<table>
<thead>
<tr>
<th>タイトル</th>
<th>Solvability of a certain integral equation and its application</th>
</tr>
</thead>
<tbody>
<tr>
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<td>HORIUCHI, Toshio</td>
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</tr>
</tbody>
</table>

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Solvability of a certain integral equation
and its application

Toshio Horiuchi*

§ 0. Introduction

The purpose of this short memoire is to study the solvability of a certain integral equation and its application. First we shall treat the following integral equation defined on a region $Q$ in the half space $\mathbb{R}^n$, which is very fundamental but involves rather complicate kernel function $h(x, y)$.

\begin{equation}
S(x, y) + h(x, y) + \int_Q h(x, z) S(z, y) dz = 0,
\end{equation}

where $Q$ is a sufficiently small region in $\mathbb{R}^n$ and $h(x, y)$ is a given kernel function belonging to the class $K^{\gamma\delta}(\mathbb{R}^n, \mathbb{R}^n)$ defined in §1. In order to solve (0.1), we shall make use of the Neumann series and verify its absolute convergence for a sufficiently small $Q$.

Secondly, as its application, we shall construct a fundamental solution for the degenerated elliptic operator which was already treated in author’s paper [3]. More precisely, in [3] we treated the operator $A$ defined on a domain $\Omega$ in $\mathbb{R}^n$ which is approximated, near the boundary, by the following simple operator $L_{\alpha}$ in the half space $\mathbb{R}^n$:

\begin{equation}
L_{\alpha} = -x_n \Delta + \alpha \partial x_n,
\end{equation}

where $\alpha$ is a complex parameter with negative real part. And we constructed a fundamental solution for $A$ except for the case $-1 < \text{Re} \alpha < 0$. So we shall give a fundamental solution in this excluded case, as an application. Here we remark that its construction will be reduced to solve (0.1) with respect to $S(x, y)$ for $h(x, y) \in K^{\gamma\delta}(\mathbb{R}^n, \mathbb{R}^n)$ with $\alpha' = -\text{Re} \alpha$.

Our main plan of this paper is as follows. In §1, we describe the general setting of the integral equation (0, 1) and state the main result Theorem 1. In this work, some estimates of the truncated convolution with respect to our kernels in $K^{\gamma\delta}$ are essential. In §2, we prepare these estimates which will be established in §§3 and 4, and we prove Theorem 1 stated in §1. Finally in §5, we mention the applications.

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§1. Results

We begin with the definition of the kernel class $K_{\gamma}^{\alpha}(R^n_+, R^n_+)$ which plays an important role throughout this memoire.

**Definition 1.1.** For $0 < \alpha \leq 1$, $1 - \alpha < \beta < n$ and $0 \leq \gamma$, a function $h(x, y)$ is said to belong to $K_{\gamma}^{\alpha}(R^n_+, R^n_+)$, if it is continuous in $R^n_+ \setminus$ diagonal set, and if, for any compact set $K \subset R^n_+$, there is a positive constant $C(K)$ such that we have

$$\|(x, y) \| \leq C(K) y_n^{\alpha-1} |x - y|^\beta |x_n - y_n|^\gamma, \tag{1.2}$$

for any $x = (x', x_n) \in K$ and $y = (y', y_n) \in K$, where

$$L(x_n, y_n) = \log(2 + |x_n - y_n|^{-1}), \tag{1.3}$$

or equivalently in $K \setminus K$,

$$L(x_n, y_n) = 1 + |\log |x_n - y_n||. \tag{1.3'}$$

We consider the following equation (1.4) with respect to a kernel $S(x, y)$:

$$S(x, y) + h(x, y) + \int_Q h(x, z)S(z, y) dz = 0, \tag{1.4}$$

where $Q$ is a sufficiently small region in $R^n_+$ and $h(x, y)$ is a given kernel function in $K_{\gamma}^{\alpha}(R^n_+, R^n_+)$. If $Q \cap |z_n = 0| = \phi$, there is no difficulty to solve (1.4) for sufficiently small $Q$. So we consider only the case $Q \cap |z_n = 0| = \phi$.

