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On the positive and integrable solution of
the delay differential equations

KATSUO TAKANO*

1. Introduction

In the paper [6], the author studied the global properties of the solution of
the differential equation

$$tx'(t) = (\lambda - 1)x(t) - \sum_{i=1}^{n} \lambda_i x(t - p_i)$$

under the conditions that

$$x(t) = 0 \text{ for } t < 0,$$
$$x(p_1) = c \ (> 0),$$
$$0 < p_1 < p_2 < \ldots < p_n < \infty,$$
$$\lambda > 0, \lambda_i > 0,$$

and proved that the solution \( x(t) \) is positive on \((0, \infty)\) and integrable over \((0, \infty)\)
if and only if \( \lambda = \sum_{i=1}^{n} \lambda_i \) holds. In this note, it will be shown that the solution
\( x(t) \) of the differential equation

$$tx'(t) = a(t)x(t) - \sum_{i=1}^{n} b_i(t)x(t - p_i) \tag{1}$$

under the several conditions is positive on \((0, \infty)\) and integrable over \((0, \infty)\). This
note was motivated by the papers [3], [4], [5] of G. Ladas and I. P. Stavroulakis,
K. Sato and M. Yamazato.

2. The existence of the continuous solution

Let us solve the differential equation (1) under the following conditions.

\( (*1) \) \( x(t) = 0 \text{ for } t < 0, \)
\( x(p_1) = c \ (> 0), \)
\( 0 < p_1 < p_2 < \ldots < p_n < \infty, \)
\( (*2) \) \( a(t) \) is continuous on \([0, \infty), \)
\( a(0) > -1, \)
\( a(t) \) is differentiable from the right side at \( t = 0. \)
\( (*3) \) \( b_i(t) \ (i = 1, 2, \ldots, n) \) are continuous on \([p_i, \infty), \)
\( b_i(t) \geq 0 \ (i = 1, 2, \ldots, n). \)

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From (*1), we have
\[ tx'(t) = a(t)x(t) \quad (0 < t < p_1). \]

By (*2),
\[ a(t) = a(0) + a'_+(0)t + t \cdot o(t) \quad (o(t) \to 0 \text{ as } t \to +0) \]
holds for some small interval \([0, \delta]\), where \(a'_+(0)\) denotes the right derivative at \(t = 0\). Set \(g(t) = \int_0^t \frac{a(s)}{s} ds\) for \(0 < t \leq p_1\). Then we obtain \(x(t) = C \exp[g(t)](0 < t \leq p_1)\). In fact, we have
\[
g(t) = -\int_t^\delta \left[ \frac{a(0)}{s} + a'_+(0) + o(s) \right] ds
= -a(0) \log \frac{\delta}{t} - \int_t^\delta (a'_+(0) + o(s)) ds
\]
and obtain
\[
x(t) = C \left( \frac{t}{\delta} \right)^{a(0)} \exp \left[ -\int_t^\delta (a'_+(0) + o(s)) ds \right]
\]
for \(0 < t \leq \delta\). This shows that \(x(t)\) is well-defined on \((0, \delta]\) and also on \([\delta, p_1]\).

By (2), we define \(x(0)\) by \(x(+0)\). Then if \(a(0) > 0\), \(x(t)\) becomes continuous on \((\infty, p_1]\) and \(x'(t)\) is continuous on \((0, p_1]\), and if \(0 \geq a(0) > -1\), \(x(t)\) is continuous on \((\infty, p_1]\) except \(t = 0\) and \(x'(t)\) is continuous on \((0, p_1]\). If \(a(0) > -1\), \(x'(t)\) is continuous on \((\infty, p_1]\) except \(t = 0\). Let
\[
x_0(t) = 0 \quad (t < 0),
= x(t) \quad (0 \leq t \leq p_1).
\]

Let \(\sigma = \min\{p_1, p_2 - p_1, \ldots, p_n - p_{n-1}\}\). To simplify the explanation of the step by step integrals, suppose that \(p_1 < p_1 + \sigma < p_2 < p_1 + 2\sigma\). Then we solve the equation
\[
x'(t) - \frac{a(t)}{t} x(t) = -\frac{b_1(t)}{t} x_0(t - p_1) \quad (p_1 < t < p_1 + \sigma)
\]
and we obtain
\[
x(t) = \left[ -\int_{p_1}^t \frac{b_1(s)}{s} x_0(s - p_1) \exp \left( -\int_{p_1}^s \frac{a(u)}{u} du \right) ds + C_0 \right] \exp \left( \int_{p_1}^t \frac{a(s)}{s} ds \right)
\]
\((p_1 \leq t \leq p_1 + \sigma)\),

