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Author(s)
MATSUDA, Ryuki

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Notes on Anderson-Anderson-Johnson questions and Bouvier questions

RYŪKI MATSUDA*

§0. Introduction.

In this paper A denotes an integral domain. A domain A is called a locally factorial ring if \( A_M \) is a factorial ring for each maximal ideal \( M \) of A. If each ideal of A is a product of primary ideals of A, then A is called a Q-ring.

Let \( X^{(1)}(A) \) be the set of height one prime ideals of A, and \( \text{Spec}(A) \) the set of prime ideals of A.

[6, p.17] asks the following question:

(B-0) Let A be a locally factorial ring with \( \text{Pic}(A) = 0 \). Is A a factorial ring?

Let A be a Krull ring. [3, §2] asks the following questions:

Are the following statements true?

(AAJ-1) Let \( P \) be a height one prime ideal of A. If \( P \cap Q = PQ \) for each \( Q \in X^{(1)}(A) \) with \( Q \neq P \), then \( P \) is an invertible ideal.

(AAJ-2) Let \( P \) be a non-zero prime ideal of A. If \( P \cap Q = PQ \) for each \( Q \in \text{Spec}(A) \) with \( \text{ht} \ Q \leq 2 \) that is incomparable with \( P \), then \( P \) is a maximal ideal or an invertible ideal of A.

(AAJ-3) A is a locally factorial ring if and only if \( P \cap Q = PQ \) for each pair of distinct height one prime ideals \( P, Q \) of A.

(AAJ-4) A is a factorial ring if and only if \( \text{Pic}(A) = 0 \) and \( P \cap Q = PQ \) for each pair of distinct height one prime ideals \( P, Q \) of A.

(AAJ-5) A is a locally factorial ring if and only if \( PQ \) is a divisorial ideal for each height one prime ideals \( P, Q \) of A.

(AAJ-6) The following conditions are equivalent:

1. A is a locally factorial ring with \( \dim A \leq 2 \).
2. \( P \cap Q = PQ \) for each incomparable prime ideals \( P, Q \) of A.
3. A is a Q-ring.

Let A be a Krull ring. Relating with the above questions, [6,(3.9)] asks the following questions:

(B-1) Are the following conditions equivalent?

1. A is a locally factorial ring.
2. \( P \cap Q = PQ \) for each distinct height one prime ideals \( P, Q \) of A.
3. \( PQ \) is a divisorial ideal for each height one prime ideals \( P, Q \) of A.

(B-2) If \( \text{Pic}(A) = 0 \) and \( P \cap Q = PQ \) for each distinct height one prime ideals \( P, Q \) of A, then is A a factorial ring?

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* Department of Mathematics, Ibaraki University, Mito, Ibaraki 310, Japan.
Let $A$ be a locally factorial domain. Then are the following conditions equivalent?

1. $\dim A \leq 2$.
2. $P \cap Q = PQ$ for each incomparable prime ideals $P, Q$ of $A$.
3. $A$ is a Q-ring.

§1. On (B-0).

(1.1). Let $Z$ be the ring of integers, $\{p_i; i = 1, 2, \cdots\}$ the set of positive prime numbers, and $X, Y$ indeterminates over $Z$. Set $A = \bigcup_{i=1}^{\infty} Z[X/p_1, Y/p_1, X/p_2, Y/p_2, \cdots, X/p_n, Y/p_n]$ (cf. [5, Example]). Then we have the following.

(a) $A$ is not a Krull domain. Thus $A$ is not a factorial ring.
(b) $A$ is a locally factorial ring with $\text{Pic}(A) = 0$.

PROOF. (a): [5, Example,(a)] shows that $A$ is not a Krull domain.
(b): Let $Q$ be the field of rational numbers, and $M$ be a maximal ideal of $A$. Then $A_M$ is a localization of $Q[X, Y]$ or $A_M$ is a localization of the polynomial ring $Z_{(p)}[X/p, Y/p]$ for a prime number $p$. Hence $A_M$ is a factorial ring. Thus $A$ is a locally factorial ring. Set $A_n = Z[X/(p_1 \cdots p_n), Y/(p_1 \cdots p_n)]$ for each natural number $n$. We have $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, and $A = \bigcup_{i=1}^{\infty} A_n$. Thus $\text{Pic}(A) = \lim \{\text{Pic}(A_n)\}$ by [8, Theorem 1.3]. Since each $A_n$ is a factorial ring, we have $\text{Pic}(A) = 0$.

