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引用
Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 19: 47-48

発行日
1987

URL
http://hdl.handle.net/10109/2988

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Remarks on minimal immersions

Kohei Hatsuse*

1. Let $M^n$ be an $n$-dimensional Riemannian manifold. Let $\mathbb{M}^m(c)$ be an $m(\geq 2)$-dimensional Riemannian manifold of constant sectional curvature $c$. A space form is an example of $\mathbb{M}^m(c)$.

Takahashi [2] and Obata [1] proved the following

**THEOREM A** ([2]). Let $x$ be an isometric immersion of $M^n$ into a space form $\mathbb{M}^m(c)$. Then $x$ is a minimal immersion if and only if $\triangle x = -ncx$.

**THEOREM B** ([1]). Let $M^n$ be compact. Then $M^n$ is isometric with an $n$-sphere $S^n(c)$ if and only if there exists a non-constant differentiable function $f$ in $M^n$ such that $\Delta f = -ncf$ and $(R \cdot \nabla f, \nabla f) = (n-1)c(\nabla f, \nabla f)$.

In this note, we deal with some minimal immersions of $M^n$ into $\mathbb{M}^m(c)$. We shall mean $C^\infty$ differentiable by "differentiable".

2. Let $\mathfrak{X}(M^n)$ be the set of all differentiable vector fields on $M^n$. Let $X \in \mathfrak{X}(M^n)$. Then there exists a maximal open subset $S_x$ of $M^n$ such that $X_p \neq 0$ at each point $p \in S_x$. The closure $\overline{S}_x$ of $S_x$ is called the support of $X$. If we put

$$\mathfrak{X}_c(M^n) = \{X \in \mathfrak{X}(M^n); \overline{S}_x \text{ is compact}\},$$

then $\mathfrak{X}_c(M^n)$ is a linear subspace of $\mathfrak{X}(M^n)$. Let $g$ be the Riemannian metric in $M^n$. If $M^n$ is oriented, then we can define an inner product $(X, Y)$ in $\mathfrak{X}_c(M^n)$ by

$$(X, Y) = \int_M g(X, Y)dV, \quad X, Y \in \mathfrak{X}_c(M^n),$$

where $dV$ denotes the volume element of $M^n$ with respect to the metric $g$.

We denote by $\mathcal{R}$ the Ricci tensor field in $M^n$. Let $T_pM^n$ be the tangent space at $p \in M^n$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM^n$. If we define a mapping $\mathcal{R}_p : T_pM^n \rightarrow T_pM^n$ by

$$\mathcal{R}_p(v) = \sum_{i=1}^n R_p(v, e_i) \cdot e_i, \quad v \in T_pM^n,$$

then $\mathcal{R}_p$ is a linear mapping. The linear mapping $\mathcal{R}_p$ is independent of choice of orthonormal basis $\{e_1, \ldots, e_n\}$. We have $\mathcal{R}(X) \in \mathfrak{X}_c(M^n)$ for $X \in \mathfrak{X}_c(M^n)$.

Received March 16, 1987. Partly supported by the Grant-in-Aid for Scientific Research (C-61540012) from the Ministry of Education.

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THEOREM 1. Let $M^n$ be oriented. Let $f$ be a minimal immersion of $M^n$ into $\overline{M}^m(c)$. Then $f$ is totally geodesic if and only if $(\bar{R}(X), X) = (n-1)c(X, X)$ for $X \in \mathfrak{X}_c(M^n)$.

PROOF. If $f$ is totally geodesic then $\bar{R}(X) = (n-1)cX$ for $X \in \mathfrak{X}_c(M^n)$. Thus we have $(\bar{R}(X), X) = (n-1)c(X, X)$.

We assume $(\bar{R}(X), X) = (n-1)c(X, X)$ for $X \in \mathfrak{X}_c(M^n)$. Let $p \in M^n$ and $U$ be a neighborhood of $p$ in $M^n$. Then, for each $v \in T_pM^n$, we can find a differentiable vector field $X \in \mathfrak{X}_c(M^n)$ such that $S_X \subset U$ and $X_p = v$. If we choose $U$ sufficiently small, then there exists an orthonormal frame $\{E_1, \ldots, E_n\}$ in $U$. We denote by $\bar{g}$ the Riemannian metric in $\overline{M}^m(c)$ and by $h$ the second fundamental form of $f$. Since $f$ is isometric and minimal, we have

$$R(X, X) = (n-1)c\bar{g}(X, X) - \sum_{i=1}^{n} \bar{g}(h(E_i, X), h(X, E_i))$$

in $U$ from the equation of Gauss. It is obvious that $(\bar{R}(X), X) = \int_M R(X, X) dV$. Therefore, we have

$$\int_M \sum_{i=1}^{n} \bar{g}(h(E_i, X), h(X, E_i)) dV = 0$$

and so $h_p = 0$. Since $p$ is arbitrary, $f$ is totally geodesic.

COROLLARY. Let $M^n$ be compact. Let $f$ be a minimal immersion of $M^n$ into $\overline{M}^m(c)$. Then $f$ is totally geodesic if and only if $(\bar{R}(X), X) = (n-1)c(X, X)$ for $X \in \mathfrak{X}(M^n)$.

We denote by $id_p$ the identity mapping $T_pM^n \to T_pM^n$. We have the following

THEOREM 2. Let $f$ be a minimal immersion of $M^n$ into $\overline{M}^m(c)$. Then $f$ is totally geodesic if and only if $\bar{R}_p = (n-1)c \cdot id_p$ at each $p \in M^n$.

PROOF. $\bar{R}_p = (n-1)c \cdot id_p$ implies $R_p = (n-1)c\bar{g}_p$. Therefore, if $f$ is totally geodesic, then $R_p = (n-1)c \cdot id_p$ at each $p \in M^n$. Conversely, we have $\langle h \rangle = 0$ from the equation of Gauss, where $\langle h \rangle$ denotes the length of $h$. Therefore, $f$ is totally geodesic.

References