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On the Pearson type VII distribution and the singular integral transformation

Katsuo Takano*

Abstract. It is shown that if \( m > 0 \)
\[
\frac{x}{(1 + x^2)^{m+1/2}} = \lim_{\varepsilon \to 0} \int_{|u| \geq \varepsilon} \frac{1}{[1 + (x - u)^2]^{m+1/2}} k_m(u) \, du
\]
holds for all \( x \) in the pointwise convergence and if \( m > \frac{1}{2} \) this equality also holds in the \( L^p \) norm convergence (\( p \geq 1 \)), where \( k_m(x) \) is a singular integral kernel, that is
\[
k_m(x) = (\text{sgn } x) 2\pi^{-2} \int_0^\infty e^{-|x|v} \frac{dv}{v(J_m^2(v) + Y_m^2(v))}.
\]
This equality is an extension of the well-known equality
\[
\frac{x}{1 + x^2} = \lim_{\varepsilon \to 0} \int_{|u| \geq \varepsilon} \frac{1}{1 + (x - u)^2} \frac{1}{\pi u} \, du.
\]

1. Introduction The density of the Pearson type VII distribution is as follows:
\[
\frac{\Gamma(m+1/2)}{\sqrt{\pi \Gamma(m)}} \frac{1}{(1 + x^2)^{m+1/2}} \quad (m > 0, \ -\infty < x < \infty),
\]
where \( \Gamma(x) \) denotes the gamma function. It is well-known that
\[
\frac{x}{1 + x^2} = \lim_{\varepsilon \to 0} \int_{|u| \geq \varepsilon} \frac{1}{1 + (x - u)^2} \frac{1}{\pi u} \, du
\]
holds for almost every \( x \) in the pointwise convergence and also holds in the \( L^p \) norm convergence for \( p > 1 \) (cf. [3], [15]). In this note it is shown that Sato-Yamazato's equation derives this equality in natural way. E. Grosswald [4] lastly proved that the Student \( t \)-distribution with any degree of freedom is infinitely divisible. M. E. H. Ismail and D. Kelker [7], D. N. Shanbhag and M. Sreehari [12], C. Halgreen [5] proved that the variance mixture of the normal distribution with a generalized gamma convolution is self-decomposable. In particular C. Halgreen [5] calculated the canonical measure of its canonical representation. K. Sato & M. Yamazato [10] studied in detail the unimodality of self-decomposable
distributions. We calculate the Lévy representation of the Pearson type VII distribution and we show that the singular integral kernel relating to the Pearson type VII distribution satisfies the fundamental conditions which the singular integral kernels have to satisfy. Throughout this note $J_m(x)$ and $Y_m(x)$ denote the Bessel functions (cf. [1]).

2. The Lévy representation of the Pearson type VII distribution

It is known that $F(x)$ is a self-decomposable distribution function if and only if the characteristic function $\phi(t)$ is as follows:

$$\phi(t) = \exp \left\{ it - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}_0} \left( e^{itu} - 1 - \frac{itu}{1+u^2} \right) \frac{k(u)}{u} du \right\}, \quad (3)$$

where $\gamma$ is real, $\sigma^2 \geq 0$, $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $k(u)$ is non-positive on $(-\infty, 0)$ and non-negative on $(0, \infty)$, $k(u)$ is non-increasing on $\mathbb{R}_0$, and

$$\int_{|u| \leq 1} uk(u) du + \int_{|u| > 1} \frac{k(u)}{u} du < \infty. \quad (4)$$

The Pearson type VII distribution (1) can be represented by the variance mixture of the normal distribution $N(0, \sigma^2)$ with the distribution

$$g(x) = \frac{1}{2^m \Gamma(m)} x^{-m-1} e^{-1/(2x)} \quad (x > 0),$$

$$= 0 \quad (x \leq 0).$$

That is,

$$\frac{\Gamma(m+1/2)}{\sqrt{\pi \Gamma(m)}} \frac{1}{(1+x^2)^{m+1/2}} = \int_0^\infty \frac{1}{\sqrt{2\pi u}} e^{-x^2/(2u)} g(u) du$$

holds. For simplicity, set

$$c_m = \Gamma \left( m + \frac{1}{2} \right) / (\sqrt{\pi \Gamma(m)}) \quad \text{and} \quad f_m(x) = (1 + x^2)^{-m-1/2}.$$ 