Thus Question (B-0) was solved negatively.

§2. Lemmas.

(2.1) ([2, Corollary 2.2]). Let $A$ be a Krull ring, $I$ a nonzero ideal of $A$. Then $I$ is an invertible ideal of $A$ if and only if $I$ is a locally principal ideal of $A$.

(2.2) (A PART OF [4]). (1) If $A$ is a Q-ring, then any quotient ring is a Q-ring.
(2) Let $(A, M)$ be a quasi-local ring. Then $A$ is a Q-ring if and only if $A$ is either $\dim A \leq 1$ or $A$ is a 2-dimensional factorial ring.
(3) $A$ is a Q-ring if and only if $A$ is Laskerian and every nonmaximal prime ideal is finitely generated and locally principal.

(2.3) (A PART OF [1, Theorem 1]). Let $A$ be a Krull ring. Then the followings are equivalent:
(1) $A$ is a locally factorial ring.
(2) Each height one prime ideal of $A$ is an invertible ideal.
(3) The product of two divisorial ideals of $A$ is a divisorial ideal.
§3. The implications of the questions.

(3.1).
(1) Let $P, Q \in \text{Spec}(A)$. Then $P \cap Q = PQ$ if and only if $PA_M \cap QA_M = PA_M \cdot QA_M$ for each $M \in \text{Max}(A)$.
(2) Let $P \in \text{Spec}(A)$ and let $P \subseteq M \in \text{Max}(A)$. Then $P \nsubseteq PM$ if and only if $PA_M \not\subseteq PA_M \cdot MA_M$.

(3.2). Let $A$ be a Krull ring and let $P, Q \in X^{(1)}(A)$ with $P \neq Q$. Then
(1) $PQ$ is a divisorial ideal of $A$ if and only if $P \cap Q = PQ$.
(2) If $PQ$ is a divisorial ideal of $A$, then $PA_M \cdot QA_M$ is a divisorial ideal of $A_M$ for each $M \in \text{Max}(A)$.

**Proof.** (1): Assume that $PQ$ is a divisorial ideal. We have $\text{div}(P \cap Q) = \sup(\text{div}P, \text{div}Q) = \text{div}P + \text{div}Q = \text{div}PQ$.
Thus $P \cap Q = (PQ)^\nu = PQ$.
(2) follows by (1).

(3.3).
(1) In (AAJ-3), the necessity holds.
(2) In (AAJ-4), the necessity holds.
(3) In (AAJ-5), the necessity holds.
(4) In (AAJ-6); (3) implies (1), and (1) implies (2).
(5) In (B-1); (1) implies (3), and (3) implies (2).
(6) In (B-3); (3) implies (1), and (1) implies (2).

**Proof.** (1) and (3) follow by (2. 3).
(4) and (6) follow by (2. 2).
(5): (1) implies (3) by (2. 3). (3) implies (2) by (3. 2).

Let (A) be a ‘yes or no’ question for domains. Assume that if (A) is yes for each quasi-local domain, then (A) is yes for each domain. Then we will say that (A) is of local property.

(3.4).
(1) (AAJ-1) is of local property.
(2) (AAJ-2) is of local property.
(3) (AAJ-3) is of local property.
(4) (AAJ-4) is of local property.
(5) (AAJ-5) is of local property.
(6) (B-1) is of local property.
(7) (B-2) is of local property.

**Proof.** (1) and (2) follow by (2. 1).
(4): Let $A$ be a Krull ring. We will show the sufficiency in (AAJ-4). By assumption, $A$ is a locally factorial ring. By (2. 3), each height one prime ideal of $A$ is an invertible ideal. Since $\text{Pic}(A) = 0$, $A$ is a factorial ring.
(5) follows by (3. 2).

(3.5). If (AAJ-2) is yes, then (AAJ-1) is yes.
PROOF. Let \((A, M)\) be a quasi-local Krull domain and let \(P \in X^{(1)}(A)\). Assume that \(P \cap Q = PQ\) for each \(Q \in X^{(1)}(A)\) with \(Q \neq P\). We will show that \(P\) is an invertible ideal.