Let us set

$$B(\rho) = |z', z_n| \mid z' < \rho, 0 \leq z_n < \rho, \quad \text{for} \quad \rho > 0. \tag{1.5}$$

Now we can state the main result.

**Theorem 1.** We suppose that a function $h(x, y)$ is in $K_{\gamma}^{\alpha}(R^n_+, R^n_+)$ and $Q = B(\rho)$ with $\rho$ being sufficiently small. Then the integral equation (1.4) has a solution $S(x, y)$ defined in $B(\rho) \times B(\rho) -$ diagonal set. Moreover, $S(x, y)$ is continuous in $B(\rho) \times B(\rho) -$ diagonal set and satisfies the following estimate,

$$|S(x, y)| \leq C(\rho) y_n^{\alpha-1} |x - y|^\beta |x_n - y_n|^\gamma, \tag{1.6}$$

where $(x, y) \in B(\rho) \times B(\rho)$, $C(\rho)$ is a positive constant depending on $\rho$, and $\gamma$ is also a non-negative constant depending on $(\alpha, \beta, \gamma)$.

This theorem is verified in §2 by making use of the Neumann series. Here we note that this theorem is a local version of theorem 3 in [3] for the case $-1 < Re \alpha < 0$, see §5.
Proof of Theorem 1

In this section we prove Theorem 1 by virtue of some detailed propositions, which will be established in §§3 and 4.

As is well known, one of solution kernels of the equation (1.4) will be given by the following formal series, if it makes sense.

\[ S(x, y) = \sum_{j=0}^{\infty} (-1)^j h^j(x, y), \]

where \( h^0(x, y) = h(x, y) \) and \( h^{j+1}(x, y) = \int_0^x h(x, z)h^j(z, y)dz \), inductively.

In order to verify this idea, we shall study the truncated convolution with respect to our kernels in \( K_{\gamma^0}^{\alpha, \beta} \).

**Definition 2.** For \( h(x, y) \in K_{\gamma^0}^{\alpha, \beta}(\mathbb{R}_+^n, \mathbb{R}_+^n) \), and \( h'(x, y) \in K_{\gamma'}^{\alpha', \beta'}(\mathbb{R}_+^n, \mathbb{R}_+^n) \), we define

\[ h \# h'(\rho)(x, y) = \int_{B(\rho)} h(x, z)h'(z, y)dz, \]

where \( B(\rho) = \{ (z, z_n) \mid |z'| < \rho, 0 \leq z_n < \rho \} \).

Then we have the following proposition.

**Proposition 1.** We suppose \( h(x, y) \in K_{\gamma^0}^{\alpha, \beta}(\mathbb{R}_+^n, \mathbb{R}_+^n) \) and \( h'(x, y) \in K_{\gamma'}^{\alpha', \beta'}(\mathbb{R}_+^n, \mathbb{R}_+^n) \). Moreover we suppose

\[ 1 - \alpha < \min(\beta, \beta') \quad \text{and} \quad \beta + \beta' < n + 1. \]

Then we have, for any \( (x, y) \in B(\rho) \times B(\rho) \),

\[ |h \# h'(\rho)(x, y)| \leq \left\{ \begin{array}{ll} C(\rho)y_{n-1}^{\beta + \beta' + a - \frac{\alpha}{2} + \frac{n}{2}}L(x_n, y_n)^{\gamma + \gamma'}, & \text{if} \ \beta + \beta' + a < n + 1, \\ C(\rho)y_{n-1}^{\beta + \beta' + a}L(x_n, y_n)^{\gamma + \gamma'}, & \text{if} \ \beta + \beta' + a \geq n + 1, \end{array} \right. \]

where \( C(\rho) \) is a positive constant depending on \( \rho, \alpha, \beta, \beta', \gamma \) and \( \gamma' \).

