タイトル
On the Le`vy representation of the characteristic function of the probability distribution Ce-|x|dx

著者
TAKANO, Katsuo

引文
Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics , 20: 61-65

日付
1988

URL
http://hdl.handle.net/10109/3004

権利
このリポジトリに収録されているコンテンツの著作権は、それぞれの著作権者に帰属します。引用、転載、複製等される場合は、著作権法を遵守してください。
On the Lévy representation of the characteristic function of the probability distribution $Ce^{-|x|}dx$

KATSUO TAKANO*

1. Introduction

$\mathbb{R}^d$ denotes $d$-dimensional Euclidean space. We write $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d)$ for the elements of $\mathbb{R}^d$. The inner product of $x, y \in \mathbb{R}^d$ is the number $xy = x_1y_1 + x_2y_2 + \ldots + x_dy_d$. Let $|x| = (x_1^2 + \ldots + x_d^2)^{1/2}$ and let $dx = dx_1dx_2\ldots dx_d$ denotes the ordinary Lebesgue measure. If $\mu$ is a probability distribution on $\mathbb{R}^d$ we define the characteristic function $\phi$ by letting

$$\phi(z) = \int_{\mathbb{R}^d} e^{izx} \mu(dx), \quad z \in \mathbb{R}^d.$$ 

It is well known that the isotropic Cauchy distribution

$$f(x)dx = \pi^{-(d+1)/2}\Gamma\left(\frac{d+1}{2}\right)(1 + |x|^2)^{-(d+1)/2}dx \quad (1.1)$$

has the characteristic function

$$\phi(z) = \int_{\mathbb{R}^d} e^{izx} f(x)dx = e^{-|z|} \quad (1.2)$$

and is a stable distribution of index 1. The constant number $C$ is chosen such that

$$\int_{\mathbb{R}^d} Ce^{-|x|}dx = 1. \quad (1.3)$$

The purpose of this note is to obtain the Lévy representation of the characteristic function of the probability distribution $Ce^{-|x|}dx$. For the case $d = 1$, the Lévy representation of the characteristic function of the probability distribution $\frac{1}{2}e^{-|x|}dx$ is known; that is

$$\phi(z) = \exp \left[ \int_{-\infty}^{\infty} (e^{izx} - 1) \frac{1}{|x|} e^{-|x|} dx \right].$$

From this we obtain the following representation;

$$\phi(z) = \exp \left[ \int_0^{\infty} 2e^{-v}dv \left( \frac{1}{2} \int_0^v (e^{izu} - 1) \frac{du}{u} + \frac{1}{2} \int_0^v (e^{-izu} - 1) \frac{du}{u} \right) \right]$$

(cf. [9. Th.3.6]).

Received 11, April 1988.

* College of general education, Ibaraki University, Mito,Ibaraki.
2. On the Lévy representation

Let $d = 2, 3, \ldots$. We shall make use of the polar coordinates;

\[
\begin{align*}
    x_1 &= r \cos \theta_1, & 0 &\leq \theta_1 \leq \pi, \\
    x_2 &= r \sin \theta_1 \cos \theta_2, & 0 &\leq \theta_2 \leq \pi, \\
    &\quad \ldots \\
    x_{d-1} &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \cos \theta_{d-1}, & 0 &\leq \theta_{d-2} \leq \pi, \\
    x_d &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \sin \theta_{d-1}, & 0 &\leq \theta_{d-1} \leq 2\pi,
\end{align*}
\]

and $\, dx = r^{d-1} dr d\xi$, $\, d\xi = \sin^{d-2} \theta_1 \sin^{d-3} \theta_2 \cdots \sin \theta_{d-2} d\theta_1 d\theta_2 \cdots d\theta_{d-1}$.

From (1.3) we have

\[
\frac{1}{C} = \int_0^\infty e^{-r^{d-1}} dr \int_{S^{d-1}} d\xi = \Gamma(d) \frac{2\pi^{d/2}}{\Gamma(d/2)},
\]

where $S^{d-1}$ denotes the $d - 1$ dimensional sphere. For simplicity, set

\[
\begin{align*}
    D &= \int_{S^{d-1}} d\xi = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \\
    E &= 2^{(d-2)/2} \pi^{1/2} \Gamma\left(\frac{d-1}{2}\right), \\
    F &= \int_0^\pi \sin^{d-2} \theta d\theta = \Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(\Gamma\left(\frac{d}{2}\right)\right)^{-1},
\end{align*}
\]

and

\[
L(u) = \frac{(d+1)F}{DE} u^{d/2} K_{d/2}(u) = (2\pi)^{-d/2} (d+1) u^{d/2} K_{d/2}(u),
\]

where $K_{d/2}(u)$ denotes the modified Bessel function.