If \(\dim A \leq 1\), then \(P\) is an invertible ideal.

Let \(\dim A \geq 2\) and let \(Q \in \text{Spec}(A)\) with \(\text{ht } Q \leq 2\) that is incomparable with \(P\). If \(\text{ht } Q \leq 1\), then \(P \cap Q = PQ\). Suppose that \(\text{ht } Q = 2\). Take \(a \in P \cap Q\) with \(a \neq 0\). Since \(AQ\) is a Krull domain and \(a \in QAQ\), there exists \(Q' \in X^{(1)}(AQ)\) such that \(a \in Q'\). \(Q'\) is of the form \(Q_0AQ\) where \(Q_0 \supseteq Q' \in X^{(1)}(A)\), \(P \neq Q_0\) and \(a \in P \cap Q_0 = PQ_0 \subseteq PQ\). Thus \(P \cap Q = PQ\). Since \((AAJ-2)\) is yes, \(P\) is a maximal ideal or an invertible ideal. Since \(\dim A \geq 2\), \(P\) is an invertible ideal of \(A\).

Let \((A')\) and \((A'')\) be two 'yes or no' questions. In the followings \((A') \Rightarrow (A'')'\) means that if \((A')\) is yes then \((A'')\) is yes.

\(3.6\).
(1) \((AAJ-2) \Rightarrow (AAJ-1) \Rightarrow (AAJ-3) \Leftrightarrow (AAJ-4)\)
\(\Leftrightarrow (AAJ-5) \Leftrightarrow (B-1) \Leftrightarrow (B-2)\).
(2) \((AAJ-6) \Rightarrow (B-3)\).

\(3.7\). If \((AAJ-2)\) is yes, then we have the followings:
(1) In \((AAJ-6)\); \((1) \Leftrightarrow (2)\).
(2) In \((B-3)\); \((1) \Leftrightarrow (2)\).
(3) \((AAJ-6) \Leftrightarrow (B-3)\).

\(3.8\). Let \(A\) be a Krull ring and let \(P \in X^{(1)}(A)\). Then \(P\) is an invertible ideal of \(A\) if and only if \(P \cap Q = PQ\) for each \(P \neq Q \in X^{(1)}(A)\) and \(P \nsubseteq PM\) for each \(P \subset M \in \text{Max}(A)\).

PROOF. The sufficiency: We may assume that \((A, M)\) is a quasi-local ring. Let \(P \in X^{(1)}(A)\). Since \(A\) is a Krull ring, \(A\) is an atomic ring. Take \(a \in P - PM\). Then \(a\) is an irreducible element of \(A\). Because, if \(a\) is a reducible element with \(a = p_1p_2 \cdots p_n\), where \(n \geq 2\) and each \(p_i\) is irreducible element, then \(p_i \in P\) for some \(i\). Thus \(a \in PM\); a contradiction.

Let \(Q \in X^{(1)}(A)\) that is distinct with \(P\). If \(a \in Q\), then \(a \in P \cap Q = PQ \subseteq PM\); a contradiction. Thus \(P\) is the only height one prime ideal containing \(a\).

Let \(v\) be the valuation with the valuation ring \(AP\). Then we have \(v(a) = v(b)\) for each \(b \in P - PM\). Because, if \(v\) has \(a \nleq v(b)\) then there exists \(x\) of \(A\) such that \(x = b/a\). Then \(b = ax \in PM\); a contradiction. Thus \(P = aA \cup PM\). It follows that \(P = aA\).

\(§4.\) The integral closure of a Noetherian domain.

A Noetherian ring \(A\) is called a local ring if \(A\) has only one maximal ideal. Let \(B\) be a Noetherian domain with quotient field \(L\), \(K\) a finite algebraic extension field of \(L\) and \(A\) the integral closure of \(B\) in \(K\). By Mori-Nagata's Integral Closure Theorem, \(A\) is a Krull ring.
(4.1) (GOTO). Let $A$ be as above. Then we have $\bigcap_{n=1}^{\infty} P^n A_P = (0)$ for each $P \in \text{Spec}(A)$.

**Proof.** Set $P \cap B = P'$. Then $A_{P'}$ is the integral closure of $B_{P'}$ in $K$. Thus we may assume that $(B, M)$ is a local ring and $P \cap B = M$.

Set $d = \dim B$. Thus we may use the induction on $d$.