**Remark 2.** If \( (\beta - 1)(\beta' - 1) \neq 0 \), we can replace \( L(x_n, y_n)^{\gamma + \gamma'} \) and \( L(x_n, y_n)^{\gamma + \gamma'} \) by \( L(x_n, y_n)^{\gamma + \gamma'} \) and \( L(x_n, y_n)^{\gamma + \gamma'} \) respectively.

If \( \beta + \beta' \geq n + 1 \) holds, we have the following.

**Proposition 2.** Let \( \rho_0 \) be any positive constant. We suppose \( 0 < \rho \leq \rho_0 \), \( h(x, y) \in K_{\gamma^0}^{\alpha, \beta}(\mathbb{R}_+^n, \mathbb{R}_+^n) \) and \( h'(x, y) \in K_{\gamma'}^{\alpha', \beta'}(\mathbb{R}_+^n, \mathbb{R}_+^n) \). Moreover we suppose
(2.6) \[ 1 - a < \min(\beta, \beta') \quad \text{and} \quad \beta + \beta' \geq n + 1. \]

Then we have, for any \((x, y) \in B(\rho) \times B(\rho),\)

(2.7) \[ |h \# h'_{\rho, \rho}(x, y)| \leq C y_n^{a-1} \rho^{a+\beta} (1 + |\log \rho|^{\gamma+\gamma+1}), \]

where \(C\) is a positive constant depending only on \((a, \beta, \gamma, a', \beta', \gamma', \rho_0)\) and independent of \(\rho.\)

Next we consider the iteration of \(h(x, y)\) with itself.

**Definition 2.8.** For \(h(x, y) \in K^{C_0}_{\gamma}(\mathbb{R}_n, \mathbb{R}_n),\) we define \(h^{(p)}(x, y)\) for \(p \geq 1\) as follows.

(2.9) \[ h^{(1)}(x, y) = h(x, y) \quad \text{and for} \quad p \geq 2 \]

\[ h^{(p)}(x, y) = \int h(x, z_1) h(z_1, z_2) \cdots h(z_{p-1}, y) dz_1 \cdots dz_{p-1}. \]

Then we have the followings immediately from the above propositions, that is to say:

**Proposition 3.** For \(h(x, y) \in K^{C_0}_{\gamma}(\mathbb{R}_n, \mathbb{R}_n)\) and \(p \beta + (p-1)(a-1) \leq n,\) we have for \((x, y) \in B(\rho) \times B(\rho),\)

(2.10) \[ |h^{(p)}(x, y)| \]

\[ \leq \begin{cases} C(\rho) y_n^{a-1} |x-y|^{p \beta + (p-1)(a-1) - n} L(x_n, y_n)^{\gamma+2}, & \text{if} \ p \beta + (p-1)(a-1) n) \\ C(\rho) y_n^{a-1} L(x_n, y_n)^{\gamma+2+1}, & \text{if} \ p \beta + (p-1)(a-1) = n \end{cases} \]

Here \(C(\rho)\) is a positive constant depending on \(\rho.\)

**Proposition 4.** Let \(p_0\) be the smallest number satisfying \(p_0 \beta + (p_0 - 1)(a-1) > n.\) For \(h(x, y) \in K^{C_0}_{\gamma}(\mathbb{R}_n, \mathbb{R}_n),\) we have for \((x, y) \in B(\rho) \times B(\rho),\)

(2.11) \[ |h^{(p_0)}(x, y)| \]

\[ \leq \begin{cases} C \rho^{p_0} y_n^{a-1} \rho^{p_0 \beta + (p_0-1)(a-1) - \gamma - n} (|\log \rho|^{\gamma+2} + 1), & \text{if} \ (p_0-1)\beta + (p_0-2)(a-1) < n) \\ C \rho^{p_0} y_n^{a-1} \rho^{p_0 \beta + (p_0-1)(a-1) - \gamma - n} (|\log \rho|^{\gamma+2+1} + 1), & \text{if} \ (p_0-1)\beta + (p_0-2)(a-1) = n \end{cases} \]

And for \(k \geq 1,\) we have for \((x, y) \in B(\rho) \times B(\rho),\)

(2.12) \[ |h^{(p_0+k)}(x, y)| \leq C \rho^{p_0} y_n^{a-1} \rho^{p_0 \beta + (p_0-1)(a-1) - \gamma - n} \]
Here $C_0$ is a positive constant independent of $\rho$, and $T$ is also a positive constant.

