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<tr>
<td>Author(s)</td>
<td>YABUTA, Kozo</td>
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<tr>
<td>Citation</td>
<td>Bulletin of the Faculty of Science, Ibaraki University. Series A,</td>
</tr>
<tr>
<td></td>
<td>Mathematics, 21: 1-7</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1989</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10109/3006">http://hdl.handle.net/10109/3006</a></td>
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Sharp function estimates
for a class of pseudo-differential operators

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1. Introduction

The purpose of this paper is to give another proof of a theorem of Chanillo and Torchinsky [3] concerning pointwise estimates, in terms of the Fefferman-Stein sharp function and the Hardy-Littlewood maximal function, for pseudo-differential operators with symbols composed of the special symbol $\exp(i|\xi|^b)$ and symbols in Hörmander's class $S_{-n/2}^{1,0}$ (the precise definition will be given below). As a merit of our approach, we can improve their result considerably as our Theorem 2.3 shows.

Let $a(x,\xi)$ be a sufficiently regular function defined on $\mathbb{R}^n \times \mathbb{R}^n$. The pseudo-differential operator with symbol $a(x,\xi)$ is defined on the Schwartz space of rapidly decreasing and infinitely differentiable functions by the formula:

$$a(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) \, d\xi,$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx$ is the Fourier transform of $f$.

We say that a symbol $a(x,\xi)$ is in the class $S_{\rho,\delta}^m$, if for $x, \xi$ in $\mathbb{R}^n$,

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

For $1 \leq p < \infty$ and for a locally integrable function $f$ we define the modified sharp maximal function $f^{p\#}(x)$ by the formula:

$$f^{p\#}(x) = \sup_{x \in Q} \inf_{c} \left( \frac{1}{|Q|} \int_Q |f(y) - c|^p \, dy \right)^{1/p},$$

where $Q$ moves over all cubes with sides parallel to the coordinate axes, containing $x$, and $c$ moves over all complex numbers. $f^{1\#}(x)$ is the sharp maximal function introduced by Fefferman and Stein, and will be denoted simply by $f^\#(x)$.

For $1 \leq p < \infty$, $M_p f(x)$ will denote the $p$-th Hardy-Littlewood maximal function, i.e.,

$$M_p f(x) = \sup_{x \in Q} \left( \frac{1}{|Q|} \int_Q |f(y)|^p \, dy \right)^{1/p}.$$
Now for a symbol \( a(x,\xi) \in S_{1,\delta}^{\frac{nb}{2}}, 0 \leq \delta < 1 \), it is shown in Journé [6] that for each \( 1 < r < +\infty \)
\[
(a(x,D)f)\#(x) \leq C_r M_r f(x), \quad x \in \mathbb{R}^n, \quad f \in C_0^\infty (\mathbb{R}^n).
\]

Similar inequalities for pseudo-differential operators are given by many authors, for example, Miller [7], Miyachi and Yabuta [11], Nishigaki [12] and Yabuta [14], etc. Recently Chanillo and Torchinsky [3] showed the following.

**Theorem A.** Let \( 0 < b < 1 \) and \( s(x,\xi) \in S_{1,\delta}^{-\frac{nb}{2}}, 0 \leq \delta < 1 \) (or its generalized space \( S_{1,\delta}^{-\frac{nb}{2}}(\kappa,\kappa') \)).

In the next section we introduce the symbol class \( S_{m,\delta}^\kappa(\kappa,\kappa') \) and state our main theorem. In Section 3 we give a lemma which shows that an operator \( T \) with singular integral kernel, transforming constants to constants, satisfies the estimate of the type: \( (Tf)^\#(x) \leq C f^\#(x) \) for some \( 1 < r < \infty \). The main theorem will be proved in Section 4.

We note that the letters \( C, C_1, \) etc will always denote positive constants, which may have different values in each occasion.

### 2. Notations and statement of main theorem

\( \mathbb{R} \) is the real line and \( \mathbb{N} \) is the set of all nonnegative integers. For a real number \( s, \lfloor s \rfloor \) denotes the integer satisfying \( s - 1 < \lfloor s \rfloor \leq s \). The letters \( \alpha, \beta \) will denote multi-indices, i.e. \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). For \( x, \xi \in \mathbb{R}^n \), \( x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n \), \( |x| = (x \cdot x)^{1/2} \) and \( \langle x \rangle = (1+|x|^2)^{1/2} \).

