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On mixtures of the normal distribution by the generalized gamma convolutions

KATSUO TAKANO*

Summary. We obtain the Lévy representation of the characteristic function of a mixture of the d-dimensional normal distribution by a generalized gamma convolution and the necessary and sufficient condition for the mixture to be self-decomposable.

1. Introduction

If μ is a probability distribution on \( \mathbb{R}^d \), we define the characteristic function \( \varphi \) by

\[
\varphi(z) = \int_{\mathbb{R}^d} e^{izx} \mu(dx), \quad z \in \mathbb{R}^d.
\]

When a distribution function \( F \) satisfies the following (a), (b), (c), the distribution function \( F \) is called a generalized gamma convolution.

(a) The probability distribution \( F \) is concentrated on \([0, \infty)\).
(b) The measure \( U \) is concentrated on \((0, \infty)\) and

\[
\int_0^1 \log w |U(dw)| < \infty, \quad \int_1^\infty \frac{U(dw)}{w} < \infty.
\]

(c) The Laplace transform of \( F \),

\[
\zeta(s) = \int_0^\infty e^{-sv} F(dv),
\]

can be written as

\[
\zeta(s) = \exp \left[ -as - \int_0^\infty \log \left( 1 + \frac{s}{w} \right) U(dw) \right],
\]

\[ a \geq 0, \ Re \ s \geq 0. \]

C. Halgreen [9] proved that any mixture of the normal distribution \( N(\tau, w) \) determined by the relation \( \tau = \mu + \beta w \), where \( \mu \in \mathbb{R} \) and \( \beta \in \mathbb{R} \) are constant parameters and \( w \) follows a generalized gamma convolution, is self-decomposable.

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In this paper, when $N_d(\tau, \Sigma)$ is the $d$-dimensional normal distribution determined by
\[\tau = \mu + w\beta \quad \text{and} \quad \Sigma = wI,\]
where $\mu = (\mu_1, \ldots, \mu_d), \beta = (\beta_1, \ldots, \beta_d)$ are constant vectors and $I$ is the $d \times d$ identity matrix, we obtain the Lévy representation of the characteristic function of a mixture of the $d$-dimensional normal distribution provided that $w$ follows a generalized gamma convolution, and we show that if $d = 2, 3, \ldots$, this mixture is self-decomposable if and only if the parameter $\beta$ is the zero vector. This fact shows a difference between the one-dimensional self-decomposable distribution and the multi-dimensional self-decomposable distribution.

2. The characteristic function of the distribution with the probability density $Ae^{-(2+|\beta|^2)^{1/2}|x| + \beta x}$

Let $d = 2, 3, \ldots$. We shall make use of the polar coordinates
\[
x_1 = r \cos \theta_1, \quad 0 \leq \theta_1 \leq \pi,
\]
\[
x_2 = r \sin \theta_1 \cos \theta_2, \quad 0 \leq \theta_2 \leq \pi,
\]
\[
\ldots
\]
\[
x_{d-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \cos \theta_{d-1}, \quad 0 \leq \theta_{d-2} \leq \pi,
\]
\[
x_d = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \sin \theta_{d-1}, \quad 0 \leq \theta_{d-1} \leq 2\pi,
\]
and
\[
dx = r^{d-1} dr \, d\xi,
\]
\[
d\xi = \sin^{d-2} \theta_1 \sin^{d-3} \theta_2 \cdots \sin \theta_{d-2} d\theta_1 d\theta_2 \cdots d\theta_{d-1}.
\]

Let $S^{d-1}$ denote the $d-1$ dimensional sphere. Let $K_\nu(v)$ denote the modified Bessel function. We shall make use of the formula
\[
K_\nu(v) = \left(\frac{\pi}{2v}\right)^{\frac{1}{2}} \frac{e^{-v}}{\Gamma(\nu + 1/2)} \int_0^\infty e^{-w} w^{\nu-1/2} (1 + \frac{w}{2v})^{-\nu-1/2} dw, \quad (\nu \geq 0) \quad (2.1)
\]
(cf. [20, p. 206]). Set
\[
L_{d/2}(v) = (2\pi)^{-d/2} v^{d/2} K_{d/2}(v),
\]
\[
L(v) = (d + 1) L_{d/2}(v),
\]
and
\[
h(v) = Me^{-v} \left(1 + \frac{1}{2}\right)^{d-1},
\]
where
\[
M = \left[2(2\pi)^{d-1/2} \Gamma\left(d + 1, \frac{1}{2}\right)\right]^{-\frac{1}{2}} \left(\int_0^1 e^{-w} w^{d-1} dw + \int_1^\infty e^{-w} w^{d-1} dw\right).
\]
Then, from (2.1), we have
\[
L_{d/2}(v) \leq h(v), \quad v > 0 \quad (2.2)
\]
and \( h(v) \) is bounded on \((0, \infty)\) and integrable over \((0, \infty)\). Let \( f(x) \) be a density of a mixture of the normal distribution such that

\[
f(x) = \int_0^\infty \frac{1}{(2\pi w)^{d/2}} e^{-\frac{1}{2}w|z-w\beta|^2} \frac{1}{\Gamma\left(\frac{d+1}{2}\right)} w^{\frac{d+1}{2}-1} e^{-w} dw \quad (2.3)
\]

\[
= \frac{1}{(2\pi)^{d/2}\Gamma\left(\frac{d+1}{2}\right)} e^{\beta x} \int_0^\infty w^{\frac{1}{2}-1} e^{-\frac{1}{2}\left(\frac{|z|^2}{w} + (2+|\beta|^2)w\right)} dw. \quad (2.4)
\]

