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On the Boundedness of Random Solutions of Nonlinear Stochastic Integral Equations

Hiroshi Onose*

In this note, we consider the stochastic integral equations of the mixed Volterra-Fredholm type. A sufficient condition for the every random solution of the equation to be bounded is given.

1. Introduction. Consider the stochastic integral equations of the mixed Volterra-Fredholm type of the form

\[ x(t; \omega) = f(t; \omega) + \int_0^t a(t, s; \omega)g(x(s; \omega))ds + \int_0^\infty b(t, s; \omega)h(x(s; \omega))ds, \quad t \geq 0, \]

where

(i) \( \omega \in \Omega \), where \( \Omega \) is the supporting set of the probability measure space \((\Omega, A, P)\);
(ii) \( x(t; \omega) \) is the unknown random variable for each \( t \in \mathbb{R}^+ = [0, \infty) \);
(iii) \( f(t; \omega) \) is called the free random variable or stochastic free term defined for each \( t \in \mathbb{R}^+ \);
(iv) \( a(t, s; \omega) \) is a stochastic kernel defined for \( t \) and \( s \) in \( \mathbb{R}^+ \) such that \( 0 \leq s \leq t < \infty \);
(v) \( g(x) \) and \( h(x) \) are the scalar functions defined for \( x \in \mathbb{R} = (-\infty, \infty) \) and \( g, h \in C(-\infty, \infty) \), where \( C \) denotes the class of continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \);
(vi) \( b(t, s; \omega) \) is called a stochastic kernel and is defined for \( t \) and \( s \) in \( \mathbb{R}^+ \).

Stochastic (or random) differential, or integral equations are investigated by many authors (Cf. [2]–[4]). In this paper, assuming existence, we propose a result on the boundedness of solutions of (1).

Here, we consider the following more general stochastic integral equation of the form

\[ x(t; \omega) = f(t; \omega) + \sum_{i=1}^m \left( \int_0^t a_i(t, s; \omega)g_i(x(s; \omega))ds \right) + \sum_{j=1}^n \left( \int_0^\infty b_j(t, s; \omega)h_j(x(s; \omega))ds \right), \quad t \geq 0, \]

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and list the following conditions:

(i)-(iii) as in (1);

(iv-1) \(a_i(t, s; \omega)\) is a stochastic kernel defined for \(t\) and \(s\) in \(\mathbb{R}_+\) such that 
\(0 \leq s \leq t < \infty\) for each \(i = 1, 2, \ldots, m\);

(v-1) \(g_i, h_j \in C(-\infty, \infty)\) for all \(i = 1, 2, \ldots, m\) and \(j = 1, 2, \ldots, n\);

and

(vi-1) \(b_j(t, s; \omega)\) is defined for \(t\) and \(s\) in \(\mathbb{R}_+\), namely for \(0 \leq s, t < \infty\), \(\omega \in \Omega\), and \(b_j(t, s; \omega) = 0\) for \(0 \leq t \leq s < \infty\), for each \(j = 1, 2, \ldots, n\), \(\omega \in \Omega\).

2. Boundedness result.

THEOREM. Suppose (i)-(iii), (iv-1), (v-1), (vi-1),

\((H_1)\) \(f(t; \omega)\) is bounded on \(\mathbb{R}_+ \times \Omega\), with probability one, and

\((H_2)\) for each \(i = 1, 2, \ldots, m,\)

\[
\sup_{t \geq 0} \int_0^t |a_i(t, s; \omega)| ds \leq A_i < \infty,\quad \text{with probability one},
\]

where \(A_i\) is a constant,

and for each \(j = 1, 2, \ldots, n,\)

\[
\sup_{t \geq 0} \int_0^\infty |b_j(t, s; \omega)| ds \leq B_j < \infty,\quad \text{with probability one},
\]

where \(B_j\) is a constant.

