6.1 applies and the proposition above is obtained. To prove this assertion, we have to evaluate the value

$$\Theta = \inf \{ \theta^\kappa_a (p_a; E) : \omega \cdot p_a \in \text{supp } f_1, p_a \in \text{supp } \chi \}$$

for $\kappa > 0$ small enough. By Lemma 7.1, we have $|p_a| > \sigma + \nu_0$ for $p_a$ as above and hence it follows that $\Theta \geq (\sigma + \nu_0)^2$. This proves (7.1). Thus the proof of the proposition is complete.

References