From the formulas

\[
\int_0^\infty w^{\lambda-1} e^{-\frac{1}{2}(\frac{z^2}{w} + \psi w)} dw = 2\left(\frac{\chi}{\psi}\right)^{\lambda/2} K_\lambda(\sqrt{\chi \psi})
\]

and

\[
v^{\frac{1}{2}} K_\frac{1}{2}(v) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-v} \quad (v > 0)
\]

(cf. [20, pp. 80, 186, 187]), we obtain

\[
f(x) = \frac{2}{(2\pi)^{d/2}\Gamma\left(\frac{d+1}{2}\right)} e^{\beta x} \left(\frac{|x|^2}{2 + |\beta|^2}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(\sqrt{2 + |\beta|^2}|x|) \quad (2.5)
\]

\[
= \frac{1}{(2\pi)^{(d-1)/2}\Gamma\left(\frac{d+1}{2}\right)} e^{-\sqrt{2+|\beta|^2}|x|+\beta x}.
\]

Set

\[
A = \frac{1}{(2\pi)^{(d-1)/2}\Gamma\left(\frac{d+1}{2}\right)}.
\]

From (2.3), we obtain the characteristic function of \( f(x) \):

\[
\phi(z) = \int_0^\infty \left(\int_{\mathbb{R}^d} \frac{1}{(2\pi w)^{d/2}} e^{-\frac{1}{2}w|z-w\beta|^2} e^{i\beta x} dx\right) \frac{1}{\Gamma\left(\frac{d+1}{2}\right)} w^{\frac{d+1}{2}-1} e^{-w} dw
\]

\[
= \int_0^\infty \frac{1}{\Gamma\left(\frac{d+1}{2}\right)} w^{\frac{d+1}{2}-1} e^{-w(1+\frac{|z|^2}{2}+i\beta x)} dw
\]

\[
= \frac{1}{(1 + \frac{|z|^2}{2} - i\beta x)^{\frac{d+1}{2}}}.
\]

When \( Z = (\sigma_1 + i\rho_1, ..., \sigma_d + i\rho_d) \) and \( y = (y_1, ..., y_d) \in \mathbb{R}^d \), let us denote the complex number \((\sigma_1 + i\rho_1)y_1 + ... + (\sigma_d + i\rho_d)y_d\) by \( Zy \). Set

\[
\frac{1}{C} = \Gamma(d) \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}.
\]

Then, from [16] and the exponential decrease of \( h(v) \), we obtain a following lemma.
Lemma 1. It holds that

$$\int_{\mathbb{R}^d} C e^{-|y|} e^{iZy} dy = \exp \left[ \int_{\mathbb{R}^d} (e^{iZy} - 1) \frac{L(|y|)}{|y|^d} dy \right], \quad (2.7)$$

for $Z = \sigma + i\rho, |ho| < 1$.

Proof. If $\rho = 0$, (2.7) holds for all $\sigma \in \mathbb{R}^d$ and if $|\rho| < 1$, both sides of (2.7) is holomorphic in $\sigma_j + i\rho_j$. From this fact and by analytic continuation, the equality (2.7) holds for all $Z = (\sigma_1 + i\rho_1, ..., \sigma_d + i\rho_d), |\rho| < 1$. q.e.d.

Lemma 2. We have

$$\phi(z) = \frac{1}{(1 + |z|^2/2 - i\beta z)^{d+1}}$$

$$= \exp \left[ \int_{\mathbb{R}^d} (e^{izx} - 1) \frac{L(\sqrt{2 + |\beta|^2|x|})}{|x|^d} dx \right]$$

for $\beta \in \mathbb{R}^d$.