Under these preparations, we can prove Theorem I stated in §1. Now the proof seems to be almost obvious, that is to say:

**Proof of Theorem 1.** From Propositions 3 and 4, if we take $\rho$ so that

\[(2.13) \quad C_0\rho^{a+\delta-1}(|\log \rho|^\gamma+1)<1,\]

then the Neumann series (2.1) is absolutely convergent and satisfies the equation (1.4).

Q.E.D.

**Remark.** If $Q$ is not bounded, the class $\mathcal{K}_C^{a,\beta}$ is too wide to solve the equation (1.4) on the whole region $Q$. So we have to impose somewhat more assumptions on a kernel $h(x,y) \in \mathcal{K}_C^{a,\beta}$. For example, the following subclass $\mathcal{K}_C^{a,\beta,\delta}$ of $\mathcal{K}_C^{a,\beta}$ is appropriate to this case.

**Definition 2.14.** For $\delta \geq 0$, a function $h(x,y) \in \mathcal{K}_C^{a,\beta}$ is said to belong to $\mathcal{K}_C^{a,\beta,\delta}$, if there is a constant $C$ such that we have

\[(2.15) \quad |h(x,y)| \leq C |x-y|^{a-n}L(x_n,y_n)^{\gamma}\exp(-\delta|x-y|)\]

for any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Then, roughly speaking, if $\delta$ is sufficiently large, we can show the absolute convergence of (2.1).

§ 3. Proof of Proposition 1

In this section, we shall prove Proposition 1 step by step. To do so, we first prepare the following auxiliary function.

**Definition 3.1.** For $\beta>0, \beta'>0$, $a>0$, $b>0$ and $w' \in \mathbb{R}^{n-1}$, we set

\[(3.2) \quad J(a, b, w', \beta, \beta') = \int_{\mathbb{R}^{n-1}} (a^2 + |w'-z'|^{2(\beta-n)/2})\]

\[\times (b^{\beta} + |z'|^{2(\beta-n)/2}) \, dz'.\]

**Lemma 3.1.** For $\beta+\beta'<n+1$, we have

\[(3.3) \quad J(a, b, w', \beta, \beta') = x(\beta, \beta') \int_0^1 v^{a-3/2}(1-v)^{\beta-3/2}\]

\[\times \left[ |w'|^2 + a^2/v + b^2/(1-v) \right]^{a+n-1/2} \, dv,\]

where $x(\beta, \beta') = \pi^{1-n/2} \Gamma\left(\frac{n-\beta}{2}\right)\Gamma\left(\frac{n-\beta'}{2}\right)\Gamma\left(\frac{n+1-\beta-\beta'}{2}\right).$
**Proof of Lemma 3.1.** We make use of the heat kernel $H(t, R)$ defined by

$$H(t, R) = (4\pi t)^{\frac{3-n}{2}} \exp\left(-\frac{R^2}{4t}\right).$$

Then we can show the relation

$$\int_0^\infty H(t, R) t^{\frac{\alpha-3}{2}} dt = C_\alpha R^{\alpha-n},$$

where $C_\alpha = \pi^{\frac{3-n}{2}} 2^{1-\alpha} \Gamma\left(\frac{n-\beta}{2}\right)$.