The characteristic function $\phi(t)$ of the Pearson type VII distribution (1) is as follows:

$$\phi(t) = \int_0^\infty e^{-ut^2/2} g(u) du. \quad (5)$$

Let us denote the Laplace transform of $g(x)$ by $\zeta(s)$, that is,

$$\zeta(s) = \int_0^\infty e^{-sx} g(x) dx \quad (Re \ s \geq 0).$$

Then by the formula (5) in [5] we obtain
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\[ \log \xi(s) = - \int_0^\infty g_m(2u) \log \left( 1 + \frac{s}{u} \right) du, \quad (6) \]

where

\[ g_m(x) = 2\pi^{-2} \{ x(J_m^2(\sqrt{x}) + Y_m^2(\sqrt{x})) \}^{-1}. \]

**RESULT 1.** The Lévy representation of the Pearson type VII distribution is as follows:

\[ \phi(t) = \int_{-\infty}^\infty e^{itx} c_m f_m(x) dx \]

\[ = \exp \left[ \int_{R_0} \left( e^{itu} - 1 - \frac{itu}{1 + u^2} \right) \frac{1}{u} k_m(u) du \right] \]

**PROOF.** From (5) and (6) we have

\[ \log \phi(t) = \log \xi \left( \frac{t^2}{2} \right) = - \int_0^\infty g_m(2u) \log \left( 1 + \frac{t^2}{2u} \right) du. \quad (7) \]

Formulas 9.1.7, 9 in Abramowitz and Stegun [1] tell that as \( x \) tends to 0, \( g_m(x) \) is asymptotically proportional to \( x^{m-1} \) for \( m > 0 \). For \( x \) tending to infinity, formulas 9.2.1, 2 tell that \( g_m(x) \) is asymptotically proportional to \( x^{-1/2} \). Hence \( g_m(2u) \cdot \log(1 + t^2/(2u)) \) is integrable over \( (0, \infty) \) in the sense of the Lebesgue integral.

We have

\[ -\log \left( 1 + \frac{t^2}{2u} \right) = \int_{R_0} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) \frac{\text{sgn} x}{x} e^{-\sqrt{2u} |x|} dx. \]

From the inequality \( |e^{itx} - 1 - itx| \leq \alpha^2/2 \) and from the Fubini theorem we see that

\[ \log \phi(t) = \int_0^\infty g_m(2u)(-1) \log \left( 1 + \frac{t^2}{2u} \right) du \]

\[ = \int_{R_0} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) \frac{\text{sgn} x}{x} \left\{ \int_0^\infty e^{-\sqrt{2u} |x|} g_m(2u) du \right\} dx \]

\[ = \int_{R_0} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) \frac{\text{sgn} x}{x} \left\{ \int_0^\infty e^{-|x| v} g_m(v^2) dv \right\} dx. \quad (8) \]

In what follows, the equality (8) is justified. If \( 0 < |x| \leq 1 \) we obtain

\[ 0 < \int_0^\infty e^{-|x| v} g_m(v^2) dv < \int_0^1 g_m(v^2) dv + M_1 \int_1^\infty e^{-|x| v} dv \]

\[ = M_0 + M_1 \frac{1}{|x|} e^{-|x|}, \quad (9) \]

where
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\[ M_0 = \int_0^1 g_m(v^2)vdv, \quad M_1 = \sup \{ g_m(v^2)v; 1 \leq v < \infty \}. \]

If \( |x| \geq 1 \) we obtain

\[
0 < \int_0^\infty e^{-|x|v}g_m(v^2)vdv \\
= \int_0^1 e^{-|x|v}v^{2m-1}(v^{1-2m}g_m(v))dv + \int_1^\infty e^{-|x|v}g_m(v^2)vdv \\
< M_2 \int_0^1 e^{-|x|v}v^{2m-1}dv + M_1 \int_1^\infty e^{-|x|v}dv \\
\leq M_3 \frac{1}{|x|^{2m}} + M_1 \frac{1}{|x|} e^{-|x|}, \tag{10}
\]

where

\[
M_2 = \sup \{ v^{2-2m}g_m(v^2); 0 < v \leq 1 \}, \quad M_3 = M_2 \int_0^\infty e^{-x}x^{2m-1}dx.
\]