Proof. Set

$$Z = \frac{z - i\beta}{\sqrt{2 + |\beta|^2}} = \sigma + i\rho.$$

By Lemma 1, we see that

$$\phi(z) = \int_{\mathbb{R}^d} A e^{-\sqrt{2 + |\beta|^2}|x| + \beta x} e^{izx} dx$$

$$= \int_{\mathbb{R}^d} A e^{-|y|} e^{i(z - i\beta)y/\sqrt{2 + |\beta|^2}} dy (2 + |\beta|^2)^{-\frac{d}{2}}$$

$$= \frac{A}{(2 + |\beta|^2)^{\frac{d}{2}} C} \int_{\mathbb{R}^d} e^{-|y|} e^{iZy} dy$$

$$= \frac{A}{(2 + |\beta|^2)^{\frac{d}{2}} C} \exp \left[ \int_{\mathbb{R}^d} (e^{iZy} - 1) \frac{L(|y|)}{|y|^d} dy \right].$$

(2.9)

For the integral of the last member of (2.9), we see that

$$\int_{\mathbb{R}^d} \left( e^{izy/\sqrt{2 + |\beta|^2}} e^{\beta y/\sqrt{2 + |\beta|^2}} - 1 \right) \frac{L(|y|)}{|y|^d} dy$$

$$= \int_{\mathbb{R}^d} \left( e^{izy/\sqrt{2 + |\beta|^2}} - 1 \right) e^{\beta y/\sqrt{2 + |\beta|^2}} \frac{L(|y|)}{|y|^d} dy$$

$$+ \int_{\mathbb{R}^d} \left( e^{\beta y/\sqrt{2 + |\beta|^2}} - 1 \right) \frac{L(|y|)}{|y|^d} dy.$$

From this, we have

$$\phi(z) = \frac{A}{(2 + |\beta|^2)^{\frac{d}{2}} C} \exp \left[ \int_{\mathbb{R}^d} \left( e^{\beta y/\sqrt{2 + |\beta|^2}} - 1 \right) \frac{L(|y|)}{|y|^d} dy \right]$$

$$\exp \left[ \int_{\mathbb{R}^d} \left( e^{izy/\sqrt{2 + |\beta|^2}} - 1 \right) e^{\beta y/\sqrt{2 + |\beta|^2}} \frac{L(|y|)}{|y|^d} dy \right].$$

(2.10)
Let us show
\[
\frac{A}{(2 + |\beta|^2)^{d/2} C} \exp \left[ \int_{\mathbb{R}^d} \left( e^{\beta y/\sqrt{2 + |\beta|^2} - 1} \frac{L(|y|)}{|y|^d} \right) dy \right] = 1.
\]

In fact, for \( Z = \sigma \in \mathbb{R}^d \), we have
\[
\frac{1}{(1 + Z^2)^{d+1/2}} = \int_{\mathbb{R}^d} C e^{-|y|} e^{iZy} dy.
\]

When \( Z = \sigma + i\rho, |\rho| < 1 \), both sides of (2.11) are holomorphic in \( \sigma_j + i\rho_j \ (j = 1, \ldots, d) \) and by analytic continuation, the equality (2.11) holds. By Lemma 1 and taking \( Z = -i\frac{\beta}{\sqrt{2 + |\beta|^2}} \), we obtain
\[
(1 - \frac{|\beta|^2}{2 + |\beta|^2})^{-d+1} = \exp \left[ \int_{\mathbb{R}^d} \left( e^{\beta y/\sqrt{2 + |\beta|^2} - 1} \frac{L(|y|)}{|y|^d} \right) dy \right].
\]

From this, let us show that
\[
\frac{A}{(2 + |\beta|^2)^{d/2} C} \left( \frac{2 + |\beta|^2}{2} \right)^{d+1} = (d - 1)! \frac{2\pi^{1/2}}{\Gamma(\frac{d}{2})2^{(d-1)/2} \Gamma(\frac{d+1}{2})2^{(d+1)/2}}
\]

equals 1. For the case \( d = 2 \), we have
\[
(2.13) = 1! \frac{2\pi^{1/2}}{\Gamma(1)^2 \Gamma(\frac{3}{2})2^{3/2}} = \frac{2\pi^{1/2}}{(\frac{3}{2} - 1)\Gamma(\frac{1}{2})2^2} = 1.
\]

For the case \( d = 3 \), we have
\[
(2.13) = 2! \frac{2\pi^{1/2}}{\Gamma(\frac{3}{2})2^{1}\Gamma(2)^2} = \frac{\pi^{1/2}}{(\frac{3}{2} - 1)\Gamma(\frac{1}{2})2} = 1.
\]

Suppose that (2.13) equals 1 for \( d = 2j + 1, \ j \geq 1 \). Then, for \( d = 2j + 3 \), we see that
\[
(2.13) = (2j + 2)! \frac{2\pi^{1/2}}{\Gamma(\frac{2j+3}{2})2^{(2j+2)/2} \Gamma(\frac{2j+4}{2})2^{(2j+4)/2}}
\]
\[
= (2j + 2)(2j + 1)(2j)! \frac{2\pi^{1/2}}{(\frac{2j+3}{2} - 1)\Gamma(\frac{2j+1}{2})2^{2j+1}(j+1)\Gamma(j+1)2^{j+2}}
\]
\[
= (2j + 2)(2j + 1)2^{2j} \frac{2\pi^{1/2}}{(2j+1)(j+1)2^{2j+2}}
\]
\[
= 1.
\]

In the same way, for \( d = 2j + 2, (j = 1, 2, \ldots) \), we can show (2.13) = 1. Hence, from the equality (2.10) and by changes of the variables, \( \sqrt{2 + |\beta|^2} x = y \), we obtain (2.8). q.e.d.
3. The Lévy representation of the characteristic function of the mixture distribution