Further suppose that

\((H_3)\) for each \(i = 1, 2, \ldots, m,\)

\[
\left( \sup_{t \geq 0} \int_0^t |a_i(t, s; \omega)| ds \right) \lim_{|x| \to \infty} \frac{g_i(x)}{|x|} < \rho_i < 1,\quad \text{with probability one},
\]

where \(\rho_i\) is a constant,

and for each \(j = 1, 2, \ldots, n,\)

\[
\left( \sup_{t \geq 0} \int_0^\infty |b_j(t, s; \omega)| ds \right) \lim_{|x| \to \infty} \frac{h_j(x)}{|x|} < \sigma_j < 1,\quad \text{with probability one},
\]

where \(\sigma_j\) is a constant with \((\sum_{i=1}^{m} \rho_i + \sum_{j=1}^{n} \sigma_j) < 1\). Then, every random solution of (2) is bounded on \(0 \leq t < \infty\), with probability one.

PROOF. Let \(x(t; \omega)\) be a random solution of (2) for \(t \geq 0\). In view of the condition \((H_3)\), we can choose \(K_i, L_j\) being sufficiently large such that for each \(i = 1, 2, \ldots, m,\) with probability one

\[
\left( \sup_{t \geq 0} \int_0^t |a_i(t, s; \omega)| ds \right) |g_i(x(t; \omega))| < \rho_i |x(t; \omega)|,\quad \text{for } |x(t; \omega)| \geq K_i,
\]

and for each \(j = 1, 2, \ldots, n,\) with probability one,

\[
\left( \sup_{t \geq 0} \int_0^\infty |b_j(t, s; \omega)| ds \right) |h_j(x(t; \omega))| < \sigma_j |x(t; \omega)|,\quad \text{for } |x(t; \omega)| > L_j.
\]
Let $K = \max \{K_1, K_2, \ldots, K_m, L_1, L_2, \ldots, L_n\}$. Define

$$I_1 = \{u : |x(u; \omega)| > K\} \quad \text{and} \quad I_2 = \{u : |x(u; \omega)| \leq K, \omega \in \Omega\}.$$

From (H$_1$), there exists $M > 0$ such that $|f(t; \omega)| \leq M$ on $0 \leq t < \infty$, $\omega \in \Omega$. Since $g_i$, for $i=1, 2, \ldots, m$, and $h_j$, for $j=1, 2, \ldots, n$, are continuous, there exist $M_1, M_2 > 0$ such that

$$|g_i(x(u; \omega))| \leq M_1 \quad \text{for all} \quad i = 1, 2, \ldots, m$$

and

$$|h_j(x(u; \omega))| \leq M_2 \quad \text{for all} \quad j = 1, 2, \ldots, n, \quad \text{for} \quad |x(u; \omega)| \leq K.$$

Now from (H$_3$), (3) and (4), we have

$$x(t; \omega) = f(t; \omega) + \sum_{i=1}^m \left( \int_{I_1} a_i(t, s; \omega)g_i(x(s; \omega))ds \right)$$

$$+ \sum_{i=1}^m \left( \int_{I_2} a_i(t, s; \omega)g_i(x(s; \omega))ds \right) + \sum_{j=1}^n \left( \int_{I_1} b_j(t, s; \omega)h_j(x(s; \omega))ds \right)$$

$$+ \sum_{j=1}^n \left( \int_{I_2} b_j(t, s; \omega)h_j(x(s; \omega))ds \right).$$

That is

$$|x(t; \omega)| \leq M + \sum_{i=1}^m \left( \sup_{t \geq 0} \int_0^t |a_i(s, u; \omega)| |x(s; \omega)| ds \right)$$

$$+ M_1 \sum_{i=1}^m \int_{I_2} |a_i(t, s; \omega)| ds + \sum_{j=1}^n \left( \sup_{t \geq 0} \int_0^t |b_j(s, u; \omega)| ds \right)$$

$$|x(s; \omega)| ds + M_2 \sum_{j=1}^n \int_{I_2} |b_j(t, s; \omega)| ds$$

$$\leq M + \left( \sum_{i=1}^m \rho_i \right) \sup_{s \in t_1, s < t} |x(s; \omega)| + M_1 \sum_{i=1}^m A_i + \left( \sum_{j=1}^n \sigma_j \right) \sup_{s \in t_1, s < t} |x(s; \omega)| + M_2 \sum_{j=1}^n B_j,$$