Differential operators are denoted by \( \partial^\alpha_x \) or \( \partial^\alpha_x \);

\[
(\partial^\alpha_x f)(x) = \left( \frac{\partial}{\partial x} \right)^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]

\( C_0^\infty (\mathbb{R}^n) \) is the space of infinitely differentiable functions with compact support. \( L^p (\mathbb{R}^n) \) is the space of all Lebesgue measurable functions \( f \) with \( \| f \|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} < +\infty \).

If \( h \in \mathbb{R}^n \) and \( f \) is a function on \( \mathbb{R}^n \), then the first and second differences are denoted by

\[
\Delta_x (h) f(x) = f(x + h) - f(x),
\]

\[
\Delta_x^2 (h) f(x) = f(x + 2h) - 2f(x + h) + f(x).
\]
Now the following symbol class is introduced by Miyachi [9].

DEFINITION 2.1. Let $m \in \mathbb{R}$, $0 \leq \delta, \rho \leq 1$, $\kappa > 0$ and $\kappa' > 0$. Let $k$ and $k'$ be the nonnegative integers satisfying $k < \kappa \leq k + 1$ and $k' < \kappa' \leq k' + 1$. Then $S^m_{\delta, \rho, \kappa, \kappa'}(\kappa, \kappa')$ denotes the set of those functions $a = a(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ which have the following estimates:

(i) if $|\beta| \leq k$ and $|\alpha| \leq k'$, then the derivatives $\partial_\xi^\beta \partial_\xi^\alpha a(x, \xi)$ exists in the classical sense and

\[ |\partial_\xi^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(\xi)^{m+|\beta|-\rho|\alpha|}, \]

(ii) if $|\beta| = k$ and $|\alpha| \leq k'$, $h \in \mathbb{R}^n$, and $|h| \leq (\xi)^{-\delta}$, then

\[ |\Delta_\xi^2(h) \partial_\xi^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(\xi)^{m+|\beta|+\rho|\alpha|} |h|^{\kappa-k}; \]

(iii) if $|\beta| \leq k$ and $|\alpha| = k'$, $\eta \in \mathbb{R}^n$, and $|\eta| \leq (\xi)^{\rho/4}$, then

\[ |\Delta_\xi^2(\eta) \partial_\xi^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(\xi)^{m+|\beta|+\rho|\alpha|} |\eta|^{\kappa-k'}; \]

(iv) if $|\beta| = k$ and $|\alpha| = k'$, $h, \eta \in \mathbb{R}^n$, and $|h| \leq (\xi)^{-\delta}$, $|\eta| \leq (\xi)^{\rho/4}$, then

\[ |\Delta_\xi^2(h) \Delta_\xi^2(\eta) \partial_\xi^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(\xi)^{m+|\beta|+\rho|\alpha|} |h|^{\kappa-k} |\eta|^{\kappa-k'}. \]

Here $A$ is a constant which does not depend on $\alpha, \beta, x, \xi, h$ and $\eta$. The smallest such constant is denoted by $\|a\|_{m, \rho, \delta, \kappa, \kappa'}$.

REMARK 2.2. If $\kappa_2 \leq \kappa_1$ and $\kappa'_2 \leq \kappa'_1$, then $S^m_{\delta, \rho, \kappa_2, \kappa'_2} \subset S^m_{\delta, \rho, \kappa_1, \kappa'_1}$ and $\|a\|_{m, \rho, \delta, \kappa_2, \kappa'_2} \leq \|a\|_{m, \rho, \delta, \kappa_1, \kappa'_1}$. Furthermore, in the above definition, if $\kappa$ (resp. $\kappa'$) is not an integer, the second difference with respect to $x$ (resp. $\xi$) can be replaced by the first difference. It should also be remarked that $|h| \leq (\xi)^{-\delta}$ and $|\eta| \leq (\xi)^{\rho/4}$ can be replaced by $h \in \mathbb{R}^n$ and $|\eta| \leq (\xi)/4$, if one modifies the constant $A$.

Now we can state our theorem.