By (9) and (10) we see that

\[
\left| \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{\text{sgn } x}{x} \right| \int_0^\infty e^{-|x|v}g_m(v^2)vdv \\
\leq \left| \left( e^{itx} - 1 - itx \right) + \frac{itx^3}{1+x^2} \right| \frac{1}{|x|} \left( M_0 + M_1 \frac{1}{|x|} e^{-|x|} \right) \leq \left( |t| + \frac{t^2}{2} \right) (M_0 + M_1)
\]

for \( 0 < |x| \leq 1 \) and

\[
\left| \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{\text{sgn } x}{x} \right| \int_0^\infty e^{-|x|v}g_m(v^2)vdv \\
\leq (2 + |t|) \frac{1}{|x|} \left( M_3 \frac{1}{|x|^{2m}} + M_1 \frac{1}{|x|} e^{-|x|} \right)
\]

for \( |x| \geq 1 \). By these facts we can apply the Fubini theorem and obtain (8), that is

\[
\log \phi(t) = \int_{\mathbb{R}_0} \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1}{x} k_m(x) dx. \tag{11}
\]

The equality (11) is the Lévy representation of the Pearson type VII distribution.

Q. E. D.

RESULT 2. The Lévy representation of the Student t-distribution is as follows:

\[
\int_{-\infty}^{\infty} e^{itx} \frac{\Gamma((n+1)/2)}{\sqrt{n\pi} \Gamma(n/2)} \frac{1}{(1+x^2/n)^{(n+1)/2}} dx \\
= \exp \left[ \int_{\mathbb{R}_0} \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1}{x} k_{n/2} \left( \frac{x}{\sqrt{n}} \right) dx \right].
\]
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**PROOF.** By changes of variables and by the fact that $k_{n/2}(u)$ is an odd function we see that

$$
\int_{-\infty}^{\infty} e^{itx} \frac{\Gamma((n+1)/2)}{\sqrt{n\pi \Gamma(n/2)}} \frac{1}{(1+x^2/n)^{(n+1)/2}} \, dx
$$

$$
= \int_{-\infty}^{\infty} e^{iy^2} \frac{\Gamma((n+1)/2)}{\sqrt{n\pi \Gamma(n/2)}} \frac{1}{(1+y^2/n)^{(n+1)/2}} \, dy
$$

$$
= \exp \left[ \int_{R_0} \left( e^{i\sqrt{n} tu} - 1 - \frac{i\sqrt{n} tu}{1+u^2} \right) \frac{1}{u} \, k_{n/2}(u) \, du \right]
$$

$$
= \exp \left[ \int_{R_0} \left( e^{itv} - 1 - \frac{itv}{1+v^2/n} \right) \frac{1}{v} \, k_{n/2} \left( \frac{v}{\sqrt{n}} \right) \, dv \right]
$$

$$
= \exp \left[ \int_{R_0} \left( e^{itv} - 1 - \frac{itv}{1+v^2/n} \right) \frac{1}{v} \, k_{n/2} \left( \frac{v}{\sqrt{n}} \right) \, dv \right]. \quad \text{Q. E. D.}
$$

3. **On an extension of the equality (2).**

For any $\varepsilon > 0$, suppose that $f(x-u)k_m(u)$ is integrable with respect to the variable $u$ over $R - (-\varepsilon, \varepsilon)$ for almost all $x$ and let

$$
T_\varepsilon f(x) = \int_{|u| \leq \varepsilon} f(x-u)k_m(u) \, du.
$$

**RESULT 3.** If $m > 0$,

$$
xf_m(x) = \lim_{\varepsilon \to 0} T_\varepsilon f_m(x) \quad (12)
$$

holds for all $x$ in the pointwise convergence. If $m > \frac{1}{2}$ and $p \geq 1$, the equality (12) also holds in the $L^p$ norm convergence.

**PROOF.** From the result 1 and from Sato & Yamazato’s equation [10. (2.1)] we obtain

$$
xc_m f_m(x) = \int_{R_0} \left( c_m f_m(x-u) - c_m f_m(x) - \frac{1}{1+u^2} \right) k_m(u) \, du.
$$

We see that

$$
xf_m(x) = \int_{0 < |u| < \varepsilon} \left( f_m(x-u) - f_m(x) + f_m(x) \frac{u^2}{1+u^2} \right) k_m(u) \, du
$$

$$
+ \int_{|u| \geq \varepsilon} \left( f_m(x-u) - f_m(x) \frac{1}{1+u^2} \right) k_m(u) \, du. \quad (13)
$$