where we use the condition (vi-1). Since this is valid for every $t \geq 0$, we have, with probability one

$$\sup_{0 \leq s \leq T} |x(t; \omega)| \leq N_1 + \left( \sum_{i=1}^m \rho_i \right) + \left( \sum_{j=1}^n \sigma_j \right),$$

where

$$N_1 = M + M_1 \sum_{i=1}^m A_i + M_2 \sum_{j=1}^n B_j,$$

for every $T > 0$. From this and $\left( \sum_{i=1}^m \rho_i \right) + \left( \sum_{j=1}^n \sigma_j \right) < 1$, we see that
Since this bound is independent of $t$, we see that
\[
\sup_{0 \leq t < \infty} |x(t; \omega)| \leq N_2
\]
with probability one and this completes the proof of the theorem.

**Example (Cf. [5], Example 1).** Consider the following random equation
\begin{align*}
(5) \quad x(t; \omega) &= f(t; \omega) + \int_0^t a_1(t, s; \omega) g_1(x(s; \omega)) ds + \int_0^t a_2(t, s; \omega) g_2(x(s; \omega)) ds,
\end{align*}
$t \geq 0$ and $\omega \in \Omega \subset \mathbb{R}_+$, where $f(t; \omega) = 2 \exp(-t/3 - \omega) - \exp(-t/5 - \omega)$, $t \geq 0$, $\omega \in \Omega$, $a_1(t, s; \omega) = \exp(-t + s - 2\omega/3)$, $0 \leq s \leq t < \infty$, $\omega \in \Omega$, $a_2(t, s; \omega) = \exp(-t + s - 4\omega/5)$, $0 \leq s \leq t < \infty$, $\omega \in \Omega$, $g_1(x) = -\frac{4}{3} x^{1/3}$ for $x \in \mathbb{R}$ and $g_2(x) = \frac{4}{5} x^{1/5}$ for $x \in \mathbb{R}$. By Theorem, every solution $x(t; \omega)$ of (5) is bounded, with probability one. In fact, the random function $x(t; \omega) = \exp(-t - \omega)$, $0 \leq t < \infty$ and $\omega \in \Omega$, is a bounded solution of (5).

**Corollary.** Suppose (i)-(v) and (vi-1)$'$

\begin{align*}
& b(t, s; \omega) \text{ is defined for each } t \text{ and } s \text{ in } \mathbb{R}_+, \text{ namely, for } 0 \leq s, t < \infty, \\
& \omega \in \Omega \text{ and } b(t, s; \omega) = 0 \text{ for } 0 \leq t \leq s < \infty, \text{ with probability one.}
\end{align*}

Further suppose that (H4) and

\begin{align*}
& (H_4) \quad \text{with probability one } \sup_{t \geq 0} \int_0^t |a(t, s; \omega)| ds \leq A_0 < \infty, \\
& \text{with probability one } \sup_{t \geq 0} \int_0^\infty |b(t, s; \omega)| ds \leq B_0 < \infty, \\
& \text{with probability one } \left( \sup_{t \geq 0} \int_0^t |a(t, s; \omega)| ds \right) \limsup_{|x| \to \infty} \left| \frac{g(x)}{x} \right| < \rho_1 < 1, \\
& \text{and with probability one } \left( \sup_{t \geq 0} \int_0^t |b(t, s; \omega)| ds \right) \limsup_{|x| \to \infty} \left| \frac{h(x)}{x} \right| < \rho_2 < 1,
\end{align*}

with $0 < \rho_1 + \rho_2 < 1$.

Then every random solution of (1) is bounded on $0 \leq t < \infty$, with probability one.

**Remark.** The Rao's theorem ([5], Theorem 3.1, p. 2 (3.2), p. 403 lines 11 to 13 from the top) seems to be inadequate for the present author, since $s$ may vary beyond $t$ in the case of $b(t, s; \omega)$. Here, we rewright them as the above theorem and corollary.
References