Let \( F(w) \) be a distribution function of a generalized gamma convolution and we consider a mixture of the normal distribution by the generalized gamma convolution \( F(w) \) such as

\[
f(x) = \int_0^\infty \frac{1}{(2\pi w)^{\frac{d}{2}}} e^{-\frac{1}{2w}|x-w\beta|^2} F(dw). \tag{3.1}
\]

**Lemma 3.** Under the condition \((b)\), the integral

\[
\int_{\mathbb{R}^d} \frac{e^{\beta y}}{(1+|y|^2)|y|^{d-2}} \left( \int_0^\infty L_{d/2}(\sqrt{2w+|\beta|^2}|y|)U(dw) \right) dy \tag{3.2}
\]

converges for all \( \beta \in \mathbb{R}^d \).

**Proof.** Let us prove the case \( d \geq 2 \). By the orthogonal transformation and by making use of the polar coordinate, we have

\[
\int_{\mathbb{S}^{d-1}} d\xi \int_0^\infty \frac{r}{1+r^2} e^{\beta|\xi|\cos \theta_1} \left( \int_0^\infty L_{d/2}(\sqrt{2w+|\beta|^2r})U(dw) \right) dr
\]

Hence, it suffices to prove that

\[
\int_0^\frac{\pi}{2} \sin^{d-2} \theta d\theta \int_0^\infty \frac{r}{1+r^2} e^{\lambda r \cos \theta} \left( \int_0^\infty L_{d/2}(\sqrt{2w+\lambda^2 r})U(dw) \right) dr
\]

converges for all \( \lambda \geq 0 \). By \((2.2)\) and by the Fubini theorem, we shall prove that

\[
\int_0^\infty U(dw) \int_0^\frac{\pi}{2} \sin^{d-2} \theta d\theta \int_0^\infty \frac{r}{1+r^2} e^{\lambda r \cos \theta} h(\sqrt{2w+\lambda^2 r}) dr \tag{3.3}
\]

converges for all \( \lambda \geq 0 \). We divide the integral \((3.3)\) into the three integrals as follows:

\[
(3.3) = \int_0^1 U(dw) \int_0^\frac{\pi}{2} \sin^{d-2} \theta d\theta \int_0^1 \frac{r}{1+r^2} e^{\lambda r \cos \theta} h(\sqrt{2w+\lambda^2 r}) dr
\]

\[
+ \int_1^\infty U(dw) \int_0^\frac{\pi}{2} \sin^{d-2} \theta d\theta \int_0^\infty \frac{r}{1+r^2} e^{\lambda r \cos \theta} h(\sqrt{2w+\lambda^2 r}) dr \tag{3.4}
\]

\[
+ \int_0^1 U(dw) \int_0^\frac{\pi}{2} \sin^{d-2} \theta d\theta \int_1^\infty \frac{r}{1+r^2} e^{\lambda r \cos \theta} h(\sqrt{2w+\lambda^2 r}) dr.
\]

By the boundedness of \( h(u) \) and by \((b)\), the first integral of \((3.4)\) converges. With the second integral of \((3.4)\), it suffices to prove that

\[
\int_1^\infty U(dw) \int_0^\infty \frac{r}{1+r^2} e^{-(\sqrt{2w+\lambda^2 r}-\lambda)r}\left(\sqrt{2w+\lambda^2 r} + \frac{1}{2}\right)^{\frac{d-1}{2}} dr \tag{3.5}
\]
converges for all $\lambda \geq 0$. By a change of the variable, $(\sqrt{2w + \lambda^2} - \lambda)r = u$, and by (b), we see that

\begin{equation}
\int_1^\infty U(dw) \int_0^\infty \frac{u}{(\sqrt{2w + \lambda^2} - \lambda)^2 + u^2} e^{-u} \left( \frac{\sqrt{2w + \lambda^2}}{\sqrt{2w + \lambda^2} - \lambda} u + \frac{1}{2} \right)^{\frac{d-1}{2}} du 
\end{equation}

\begin{equation}
\leq \int_1^\infty \frac{U(dw)}{(\sqrt{2w + \lambda^2} - \lambda)^2} \int_0^\infty u(M(\lambda)u + \frac{1}{2})^{\frac{d-1}{2}} e^{-u} du < \infty,
\end{equation}

where

$$M(\lambda) = \sup\{ \frac{\sqrt{2w + \lambda^2}}{\sqrt{2w + \lambda^2} - \lambda} : w \geq 1\}.$$ 