THEOREM 2.3. Let $0 < b < 1$, $1 \leq p \leq 2$, $0 \leq \delta < 1$, $\kappa > 0$, and $\kappa' > \frac{n}{p} + \delta$. Then for any $r \geq p$ ($r > p$ if $p = 1$), any $1 \leq s \leq r$ with $s \leq 2$, and any $\sigma(x, \xi) \in S^{\frac{-nb}{2}}_{1, \delta} (\kappa, \kappa')$ there exists a constant $C > 0$ such that

\[ (a(x, D)f)^\#(x) \leq CMr f(x), \quad x \in \mathbb{R}^n, \quad f \in C_{0}^{\infty}(\mathbb{R}^n), \]

where $a(x, \xi) = \sigma(x, \xi) \exp(i|\xi|^b)$, and where $C$ depends only on $p, r, s, n, \kappa, \kappa'$ and the norm of $\sigma(x, \xi)$.

This theorem clearly implies Theorem A as a special case ($\delta = 0$, and $s = p = 1$).
3. Estimates for singular integral operators

For \( 1 \leq r < \infty \) and \( t > 0 \) we define

\[
\Omega_r f(x, t) = \sup_B \inf_c \left( \frac{1}{|B|} \int_B |f(y) - c|^r \, dy \right)^{1/r},
\]

where \( B \) moves over all balls with radius \( t \) and containing \( x \), and \( c \) moves over all complex numbers.

**Lemma 3.1.** Let \( 1 \leq p < \infty \) and let \( \varphi(t) \) be a positive and nondecreasing function on \((0, \infty)\) such that \( \int_0^1 \varphi(t) t^{-1} \, dt < +\infty \). Suppose \( T \) is a bounded linear operator on \( L^p(\mathbb{R}^n) \), transforms constants to constants, and has the kernel \( K(x, y) \) satisfying the following conditions: For any \( f \in L^p(\mathbb{R}^n) \) with compact support

\[
|K(x, y) - K(z, y)|^{p'} \, dy \leq C_0 \, d^{-n/p} \varphi \left( \frac{|x - z|}{d} \right), \quad \text{for } |x - z| < \frac{d}{2}, \quad d > 0;
\]

where \( C_0 \) is independent of \( d, x \) and \( z \).

Then, if \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) satisfies

\[
H_p f(x, t) = \sum_{j=0}^\infty \sum_{l=0}^j \Omega_p f(x, 2^{l+1}t) \omega (2^{-j}) < \infty,
\]

for all \( x \in \mathbb{R}^n \) and \( t > 0 \), one can well define \( Tf \) modulo constants so that

\[
\Omega_p(Tf)(x, t) \leq C H_p f(x, t), \quad x \in \mathbb{R}^n \text{ and } t > 0,
\]

where \( C \) is independent of \( f \).

**Proof.** This is the special case of Lemma 3.2 in Yabuta [16] (the case \( k = 1 \)). Note also that "\( H_p f(x, t) < \infty \) for almost all \( x \in \mathbb{R}^n \) and for almost all \( t > 0 \)" implies "\( H_p f(x, t) < \infty \) for all \( x \in \mathbb{R}^n \) and for all \( t > 0 \)."

As a direct consequence of Lemma 3.1 we have

**Corollary 3.2.** Let \( \varphi(t) \) be a positive and nondecreasing function on \((0, \infty)\) such that \( \int_0^1 \varphi(t) \log \left( 1 + \frac{1}{t} \right) t^{-1} \, dt < +\infty \). Let \( p, p', T \) and \( K \) be the same as in Lemma 3.1. Then, if \( f^{p#}(x) < \infty \) a.e., one can define \( Tf \) modulo constants so that

\[
(Tf)^{p#}(x) \leq C f^{p#}(x), \quad x \in \mathbb{R}^n.
\]
We note that similar results are discussed in Jawerth-Torchinsky [5, Theorem 4.7, p. 258], relating to the sharp maximal function $M^\#_0 f$, introduced by F. John and J. O. Strömberg. However, we mention here the following. In their theorem one must impose the additional assumption that $T$ maps constants to constants, like our assumption in the above Corollary.

We need one more lemma.