because the integral including \(x_0(t)\) of the above solution exists from (2). To make \(x(t)\) continuous on \((0, \infty)\), we let \(x(p_1) = C_0 = x_0(p_1)\). \(x'(t)\) is possibly discontinuous at \(t = p_1\). Next, we define \(x_1(t)\) by
\[
x_1(t) = x_0(t) \quad (t \leq p_1),
= x(t) \quad (p_1 \leq t \leq p_1 + \sigma),
\]
and solve

\[ x'(t) - \frac{a(t)}{t} x(t) = -\frac{b_1(t)}{t} x_1(t - p_1) \quad (p_1 + \sigma < t < p_2), \]

\[ x'(t) - \frac{a(t)}{t} x(t) = -\frac{b_1(t)}{t} x_1(t - p_1) - \frac{b_2(t)}{t} x_1(t - p_2) \quad (p_2 < t < p_2 + \sigma), \]

and we obtain the continuous solution

\[ x(t) = \left[ -\int_{p_1+\sigma}^{t} \frac{1}{s} (b_1(s)x_1(s - p_1) + b_2(s)x_1(s - p_2)) \exp \left( -\int_{p_1+\sigma}^{s} \frac{a(u)}{u} du \right) ds \right. 
\[ + C_1 \exp \left( \int_{p_1+\sigma}^{t} \frac{a(s)}{s} ds \right) \]

\[ (p_1 + \sigma \leq t \leq p_1 + 2\sigma), \]

because the integral including the solution \( x_1(s - p_2) \) exists from (2). To make \( x(t) \) continuous, let \( x(p_1 + \sigma) = C_1 = x_1(p_1 + \sigma) \). \( x'(t) \) is possibly discontinuous at \( t = p_2 \). Repeating this procedure, we obtain the continuous solution

\[ x(t) = \left[ -\int_{p_1+(m-1)\sigma}^{t} \frac{1}{s} \left( \sum_{i=1}^{n} b_i(s)x_{m-1}(s - p_i) \right) \exp \left( -\int_{p_1+(m-1)\sigma}^{s} \frac{a(u)}{u} du \right) ds \right. 
\[ + x_{m-1}(p_1 + (m - 1)\sigma) \exp \left( \int_{p_1+(m-1)\sigma}^{t} \frac{a(s)}{s} ds \right) \]

\[ (p_1 + (m - 1)\sigma \leq t \leq p_1 + m\sigma), \]

of the equation

\[ x'(t) - \frac{a(t)}{t} x(t) = -\frac{1}{t} \sum_{i=1}^{n} b_i(t)x_{m-1}(t - p_i) \quad (p_1 + (m - 1)\sigma < t < p_1 + m\sigma). \]

\( x'(t) \) is possibly discontinuous at \( t = 0, p_1, \ldots, p_n \).

3. The positive and integrable solution

**Theorem.** The solution \( x(t) \) of the equation (1) under the conditions (*1), (*2), (*3) is positive on \((0, \infty)\) and integrable over \((0, \infty)\) if the following conditions are satisfied.

(*4)

\[ \sum_{i=1}^{n} \int_{t-p_i}^{t} b_i(u + p_i)du > 0 \quad (t \geq p_1). \]
For a sufficiently large $T$, there exists a positive number $D$ such that

$\displaystyle a(t) - \sum_{i=1}^{n} b_i(t) \exp \left( -\int_{t-p_i}^{t} \frac{a(u)}{u} \, du \right) < -D \quad (t > T).$

Hence $x(t)$ is strictly decreasing on $(T, \infty)$ and

$\lim_{t \to \infty} x(t) = A$ exists. If $A > 0$, we see that

$\displaystyle x'(t) \leq -D(A - \delta_1) \frac{1}{t} \quad (t \geq t_1 > T, \delta_1 > 0, A - \delta_1 > 0)$

holds for a sufficiently large $t_1$. Hence we have

$\displaystyle x(t) - x(t_1) \leq -D(A - \delta_1) \log \frac{t}{t_1} \to -\infty \quad (t \to \infty).$
This means that $x(t) \to -\infty$ ($t \to \infty$) and this contradicts that $x(t)$ is positive on $(0, \infty)$ and so, $A = 0$. Next, let us show that $x(t)$ is integrable over $(0, \infty)$. By the equality

$$tx(t) = \sum_{i=1}^{n} \int_{t-p_i}^{t} b_i(u+p_i)x(u)du,$$

we obtain

$$0 < x(t) \leq \frac{B}{t} \sum_{i=1}^{n} \int_{t-p_i}^{t} b_i(u+p_i)du,$$

because $0 < x(t) \leq B$ for $t \geq T$ and, substituting this into (3),

$$0 < x(t) \leq \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{B}{t} \int_{t-p_j}^{t} \frac{b_j(u+p_j)}{u} \left( \int_{u-p_i}^{u} b_i(v+p_i)dv \right)du.$$

Hence, by (*7), we see that $x(t)$ is integrable over $(0, \infty)$. q.e.d.