Let us prove that the third integral of (3.4),

\begin{equation}
\int_0^1 U(dw) \int_0^{\frac{\pi}{2}} \sin^{d-2} \theta d\theta
\end{equation}

\begin{equation}
\int_1^\infty \frac{r}{1 + r^2} e^{-(\sqrt{2w + \lambda^2} - \lambda \cos \theta)r} (\sqrt{2w + \lambda^2} r + \frac{1}{2})^{\frac{d-1}{2}} dr,
\end{equation}

converges for all $\lambda \geq 0$. Suppose $\lambda = 0$. Then, by (b), we see that

\begin{equation}
\int_0^1 U(dw) \int_1^\infty \frac{r}{1 + r^2} e^{-\sqrt{2wr}r} (\sqrt{2wr}r + \frac{1}{2})^{\frac{d-1}{2}} dr 
\end{equation}

\begin{equation}
= \int_0^1 U(dw) \int_1^\infty \frac{u}{2w + u^2} e^{-u}(u + \frac{1}{2})^{\frac{d-1}{2}} du 
\end{equation}

\begin{equation}
\leq \int_0^1 \left( \sup\{e^{-u}(u + \frac{1}{2})^{\frac{d-1}{2}} : 0 < u \leq 2\} \int_2^\infty \frac{du}{u} 
+ \int_2^\infty \frac{1}{u} e^{-u}(u + \frac{1}{2})^{\frac{d-1}{2}} du \right) U(dw) < \infty.
\end{equation}

From this fact and by the Fubini theorem, the integral (3.6) converges for $\lambda = 0$. Suppose $\lambda > 0$. We divide the integral (3.6) into the two integrals as follows:

\begin{equation}
(3.6) = \int_0^1 U(dw) \int_0^{\frac{\pi}{2}} \sin^{d-2} \theta d\theta
\end{equation}

\begin{equation}
\int_1^\infty \frac{r}{1 + r^2} e^{-(\sqrt{2w + \lambda^2} - \lambda \cos \theta)r} (\sqrt{2w + \lambda^2} r + \frac{1}{2})^{\frac{d-1}{2}} dr 
\end{equation}

+ \int_0^1 U(dw) \int_{\frac{\pi}{2}}^{\delta} \sin^{d-2} \theta d\theta
\end{equation}

\begin{equation}
\int_1^\infty \frac{r}{1 + r^2} e^{-(\sqrt{2w + \lambda^2} - \lambda \cos \theta)r} (\sqrt{2w + \lambda^2} r + \frac{1}{2})^{\frac{d-1}{2}} dr, \quad 0 < \delta < 1.
\end{equation}
When $0 < w < 1, \delta \leq \theta \leq \frac{\pi}{2}$, we have
\[
\sqrt{2w + \lambda^2} - \lambda \cos \theta \geq \lambda - \lambda \cos \delta > 0
\]
and hence, the second integral of (3.7) converges. With the first integral of (3.7), we see that

\[(\text{the first integral of (3.7)})
\]
\[
= \int_0^1 U(dw) \int_0^\theta \sin^{d-2} \theta d\theta \int_{\sqrt{2w + \lambda^2} - \lambda \cos \theta}^\infty \frac{u}{(\sqrt{2w + \lambda^2} - \lambda \cos \theta)^{\frac{d-1}{2}}} e^{-u} \left(\frac{\sqrt{2w + \lambda^2}}{\sqrt{2w + \lambda^2} - \lambda \cos \theta} \right)^{\frac{d-1}{2}} du
\]
\[
= \int_0^1 U(dw) \int_0^\theta \sin^{d-2} \theta \int_{\sqrt{2w + \lambda^2} - \lambda \cos \theta}^\infty \frac{u^{\frac{d-1}{2}}}{(\sqrt{2w + \lambda^2} - \lambda \cos \theta)^{\frac{d-1}{2}}} e^{-u} \left(\frac{2w + \lambda^2}{2u} - \lambda \cos \theta\right)^{\frac{d-1}{2}} du
\]
\[
\leq \int_0^1 U(dw) \int_0^\theta \sin^{d-2} \theta \int_{\sqrt{2w + \lambda^2} - \lambda \cos \theta}^\infty u^{\frac{d-1}{2}} e^{-u} \left(2 + \frac{\lambda^2}{2} + \frac{1}{2}\right)^{\frac{d-1}{2}} du
\]
\[
\leq (2^{\frac{d-1}{2}} - 1)^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right) \int_0^1 U(dw) \int_0^\theta \sin^{d-2} \theta \int_{\sqrt{2w + \lambda^2} - \lambda \cos \theta}^\infty u^{\frac{d-1}{2}} e^{-u} du
\]