**Lemma 3.3.** Let $\omega(t)$, $p$ and $p'$ be the same as in Lemma 3.1. Suppose $T$ is a bounded linear operator on $L^p(\mathbb{R}^n)$, and has the kernel $K(x,y)$ satisfying (3.1) for all $f \in C^\infty_0(\mathbb{R}^n)$ and (3.2). Then it holds that

$$ (Tf)^\#(x) \leq C M_p f(x), \quad x \in \mathbb{R}^n, \quad f \in C^\infty_0(\mathbb{R}^n). $$

**Proof.** The proof of Proposition 1.4 in Yabuta [14] essentially shows this fact. q.e.d.

**Remark 3.4.** The condition (3.2) is essentially equivalent to the condition $(D'p')$ in Rubio de Francia, Ruiz and Torrea [13, p. 30].

### 4. Proof of the main theorem

Clearly it suffices to prove the case $r = s \leq 2$. Take a radial function $\varphi(\xi) \in C^\infty(\mathbb{R}^n)$ such that $\varphi(\xi) = 0$ if $|\xi| < 1/2$ and $\varphi(\xi) = 1$ if $|\xi| \geq 1$. Put

$$\begin{align*}
\tau(\xi) &= \varphi(\xi)|\xi|^{-n/2} \exp (i|\xi|^b), \\
\sigma_1(x,\xi) &= \sigma(x,\xi)|\xi|^{n/2} \varphi(\xi), \\
a_1(x,\xi) &= \sigma_1(x,\xi)\tau(\xi), \\
\sigma_2(x,\xi) &= (1 - \varphi^2(\xi))\sigma(x,\xi), \\
a_3(x,\xi) &= (1 - \varphi^2(\xi)) \left( \exp (i|\xi|^b) - 1 \right) \sigma(x,\xi).
\end{align*}$$

Then we have

$$ a(x, D) = \sigma_1(x, D)\tau(D) + a_2(x, D) + a_3(x, D). $$

We see that $a_2(x, \xi) \in S^0_{0,\delta}(\kappa, \kappa')$. Since $\kappa > 0$ and $\kappa' > \frac{n}{p} + \delta$, we see by Propositions 2 and 3 in Yabuta [15] that the distributional kernel $K_2(x,y)$ of $a_2(x, D)$ satisfies

$$\begin{align*}
\left( \int_{d <|x-y|< 2d} \left| K_2(x,y) \right|^{p'} \, dy \right)^{1/p'} &\leq C d^{-n/p}, \quad d > 0, \\
\left( \int_{d <|x-y|< 2d} \left| K_2(x,y) - K_2(z,y) \right|^{p'} \, dy \right)^{1/p'} &\leq C d^{-n/p} \left( \frac{|x-z|}{d} \right)^\gamma, \quad |x-z| < \frac{d}{2}, \quad d > 0,
\end{align*}$$

where $\gamma = \frac{n}{2} - \frac{\delta}{2}$.
where $\gamma$ is a positive number smaller than $\min(1, \kappa, \kappa' - \delta - \frac{n}{p})$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (take $g(t) = t^\gamma$ in Example 1 in [15]). Since $r \geq p$, we see that (4.2) and (4.3) hold with $p$ replaced by $r$. Now we may assume $\kappa'$ is not an integer. Take $\gamma_1 < \min(\kappa, \kappa' - \delta - \frac{n}{p}, b)$. Then we can choose $\zeta > \gamma_1^2$ so that $0 < \zeta < b$ and $\zeta/\gamma_1 < \min(1, \kappa' - \delta - \frac{n}{p})$ (this can be done). Then we have

$$|\partial_\xi a_3(x, \xi)| \leq C|\xi|^{-|\alpha|+\zeta}, \quad |\alpha| \leq |\kappa'|;
$$

$$|\Delta_\xi(\eta)\partial_\xi a_3(x, \xi)| \leq C|\xi|^{-\kappa'+\zeta}|\eta|^{|\kappa'|-|\kappa'|}, \quad |\eta| < \frac{|\xi|}{2}, |\alpha| = |\kappa'|;
$$

$$|\Delta_\zeta(h)\partial_\xi a_3(x, \xi)| \leq C|\xi|^{-|\alpha|+\zeta}|h|^{\gamma_1}, \quad |\alpha| \leq |\kappa'|;
$$

$$|\Delta_\zeta(h)\Delta_\xi(\eta)\partial_\xi a_3(x, \xi)| \leq C|\xi|^{-\kappa'+\zeta}|\eta|^{|\kappa'|-|\kappa'|}|h|^{\gamma_1}, \quad |\eta| < \frac{|\xi|}{2}, |\alpha| = |\kappa'|.
$$