RESULT 1. The Lévy representation of the characteristic function of the probability distribution $Ce^{-|x|} dx$ is as follows;

\[
\phi(z) = \exp \left[ \int_{\mathbb{R}^d} \left( e^{izx} - 1 \right) \frac{L(|x|)}{|x|^d} \, dx \right].
\]  

PROOF. By the formula

\[
K_{\nu}(v) = \left( \frac{\pi}{2v} \right)^{\frac{1}{2}} \frac{e^{-v}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-w} w^{\nu-1/2} (1 + \frac{w}{2v})^{\nu-1/2} \, dw
\]

(cf. [11. p. 206]), we see that

\[
\int_{\mathbb{R}^d} \frac{|x|}{1+|x|} \frac{L(|x|)}{|x|^d} \, dx = \int_{S^{d-1}} d\xi \int_0^\infty \frac{r}{1+r} \frac{L(r)}{r^{d-1}} \, dr
\]

\[
= D \int_0^\infty \frac{1}{1+r} L(r) \, dr < \infty.
\]
This shows that the measure
\[ \frac{L(|x|)}{|x|^d} \, dx \]
on \mathbb{R}^d is the Lévy measure and hence (2.1) is a characteristic function of an infinitely divisible distribution. Making use of the polar coordinates, we obtain from (2.1) that
\[ \phi(z) = \exp \left[ \int_{S^{d-1}} d\xi \int_0^\infty (e^{iu\xi} - 1) \frac{L(u)}{u} \, du \right]. \tag{2.3} \]
If we take \( \cos \theta_1 \) as the projection of the vector \( \xi \) to the vector \( z \), from (2.3) and by the Fubini theorem we obtain
\[ \log \phi(z) = \frac{D}{F} \int_0^\pi \sin^{d-2} \theta_1 \, d\theta_1 \int_0^\infty (e^{iu|z|\cos \theta_1} - 1) \frac{L(u)}{u} \, du \]
\[ = \frac{D}{F} \int_0^\infty \left[ \int_0^\pi (\cos(u|z|\cos \theta_1) - 1) \sin^{d-2} \theta_1 \, d\theta_1 \right] \frac{L(u)}{u} \, du. \tag{2.4} \]
By using the Bessel function \( J_{(d-2)/2}(z) \) and by the fact that
\[ F = \lim_{\varepsilon \to 0} E \, e^{(2-d)/2} J_{(d-2)/2}(\varepsilon), \]
we have
\[ \log \phi(z) = \frac{D}{F} \int_0^\infty \left[ E(u|z|)^{(2-d)/2} J_{(d-2)/2}(u|z|) - F \right] \frac{L(u)}{u} \, du. \]
By (1.1), (1.2) and by the Fourier inversion formula we see that
\[ \int_{\mathbb{R}^d} e^{izx} e^{-|x|} \, dx = C(2\pi)^d \pi^{-(d+1)/2} \Gamma\left( \frac{d+1}{2} \right) \frac{1}{(1+|z|^2)^{(d+1)/2}} \]
\[ = \frac{1}{(1+|z|^2)^{(d+1)/2}}. \tag{2.5} \]
Therefore let us show that
\[ \log \frac{1}{(1+r^2)^{(d+1)/2}} = \frac{D}{F} \int_0^\infty \left[ E(ur)^{(2-d)/2} J_{(d-2)/2}(ur) - F \right] \frac{L(u)}{u} \, du \tag{2.6} \]
holds for all non-negative \( r \). At first, from (2.4), (2.6) holds for \( r = 0 \) and furthermore we see that both sides of (2.6) converge to 0 as \( r \to 0 \). By differentiating formally both sides of (2.6) we have
\[ (d+1) \frac{r}{1+r^2} = \frac{D}{F} \int_0^\infty (ur)^{-d/2} \left[ \frac{d-2}{2} J_{(d-2)/2}(ur) - ur J'_{(d-2)/2}(ur) \right] L(u) \, du \]
\[ = \frac{D}{F} \int_0^\infty (ur)^{-d/2} ur J_{d/2}(ur) L(u) \, du. \tag{2.7} \]
for \( r > 0 \). From the formula

\[
\int_0^\infty u^{1/2} K_{d/2}(ur)(ur)^{1/2} J_{d/2}(ur)du = \frac{r^{(d+1)/2}}{1 + r^2}
\]

(cf. [7. p. 108. 12.5]), the equality (2.7) holds for \( r > 0 \). Both sides of (2.7) are continuous on \((0, \infty)\) and the function \((ur)^{-d/2} J_{d/2}(ur)\) is bounded on \((0, \infty)\) and \(uL(u)\) is integrable on \((0, \infty)\). Hence the differentiation of the right side of (2.6) is justified. By integrating both sides of (2.7) from \( \varepsilon(>0) \) to \( r \) and by letting \( \varepsilon \to 0 \) we see that (2.6) holds for all \( r > 0 \).

RESULT 2. The characteristic function of the probability distribution \( Ce^{-|x|^2} dx \) is also represented as follows;

\[
\phi(z) = \exp\left[ \int_0^\infty \frac{(d + 1)F}{E} v^{d/2} K_{(d-1)/2}(v)dv \int_{S_{d-1}} \frac{\xi}{D} \int_0^v \left( e^{iux} - 1 \right) \frac{du}{u} \right].
\]

(2.8)

PROOF. From (2.2),

\[
\frac{(d + 1)F}{DE} v^{d/2} K_{(d-1)/2}(v) > 0 \quad \text{on} \quad (0, \infty).
\]

By the Fubini theorem and by a change of the order of integration and by the relation

\[
u K_{d/2}'(u) + \frac{d}{2} K_{d/2}(u) = -uK_{d/2-1}
\]

and by the fact that

\[L(u) = \frac{(d + 1)F}{DE} \int_u^\infty v^{d/2} K_{(d-1)/2}(v)dv,
\]

we see that (2.8) equals (2.3).

References