With the integral of (3.8), we see that
\[
\int_0^\theta \sin^{d-2} \theta \int_{\sqrt{2w + \lambda^2} - \lambda \cos \theta}^\infty u^{\frac{d-1}{2}} e^{-u} du
\]
\[
\leq \int_0^\theta \sin^{d-2} \theta \int_{\lambda(1 - \cos \theta)}^{2^{\frac{d-1}{2}} \lambda(1 - \cos \theta)} \left(\sqrt{2w + \lambda^2} - \lambda \cos \theta\right)^{\frac{d-1}{2}} d\theta
\]
\[
= \int_0^\theta \left(\frac{2 \cos \theta}{\theta} \sin \theta \right)^{d-2} \left(\sqrt{2w + \lambda^2} - \lambda (1 - \frac{\theta^2}{2})\right)^{\frac{d-1}{2}} \left(\sqrt{2w + \lambda^2} - \lambda \cos \theta\right)^{\frac{d-1}{2}} d\theta
\]
\[
\leq (2\lambda)^{\frac{d-1}{2}} \sup_{\theta < \delta} \left[\left(\sqrt{2w + \lambda^2} - \lambda (1 - \frac{\theta^2}{2})\right)^{\frac{1}{2}} \left(\sqrt{2w + \lambda^2} - \lambda \cos \theta\right)^{\frac{1}{2}} : 0 < \theta < \delta\right] \int_0^\theta \left(\frac{1}{2} \theta^2 + \sqrt{2w + \lambda^2} - \lambda\right)^{\frac{1}{2}} d\theta
\]

For $0 < w < 1, 0 < \theta < \delta < 1$, we obtain
\[
\frac{\sqrt{2w + \lambda^2} - \lambda \cos \theta}{\sqrt{2w + \lambda^2} - \lambda (1 - \frac{\theta^2}{2})} = 1 + \frac{\lambda(1 - \frac{\theta^2}{2} - \cos \theta)}{\sqrt{2w + \lambda^2} - \lambda + \frac{\theta^2}{2}}
\]
\[
\geq 1 - 2\frac{\cos \theta - 1 + \frac{\theta^2}{2}}{\theta^2} \geq 1 - \frac{\delta^2}{12} > 0.
\]
For $0 < w < 1$, we have
\[
\int_{0}^{\delta} \frac{d\theta}{(\lambda \frac{\theta^2}{2} + \sqrt{2w + \lambda^2} - \lambda)^{\frac{1}{2}}}
= \left(\frac{2}{\lambda}\right)^{\frac{1}{2}} \left\{ \log \left( \delta + \left(\delta^2 + \frac{2}{\lambda} (\sqrt{2w + \lambda^2} - \lambda)\right)^{\frac{1}{2}} \right) - \log \left(\frac{2}{\lambda} (\sqrt{2w + \lambda^2} - \lambda)\right)^{\frac{1}{2}} \right\}
\]
and by (b),
\[
\int_{0}^{1} |\log(\sqrt{2w + \lambda^2} - \lambda)|U(dw) < \infty,
\]
and hence, the integral of (3.8) converges for all $\lambda > 0$. Combining the above facts, we obtain that the integral of (3.3) converges for all $\lambda \geq 0$. In the case $d = 1$, we have to prove that
\[
\int_{0}^{\infty} \frac{r}{1 + r^2} e^{\lambda r} \left( \int_{0}^{\infty} e^{-\sqrt{2w + \lambda^2} r} U(dw) \right) dr
\]
converges for all $\lambda \geq 0$. It suffices to prove that
\[
\int_{0}^{1} U(dw) \int_{1}^{\infty} \frac{r}{1 + r^2} e^{-\left(\sqrt{2w + \lambda^2} - \lambda\right) r} dr
\]
converges for all $\lambda \geq 0$. By a change of the variable, $\left(\sqrt{2w + \lambda^2} - \lambda\right) r = u$, and by (b), we see that
\[
\int_{0}^{1} U(dw) \int_{\sqrt{2w + \lambda^2} - \lambda}^{\infty} \frac{u}{\sqrt{2w + \lambda^2} - \lambda} e^{-u} du
\]
\[
\leq \int_{0}^{1} U(dw) \int_{\sqrt{2w + \lambda^2} - \lambda}^{\infty} \frac{1}{u} e^{-u} du
\]
\[
\leq \int_{0}^{1} U(dw) \int_{\sqrt{2w + \lambda^2} - \lambda}^{\infty} \frac{1}{u} du + \int_{0}^{1} U(dw) \int_{\sqrt{2w + \lambda^2} - \lambda}^{\infty} \frac{1}{u} e^{-u} du < \infty.
\]
q.e.d.

We obtain the following theorem which extends Halgeen's result of the one-dimensional case to the multi-dimensional case (cf. [9]).

**Theorem.** The mixture (3.1) of the normal distribution by a generalized gamma convolution has the following Lévy's representation:

\[
\phi(z) = \exp \left[ i(a\beta + \gamma)z - \frac{a|z|^2}{2} \right] + \int_{\mathbb{R}^d} \left( e^{izy} - 1 - \frac{izy}{1 + |y|^2} \right) \frac{2e^{\beta y}}{|y|^d} \left( \int_{0}^{\infty} L_{d/2}(\sqrt{2w + |\beta|^2 |y|}) U(dw) \right) dy.
\]
The characteristic function of the mixture is as follows:

\[ \phi(z) = \exp \left[ -a \frac{|z|^2}{2} - i\beta z - \int_0^\infty \log \left( 1 + \frac{1}{w} \left( \frac{|z|^2}{2} - i\beta z \right) \right) U(dw) \right]. \]  

(3.11)