**Corollary..** The solution $x(t)$ of the equation (1) is positive on $(0, \infty)$ and integrable over $(0, \infty)$ under the conditions (*1), (*2), (*3), (*4), (*5) and the following conditions (**6), (**7).

(**6) For a sufficiently large $T$, there exists a positive number $D$ such that

$$\sum_{i=1}^{n} (b_i(t+p_i) - b_i(t)) - 1 < -D \quad (t > T).$$

(**7) $b_i(t)$ are bounded on $[p_i, \infty)$.

**4. A simple equation**

Let us construct the positive and integrable solution which has at least two maximal values. Consider the following equation

$$tx'(t) = a(t)x(t) - b(t)x(t-1) \quad (t > 0)$$

under the conditions that $x(t) = 0$ ($t < 0$), $x(1) = c > 0$, and

$$a(t) + 1 = 2 \quad (0 \leq t \leq m_1)$$

$$= 3(k - 2)(t - m_1) + 2 \quad (k > 2, m_1 \leq t \leq m_1 + \frac{1}{3})$$

$$= k \quad (m_1 + \frac{1}{3} \leq t \leq m_1 + \frac{2}{3})$$

$$= -3(k - 2)(t - m_1 - \frac{2}{3}) + k \quad (m_1 + \frac{2}{3} \leq t \leq m_1 + 1)$$

$$= 2 \quad (m_1 + 1 \leq t),$$

and $a(t) + 1 = b(t + 1)$. By the step by step integrals in the section 2, we obtain the solution.
\[
x(t) = \left[ - \int_{m}^{\frac{1}{s}} b(s) x_{m-1}(s-1) \exp \left( - \int_{m}^{s} \frac{a(u)}{u} du \right) \right] ds \\
+ x_{m-1}(m) \right] \exp \left( \int_{m}^{t} \frac{a(s)}{s} ds \right) \\
(m \leq t \leq m+1),
\]
and the derivative
\[
x'(t) = \left[ - \int_{m}^{\frac{1}{s}} b(s) x_{m-1}(s-1) \exp \left( - \int_{m}^{s} \frac{a(u)}{u} du \right) \right] ds \\
+ x_{m-1}(m) \right] \exp \left( \int_{m}^{t} \frac{a(s)}{s} ds \right) \frac{a(t)}{t} - \frac{b(t)}{t} x_{m-1}(t-1)
\]
(\(m < t < m+1\)).

Since \(x(t)\) is positive on \((0, \infty)\),
\[- \int_{m}^{\frac{1}{s}} b(s) x_{m-1}(s-1) \exp \left( - \int_{m}^{s} \frac{a(u)}{u} du \right) ds + x_{m-1}(m)\]
is positive on \([m, m+1]\) and determined by \(x_{m-1}(s-1)\). \(x(t) = ct\) \((0 < t < 1)\) and if \(m_1\) is sufficiently large,
\[a(t) - b(t) \exp(- \int_{t-1}^{t} \frac{a(u)}{u} du) = 1 - 2 \exp(- \int_{t-1}^{t} \frac{1}{u} du) < 0\]
holds on \((m_1 - 1, m_1)\). Hence \(x'(t) < 0\) on \((m_1 - 1, m_1)\) and \(x(t)\) has at least one maximal value on \((0, m_1)\). Since \(b(t)\) is a delayed function of \(a(t)\), that is, \(a(t) + 1 = b(t + 1)\) and
\[\exp \left( \int_{m_1}^{t} \frac{a(s)}{s} ds \right) \frac{a(t)}{t} \to \infty \quad (k \to \infty)\]
for any \(t\) in \((m_1 + \frac{1}{2}, m_1 + \frac{2}{3})\), by (4), we can make \(x'(t)\) positive on the interval \((m_1 + \frac{1}{2}, m_1 + \frac{2}{3})\) if we take a sufficiently large \(k\). But, for a sufficiently large \(m_2\), \(x(t)\) is strictly decreasing on \((m_2, \infty)\) and hence, \(x(t)\) has at least one maximal value on the interval \((m_1, m_2 + 1)\).

\section*{References}


[7] —, On solution of $xf'(x) = (\nu - 1)f(x) - \sum_{i=1}^{n} \lambda_if(x - p_i) - \sum_{j=1}^{m} \mu_jf(x + q_j)$, Bull. Fac. Sci. Ibaraki Univ., Math., 15 (1983), 29–37.