Using (3.5) for $R = (a^2 + |w'-z'|^2)^{1/2}$ and $R = (b^2 + |z'|^2)^{1/2}$, we have

$$J(a, b, \omega', \beta, \beta') = (C_\alpha C_\beta)^{-1} \int_0^\infty \int_0^\infty s^{\frac{\alpha-3}{2}} t^{\frac{\alpha-3}{2}} dsdt$$

$$\times \int_{\mathbb{R}^n} H(s, (a^2 + |w'-z'|^2)^{1/2})$$

$$\times H(t, (b^2 + |z'|^2)^{1/2}) dz'.$$

We can compute the last integral and rewrite (3.6) to obtain

$$J(a, b, \omega', \beta, \beta') = (4\pi)^{\frac{3-n}{2}} (C_\alpha C_\beta)^{-1}$$

$$\int_0^\infty \int_0^\infty s^{\frac{\alpha-3}{2}} t^{\frac{\alpha-3}{2}} (s+t)^{1-n/2}$$

$$\times \exp\left[-a^2/s + b^2/t + |\omega'|^2/(s+t)/4\right] dsdt.$$

Carrying out a change of variables defined by

$s + t = u$  and  $s = uv$

and integrating with respect to $u$, we get (3.3).  Q.E.D.

By this lemma and (1.2), we have

$$|h \# h_{m,\omega}(x, y)| \leq C y_n^{-1} \int_0^\infty z_n^{-1} J(a, b, \omega', \beta, \beta')$$

$$\times \left[L(x_n, z_n) + L(z_n, y_n)\right]^{\gamma + \gamma} dz_n,$$

where $a = |x_n - z_n|$, $b = |z_n - y_n|$, $\omega = x' - y'$ and $C$ is a positive constant independent of $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

In order to estimate (3.8), we begin with the majoration of $J(a, b, \omega', \beta, \beta')$. Without loss of generality, we may assume $\beta \geq \beta'$, and we classify $J$ by the values of $\beta$ and $\beta'$, that is to say:

$$\begin{cases}
\text{Case 1, } & \beta \geq \beta' \geq 1. \\
\text{Case 2, } & \beta \geq 1 > \beta'. \\
\text{Case 3, } & 1 > \beta \geq \beta'.
\end{cases}$$
Then we can show the following majorations.

**Lemma 3.2.** For $\beta + \beta' < n + 1$, we have the following majorations.

\[
\begin{align*}
\text{Case 1,} & \quad J \leq C_1 Q^{\alpha+\beta - n - 1/2} \left[ \log \frac{Q}{(ab)} + 1 \right]. \\
\text{Case 2,} & \quad J \leq C_2 Q^{\alpha-n/2} b^{\beta'} \left[ \log \left( \frac{Q}{a^2} \right) + 1 \right]. \\
\text{Case 3,} & \quad J \leq C_3 \max \left\{ Q^{\alpha-n/2} b^{\beta - 1}, \ Q^{\beta-n/2} a^{\alpha - 1} \right\}.
\end{align*}
\]

Here $J = J(a, b, w', \beta, \beta')$, $Q = |w'|^2 + 2(a^2 + b^2)$ and $C_i (i = 1, 2, 3)$ is a positive constant independent of $(x, y)$. Here we note that we can neglect the logarithmic terms if $(\beta - 1)(\beta' - 1) \neq 1$.

To prove this lemma, we prepare the following.

Let us set, for $p > 0$, $0 \leq r \leq 1$, $s \geq 0$ and $q, a, b \in \mathbb{R}$,

\[
G(R, p, q, r, s) = \int_0^R t^{p-1}(t+1)^s |t-a|^{r-1} \log |t-b| t^s dt,
\]

where $|a| \leq 1$, $|b| \leq 1$, $p+r > 1$ and $R > 0$.

Then we immediately have the majoration as follows.

**Lemma 3.3.** $G(R, p, q, r, s)$ defined by (3.11) is majorated by

\[
C |R/(R+1)|^{p+r-1} \left[ \log (2+1/R)^{q} + (1+R)^{p+q+r-1} \log (2+R)^s \right],
\]

where $s^* = s$ for $p+q+r \neq 1$ and $s^* = s+1$ for $p+q+r = 1$, and $C$ is a positive constant independent of $R$.