From Lemma 2 and by changes of variables, we see that

\[
- \log \left( 1 + \frac{|z|^2}{w} - i\beta z \right)
= \int_{\mathbb{R}^d} \left( e^{izx}/\sqrt{w} - 1 \right) 2e^{i\beta x}/\sqrt{w} \frac{L_{d/2}(\sqrt{2+w}|x|)}{|x|^d} dx
= \int_{\mathbb{R}^d} \left( e^{izw} - 1 \right) 2e^{i\beta w} \frac{L_{d/2}(\sqrt{2+w}|y|)}{|y|^d} dy.
\]

(3.12)

The integrand of (3.12) is absolutely integrable over \( \mathbb{R}^d \) for each \( w \in (0, \infty) \). If \( \beta = 0 \), we have

\[
\int_{\mathbb{R}^d} \frac{|y|}{1 + |y|^2} \frac{L_{d/2}(\sqrt{2+w}|y|)}{|y|^d} dy = \int_{S^{d-1}} d\xi \int_0^\infty \frac{1}{1 + r^2} \frac{L_{d/2}(\sqrt{2+wr})}{L_{d/2}(\sqrt{2w})} dr
\leq \int_{S^{d-1}} d\xi \int_0^\infty \frac{1}{1 + r^2} h(\sqrt{2wr}) dr < \infty
\]

(3.13)

and

\[
\int_{\mathbb{R}^d} \frac{y_k}{1 + |y|^2} \frac{L_{d/2}(\sqrt{2+w}|y|)}{|y|^d} dy = 0.
\]

Hence, from (3.12), we have

\[
- \log \left( 1 + \frac{|z|^2}{2w} \right) = \int_{\mathbb{R}^d} \left( e^{izw} - 1 - \frac{izw}{1 + |y|^2} \right) 2L_{d/2}(\sqrt{2+w}|y|) \frac{dy}{|y|^d}.
\]

Substituting this into (3.11), and then, by Lemma 3 and by the Fubini theorem, changing the order of the integrations, we obtain

\[
\log \phi(z) = -a \frac{|z|^2}{2} + \int_{\mathbb{R}^d} \left( e^{izw} - 1 - \frac{izw}{1 + |y|^2} \right) \frac{2L_{d/2}(\sqrt{2+w}|y|)}{|y|^d} \left( \int_0^\infty L_{d/2}(\sqrt{2+w}|y|) U(dw) \right) dy.
\]
If $\beta \neq 0$, it suffices to prove that

\begin{align}
\int_{\mathbb{R}^d} |e^{izy} - 1 - \frac{izy}{1 + |y|^2}| \frac{e^{\beta y}}{|y|^d} L_{d/2}(\sqrt{2w + |\beta|^2|y|})\,dy < \infty, \quad (3.14) \\
\int_{\mathbb{R}^d} \frac{|y|}{1 + |y|^2} L_{d/2}(\sqrt{2w + |\beta|^2|y|})\,dy < \infty, \quad (3.15) \\
\int_{\mathbb{R}^d} \frac{|y_k(1 - e^{-\beta y})|}{(1 + |y|^2)|y|^d} e^{\beta y} L_{d/2}(\sqrt{2w + |\beta|^2|y|})\,dy < \infty, \quad (3.16)
\end{align}

for each $w \in (0, \infty)$ and

\begin{equation}
\int_0^\infty U(dw) \int_{\mathbb{R}^d} \frac{|y|}{1 + |y|^2} e^{\beta y} L_{d/2}(\sqrt{2w + |\beta|^2|y|})\,dy < \infty. \quad (3.17)
\end{equation}

Then, from (3.14), (3.15), (3.16) and by the fact that

\begin{equation}
\int_{\mathbb{R}^d} \frac{y_k}{1 + |y|^2} \frac{L_{d/2}(\sqrt{2w + |\beta|^2|y|})}{|y|^d} dy = 0,
\end{equation}

we obtain

\begin{align*}
&- \log(1 + \frac{1}{w}(\frac{|z|^2}{2} - i\beta z)) \\
&= \int_{\mathbb{R}^d} (e^{izy} - 1 - \frac{izy}{1 + |y|^2}) \frac{2e^{\beta y}}{|y|^d} L_{d/2}(\sqrt{2w + |\beta|^2|y|})\,dy \\
&+ i \int_{\mathbb{R}^d} \frac{zy(1 - e^{-\beta y})}{(1 + |y|^2)|y|^d} 2e^{\beta y} L_{d/2}(\sqrt{2w + |\beta|^2|y|})\,dy.
\end{align*}