If $d = 0, 1$, then the statement is clear. Suppose that $d \geq 2$ and the statement holds for all positive integer lower than $d$. Set $n = [K : L]$. Then $K = \sum_{i=1}^{n} Lx_i$ for some $x_1, x_2, \ldots, x_n \in A$. Set $M' = P \cap B[x_1, \ldots, x_n]$. We may replace $B$ with $B[x_1, x_2, \ldots, x_n]$, and $K$ with $L$.

Then $A$ is the integral closure of $B$ in $K$.

We may assume that $ht P \geq 2$. Take $Q \in X^{(1)}(A)$ with $Q \subset P$. Set $Q \cap B = Q'$. Then $Q' \neq (0)$, $B/Q' \subset A/Q$ and the quotient field of $A/Q$ is a finite algebraic extension of $B/Q'$ by Mori-Nagata's Theorem. Let $R$ be the integral closure of $B/Q'$ in the quotient field of $A/Q$. We have $\dim (B/Q') < d$.

Take $N \in \text{Spec}(R)$ with $N \cap (A/Q) = P/Q$. Since $\dim R < d$, we have

$$0 = \bigcap_{n=1}^{\infty} N^n R_N \supset \bigcap_{n=1}^{\infty} P^n (A_P/Q A_P).$$

Thus

$$\bigcap_{n=1}^{\infty} P^n A_P \subset \bigcap_{Q \in X^{(1)}(A)} \left( \bigcap_{Q \subset P} Q A_P \right).$$

Since $\dim A_P \geq 2$, it follows that $\bigcap_{Q \in X^{(1)}(A)} Q A_P = (0)$.

Thus $\bigcap_{n=1}^{\infty} P^n A_P = (0)$.

(4.2). Let $B$ be a Noetherian domain with quotient field $L$. Let $K$ be a finite algebraic extension field of $L$, and $A$ the integral closure of $B$ in $K$. Then (AAJ-1) is yes for $A$.

**Proof.** Let $P \in X^{(1)}(A)$. Assume that $P \cap Q = P Q$ for each $Q \in X^{(1)}(A)$ with $P \neq Q$. Let $P \subset M \in \text{Max}(A)$. If $P = PM$, then $P A_M = P M A_M = P M^2 A_M = P M^3 A_M = \cdots = (0)$ by (4.1); a contradiction. Therefore $P \nsubseteq PM$. (3.8) implies that $P$ is an invertible ideal.

(4.3). Let $A$ be as (4.2). Then for $A$ we have

$$(AAJ - 6) \iff (B - 3).$$

**Proof.** The sufficiency: we assume (2) of (AAJ-6). We will prove (3) of (AAJ-6). Each height one prime ideal of $A$ is an invertible ideal by (4.2). Therefore $A$ is a locally factorial ring. Since (B-3) is yes, $A$ is a Q-ring.

(4.4). Let $A$ be as (4.2). Then for $A$ we have

$$(AAJ - 2) \implies (B - 3).$$

**Proof.** We assume moreover that $A$ is locally factorial. Assume (2) of (B-3). We will show (3) of (B-3). Since (AAJ-2) is yes for $A$, we have $\dim A \leq 2$.

By Nagata's Theorem, $A$ is a Noetherian ring. By [3, Corollary (2.8)], $A$ is a Q-ring.
§5. An example

Relating with (4.1), there naturally arises the following question:
Let \((A, M)\) be a quasi-local Krull domain. Then is \(\bigcap_{n=1}^{\infty} M^n = (0)\)?
If the answer to this question is yes, then (AAJ-1) is yes.

(5.1). There exists a quasi-local factorial ring \((A, M)\) with \(M \neq (0)\) such that \(\bigcap_{n=1}^{\infty} M^n = M\).

In fact, let \(A\) be a factorial ring, \(X, U_1, X_1, Y_1\) be indeterminates. Set \(V_1 = (X - U_1X_1)/Y_1, D' = A[U_1, X_1, V_1, Y_1]\) and \(D = A[X]\). Then we have the following,

1. \(ht (P' \cap D) \leq 1\) for each \(P' \in X^{(1)}(D')\).
2. For each \(P \in X^{(1)}(D)\) there exists a unique \(P' \in X^{(1)}(D')\) such that \(P' \cap D = P\). Further \(P'D' p = PD' p\).