$|f_m(x-u) - f_m(x)| \leq (2m+1)|u|$ holds and $|f_m'(x)|$, $f_m(x)$ are bounded and $|uk_m(u)|$ is bounded on $0 < |u| \leq 1$. Hence we see that (the first term of the right hand side of (13))
From (9) and (10) we obtain

\[(\text{the second term of the right hand side of (13)}) = \int_{|u| \geq \varepsilon} f_m(x-u)k_m(u)du \]

\[ = T_\varepsilon f_m(x) \]

and hence

\[ xf_m(x) = \lim_{\varepsilon \to 0} T_\varepsilon f_m(x) \]

for all \( x \) in the pointwise convergence. Next, when \( m > \frac{1}{2} \) and \( p \geq 1 \), let us show that as \( \varepsilon \to +0 \) \( T_\varepsilon f_m(x) \) converges in \( L^p \) norm. By the Minkowski inequality we see that

\[
\left( \int_{-\infty}^{\infty} |T_\varepsilon f_m(x) - xf_m(x)|^p dx \right)^{1/p} \\
\leq \left\{ \int_{-\infty}^{\infty} \left( \int_{0 < |u| < \varepsilon} |f_m(x-u) - f_m(x)| |k_m(u)| du \right)^p dx \right\}^{1/p} \\
+ \left\{ \int_{-\infty}^{\infty} \left( \int_{0 < |u| < \varepsilon} f_m(x) \frac{u^2}{1+u^2} |k_m(u)| du \right)^p dx \right\}^{1/p}.
\]

Since \( |uk_m(u)| \) is bounded on \( 0 < |u| \leq 1 \), the second term of the right hand side of (14) tends to zero as \( \varepsilon \to +0 \). By the Hölder inequality we see that (the first term of the right hand side of (14))

\[
= \left\{ \int_{-\infty}^{\infty} \left( \int_{0 < |u| < \varepsilon} |uk_m(u)| |f'_m(x-\theta u)| du \right)^p dx \right\}^{1/p} \quad (0 < \theta < 1) \\
\leq \left( \int_{0 < |u| < \varepsilon} |uk_m(u)|^q du \right)^{1/q} \left\{ \int_{-\infty}^{\infty} \int_{0 < |u| < \varepsilon} |f'_m(x-\theta u)|^p du dx \right\}^{1/p} \quad (0 < \varepsilon < \frac{1}{2} \text{ and } 0 < |u| < \varepsilon)
\]

When \( 0 < \varepsilon < \frac{1}{2} \) and \( 0 < |u| < \varepsilon \) we have

\[
|f'_m(x-\theta u)| \leq \frac{2m+1}{(1+(x-1/2)^2)^{m+1}} (x > 1) \quad \text{or} \quad \frac{2m+1}{(1+(x+1/2)^2)^{m+1}} (x < -1).
\]

From this inequality and from the fact that \( |uk_m(u)| \) is bounded on \( 0 < |u| \leq 1 \), we see that the last member of (15) tends to zero as \( \varepsilon \to +0 \). Q. E. D.
4. On the singular integral kernel \( k_m(x) \).

By (9), (10) and by the same calculations as (9), (10) we obtain the following

**RESULT 4.** Let \( m > 0 \).

a) \[ |k_m(x)| \leq \frac{C_1}{|x|}, \quad 0 < |x| \leq 1, \]
\[ \leq \frac{C_1}{|x|^{2m}}, \quad 1 < |x|, \]

b) \( k_m(x) \) is an odd function on \( R_0 \),

c) \[ |k_m(x)| \leq \frac{C_2}{|x|^2}, \quad 0 < |x| \leq 1, \]
\[ \leq \frac{C_2}{|x|^{1+2m}}, \quad 1 < |x|, \]

where \( C_1, C_2 \) are constant numbers.

When \( m = 1/2 \), by [16] we have

\[ J_{1/2}(v) = \left( \frac{2}{\pi v} \right)^{1/2} \sin v, \quad Y_{1/2}(v) = J_{-1/2}(v) = \left( \frac{2}{\pi v} \right)^{1/2} \cos v, \quad (v > 0). \]

Hence \( k_{1/2}(x) = \frac{1}{\pi x} \).

From the result 4 and Theorem 2 in [13. p. 35] we obtain the following

**RESULT 5.** If \( m \geq \frac{1}{2} \) and \( p > 1 \), then for \( f \) in \( L^p(R) \)

\[ \| T_{\varepsilon} f \|_p \leq A_p \| f \|_p \]

holds with a constant \( A_p \) independent of \( f \) and \( \varepsilon \), and as \( \varepsilon \to +0 \), \( T_{\varepsilon} f \) converges in \( L^p \) norm.

**References**