Substituting this into (3.11), and then, by Lemma 3 and (3.17) and by using the Fubini theorem, we obtain (3.9). Let us show (3.14), ..., (3.17). By making use of the inequality, $|e^{izy} - 1 - izy| \leq (|z||y|)^2/2$, $|y| < 1$ and from the proof of Lemma 3, we see that (3.14) converges for each $w \in (0, \infty)$. In the same way as the calculation of (3.13), we obtain (3.15). Let us show (3.17). By the orthogonal transformation and by making use of the polar coordinates, it suffices to prove that

\begin{equation}
\int_0^\infty U(dw) \int_0^\pi \sin^{d-2} \theta d\theta \int_0^\infty \frac{1 - e^{-r|\beta|\cos \theta}e^{r|\beta|\cos \theta}}{1 + r^2} \frac{L_{d/2}(\sqrt{2w + |\beta|^2r})}{dr} L_{d/2}(\sqrt{2w + |\beta|^2r})dr
\end{equation}

converges. If we set $|\beta| = \lambda$, we have

\begin{align}
&\int_0^\infty U(dw) \int_0^{\frac{\pi}{2}} \sin^{d-2} \theta d\theta \int_0^\infty \frac{1 - e^{-r\lambda \cos \theta}e^{r\lambda \cos \theta}}{1 + r^2} \frac{L_{d/2}(\sqrt{2w + \lambda^2r})}{dr} L_{d/2}(\sqrt{2w + \lambda^2r})dr \\
&+ \int_0^\infty U(dw) \int_0^{\frac{\pi}{2}} \sin^{d-2} \theta d\theta \int_0^\infty \frac{(e^{r\lambda \cos \theta} - 1)e^{-r\lambda \cos \theta}}{1 + r^2} \frac{L_{d/2}(\sqrt{2w + \lambda^2r})}{dr} L_{d/2}(\sqrt{2w + \lambda^2r})dr.
\end{align}
By the fact that \((1 - e^{-r\lambda\cos \theta})/r\) is bounded on \((0, \infty) \times (0, \pi)\) and by Lemma 3, we see that the two integrals of (3.19) are converge. In order to prove (3.16), it suffices to prove that the integrals

\[
\int_0^{\pi/2} \sin^{d-2} \theta d\theta \int_0^\infty \frac{1}{1 + r^2} e^{\lambda \cos \theta} L_{d/2}(\sqrt{2w + \lambda^2 r}) dr, \\
\int_0^{\pi/2} \sin^{d-2} \theta d\theta \int_0^\infty \frac{1}{1 + r^2} L_{d/2}(\sqrt{2w + \lambda^2 r}) dr
\]

(3.20) (3.21)

converge for each \(w \in (0, \infty)\). By making use of the function \(h(v)\), we see that (3.20) and (3.21) converge for each \(w \in (0, \infty)\). q.e.d.

**COROLLARY.** When \(d = 1\), the mixture (3.1) of the normal distribution by a generalized gamma convolution is self-decomposable for all \(\beta \in \mathbb{R}\). When \(d = 2, 3, \ldots\), the mixture (3.1) of the normal distribution by a generalized gamma convolution is infinitely divisible for all \(\beta \in \mathbb{R}^d\) and is self-decomposable if and only if \(\beta = 0\).

Proof. Let \(d = 2, \ldots\). By using the polar coordinates, we have the following representation:

\[
\log \phi(z) = i(a\beta + \gamma)z - \frac{d|z|^2}{2} + \int_{\mathbb{S}^{d-1}} d\xi \int_0^\infty \left(\frac{e^{iux\xi} - 1}{1 + u^2}\right) du.
\]

(3.22)

If \(\beta\) is not zero, we can take a vector \(\xi_0 \in \mathbb{S}^{d-1}\) such as \(\beta\xi_0 > 0\). If we set

\[
k_\xi(u) = 2e^{u\beta\xi} \int_0^\infty L_{d/2}(\sqrt{2w + |\beta|^2 u})U(dw),
\]

by Lemma 3, \(uk_\xi(u)/(1 + u^2)\) is integrable over \((0, \infty)\) for almost all \(\xi\), and we have

\[
\int_0^\infty (k_\xi(u) - k_{-\xi}(u)) \frac{u}{1 + u^2} du > 0
\]

on a neighborhood \(N(\xi_0)\) of \(\xi_0\), which satisfies \(\int_{N(\xi_0)} d\xi > 0\). If \(\beta = 0\), the function \(k_\xi(u)\) is monotonously decreasing in \(u\) for each \(\xi \in \mathbb{S}^{d-1}\). In fact, from (2.1), \(v^{d/2}K_{(d-2)/2}(v)\) is positive on \((0, \infty)\), and by the relation

\[
vK_{d/2}(v) + \frac{d}{2} K_{d/2}(v) = -vK_{(d-2)/2}(v),
\]

we have

\[
L_{d/2}(u) = (2\pi)^{-d/2} \int_0^\infty v^{d/2}K_{(d-2)/2}(v) dv.
\]

\[
\int_0^\infty k_\xi(u) \frac{u}{1 + u^2} du = c > 0
\]

holds for all \(\xi \in \mathbb{S}^{d-1}\) and hence, by Sato's representation of the characteristic function for the self-decomposable distribution (cf. [14, Th. 3.1]), we see that the mixture of the normal distribution \(N(\tau, \Sigma)\) by a generalized gamma convolution is self-decomposable if and only if \(\beta = 0\).
References

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