Let \(k\) be a field and let \(\{X(i_1 i_2 \cdots i_n); n, i_j \in \mathbb{N}, 1 \leq i_j \leq 4\}\) be a system of indeterminates. Set

\[ Y(i_1 \cdots i_n) = X(i_1 \cdots i_n) - X(i_1 \cdots i_n 1)X(i_1 \cdots i_n 2) - X(i_1 \cdots i_n 3)X(i_1 \cdots i_n 4). \]

Let \(k[X(i_1 \cdots i_n); i_1, \ldots, i_n]\) be a polynomial ring over \(k\) of the infinite number of indeterminates. Let \(Y(i_1 \cdots i_n); i_1, \ldots, i_n\) be the ideal of \(k[X(i_1 \cdots i_n); i_1, \ldots, i_n]\) generated by the elements \(\{Y(i_1 \cdots i_n); i_1, \ldots, i_n\}\). Then \(Y(i_1 \cdots i_n); i_1, \ldots, i_n\) is a prime ideal of \(k[X(i_1 \cdots i_n); i_1, \ldots, i_n]\). Set \(D = k[X(i_1 \cdots i_n); i_1, \ldots, i_n]/(Y(i_1 \cdots i_n); i_1, \ldots, i_n)\). Let \(x(i_1 \cdots i_n)\) be the image of \(X(i_1 \cdots i_n)\) under the natural homomorphism of \(k[X(i_1 \cdots i_n); i_1, \ldots, i_n]\) to \(D\). Then we have \(D = \bigcup_{i=1}^{\infty} k[x(i_1 \cdots i_n); n \leq i, 1 \leq i_j \leq 4]\), which we denote by \(k[x(i_1 \cdots i_n); i_1, \ldots, i_n]\). We have

\[ x(i_1 \cdots i_n) = x(i_1 \cdots i_n 1)x(i_1 \cdots i_n 2) + x(i_1 \cdots i_n 3)x(i_1 \cdots i_n 4). \]

Set

\[ D_0 = k[x(1), x(2), x(3), x(4)], \]
\[ D_1 = D_0[x(11), x(12), x(13), x(14)], \]
\[ D_2 = D_1[x(21), x(22), x(23), x(24)], \]
\[ D_3 = D_2[x(31), x(32), x(33), x(34)], \]
\[ D_4 = D_3[x(41), x(42), x(43), x(44)], \]
\[ D_5 = D_4[x(111), x(112), x(113), x(114)], \]
\[ D_6 = D_5[x(121), x(122), x(123), x(124)], \]
\[ \vdots \]

Then we have \(D = \bigcup_{i=1}^{\infty} D_i\).

Each \(D_i\) is a polynomial ring over \(k\). Hence each \(D_i\) is a factorial ring. For each \(n \in \mathbb{N}\), \(D_n\) and \(D_{n+1}\) are of the following forms:

\[ D_{n+1} = A[x(a1), x(a2), x(a3), x(a4)], \]
\[ D_n = A[x(a)], \] where \(A\) is some factorial ring and \(x(a) = x(a1)x(a2) + x(a3)x(a4)\). Thus \(ht(P' \cap D_n) \leq 1\) for each \(P' \in X^{(1)}(D_{n+1})\) by (1).
Let $m \geq n$ and let $P_m$ be a height one prime ideal of $D_m$ such that $P_m \cap D_n = P_n$ for some $P_n \in X^{(1)}(D_n)$. Then we have $P_n(D_m)_{P_m} = P_m(D_m)_{P_m}$ by (2). Thus $P_n \not\subseteq (P_m)^{(2)}$ (the 2-nd symbolic power of $P_m$). Thus $D$ is a Krull domain by [7, Theorem 3]. By [7, Remark 2] we have $X^{(1)}(D) = \bigcup_{n=1}^{\infty} X^{(1)}(D_n)$. Hence $D$ is a factorial domain. Set $N = (x(i_1 \cdots i_n); i_1, \cdots, i_n)$. $N$ is a maximal ideal of $D$. We have $N = N^2 = N^3 = \cdots$. Set $A = D_N$. Then $A$ is a quasi-local factorial domain with maximal ideal $M = ND_N$. Moreover we have $\bigcap_{n=1}^{\infty} M^n = M$.

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