Hence, noting $a_3(x, \xi) = 0$ if $|\xi| \geq 1$, we have by Propositions 2 and 4 in Yabuta [15] that the distributional kernel $K_3(x, y)$ of $a_3(x, D)$ satisfies the same inequalities as (4.2) and (4.3) with $\gamma$ replaced by $\gamma_1$ (and hence with $p$ replaced by $r$). Now by the Hörmander-Mihlin multiplier theorem, the operator $(1 - \varphi^2(D)) \exp(i|D|^b)$ is bounded on $L^q(\mathbb{R}^n)$ ($1 < q < \infty$). Since $\sigma_1(x, \xi) \in S_{0,\delta}(\kappa, \kappa')$, $\kappa > 0$, $\kappa' > \frac{n}{p} + \delta$, and $1 \leq p \leq 2$, we see that $\sigma_1(x, D)$ and $\sigma(x, D)$ are bounded on $L^q(\mathbb{R}^n)$ ($p \leq q < \infty$ if $p > 1$, and $1 < q < \infty$ if $p = 1$), see e.g. Miyachi and Yabuta [10]. So, $a_j(x, D)$ ($j = 2, 3$) are also bounded on $L^r(\mathbb{R}^n)$. Hence by Lemma 3.3 we see that $a_2(x, D)$ and $a_3(x, D)$ satisfy the estimates (2.1) with $s = r$. Next, since $\sigma_1(x, \xi) \in S_{1,\delta}(\kappa, \kappa')$, $\kappa > 0$, and $\kappa' > \frac{n}{p} + \delta$, we have the same estimates as (4.2) and (4.3) (and hence with $p$ replaced by $r$) for the distributional kernel $K_1(x, y)$ of $\sigma_1(x, D)$. Since $\sigma_1(x, \xi) = 0$ for $|\xi| \leq \frac{1}{2}$, we see easily that $\sigma_1(x, D)(1) = 0$. Hence by our Corollary 3.2 we have

$$(\sigma_1(x, D)g)r#(x) \leq Cg^r#(x), \quad x \in \mathbb{R}^n.
$$

Hence using this and Lemma 4.1 below we have

$$(a_1(x, D)f)r#(x) \leq C(\tau(D)f)r#(x) \leq CMrf(x), \quad x \in \mathbb{R}^n, \ f \in C_0^\infty(\mathbb{R}^n).
$$

This finishes the proof of Theorem 2.3.

**Lemma 4.1.** Let $\tau(\xi)$ be the same as in the beginning of this section. Then, if $1 < r \leq 2$, we have

$$(\tau(D)f)r#(x) \leq CMrf(x), \quad x \in \mathbb{R}^n, \ f \in C_0^\infty(\mathbb{R}^n).
$$

**Proof.** The kernel for the operator $\tau(D)$ is given by

$$C|x|^{-n}(1 - \varphi(x)) \exp(ic|x|^{b/(1-b)}) + K(x),
$$

where $C$ and $c$ are constant numbers, $\varphi(x)$ is as in the proof of Theorem 2.3, and $K(x)$ is a good kernel satisfying the assumptions in Lemma 3.3 with $\omega(t) = t$ for every $1 < p < \infty$, see e.g. Miyachi [8, Proposition 5.1, p. 289]. For the second kernel we obtain the desired estimate by Lemma 3.3. As for the first kernel, the proof of Lemma 2.15 in Chanillo [2] can be modified to get the desired estimate. Here we note that at (2.17) in [2] $\tilde{K}_{b,\nu}$ can be replaced by $\tilde{K}_{b,\nu}$ with $0 < \epsilon < \delta$, and hence the restriction $(b + 2)/\nu' < 1$ can be dropped. q.e.d.
Sharp function estimates for pseudo-differential operators

References