**Proof.** It suffices to consider the case where $a = b = 0$, since $0 \leq r \leq 1$ and $|a|, |b| \leq 1$ hold. Both of the following inequalities are valid:

\[
\begin{align*}
G(R, p, q, r, s) & \leq C R^{q} \log (1+1/R)^s, \quad \text{if} \quad 0 < R \leq 2, \\
G(R, p, q, r, s) & \leq C |1+R|^{p+q+r-1} \log R, \quad \text{if} \quad 2 < R.
\end{align*}
\]

**Proof of Lemma 3.2.** Case 1, $(\beta \geq \beta' \geq 1)$. We set

\[
\chi(\beta, \beta')^{-1} J = \int_0^{1/2} + \int_{1/2}^1 = J_1 + J_2.
\]

For $J_1$, we have

\[
J_1 \leq C \int_a^1 \nu^{\alpha-3/2} \left| w' \right|^2 + b^2 + a^2 / \nu^{\beta+n-1/2} d\nu
\]

\[
= C \int_a^1 \nu^{n-\beta} |a^2 + \nu(b^2 + |w'|^2)\nu^{\alpha+n-1/2} d\nu.
\]

From Lemma 3.3, we can show
Similarly we have
\[ J_1 \leq C Q^{\sigma + \rho^{n-1}/2} [1 + \log (Q/a^3)]. \]

Thus the required inequality (3.10) in the case 1 follows. Here we note that the logarithmic factor \("\log (2 + R)\)" in Lemma 3.3 can be neglected if \(s = 0, r = 1\) and \(p + q \neq 0\). Hence we can also neglect the logarithmic factor in the above majorations of \(J_1\) and \(J_2\) if \((\beta - 1)(\beta' - 1) \neq 0\). By this remark, we can show the remaining inequalities (3.10) in the cases 2 and 3 similarly. Q.E.D.

Under these preparations, we shall estimate (3.8) itself from now on. For the sake of simplicity we use the following notations;
\begin{equation}
(3.13) \quad L = (x_n + y_n)/2, \quad M = |x - y|/2 \quad \text{and} \quad N = (x_n - y_n)/2.
\end{equation}

Now we carry out a change of variables defined by
\begin{equation}
(3.14) \quad z_n = L(t + 1),
\end{equation}
and making use of the relation
\begin{equation}
(3.15) \quad Q = |w|^2 + 2(a^2 + b^2) = 4(M^2 + (z_n - L)^2),
\end{equation}
we have
\begin{equation}
(3.16) \quad |h \# h_{\text{mod}}(x, y)| \leq C \int_{\tau}^{\rho/L-1} f(t, x_n, y_n) dt,
\end{equation}
where \(f(t, x_n, y_n)\) is defined by
\begin{equation}
(3.17) \quad f(t, x_n, y_n) = g_n^{-1}(1 + t)^{\sigma - 1} g(t, x_n, y_n)
\end{equation}
\begin{equation}
\times \begin{cases}
(L|t| + M)^{\beta + \rho^{n-1}}, & \text{Case 1.} \\
(L|t| + M)^{\rho^{n-1}} |t + \tau|^{\sigma - 1} L^{ho^{-1}}, & \text{Case 2.} \\
\max [(L|t| + M)^{\rho^{n-1}} |t - \tau|^{\sigma - 1} L^{ho^{-1}}], & \text{Case 3.}
\end{cases}
\end{equation}
And
\[ g(t, x_n, y_n) = [L(x_n, y_n) + |\log t + \tau| + |\log t - \tau|]^{\gamma + \gamma'}, \]
here
\[ \tau = N/L. \]

**Remark.** If \((\beta - 1)\beta' - 1 \neq 0\), we can replace \(\gamma + \gamma' + 1\) in the definition of \(g\) by \(\gamma + \gamma'\).

In order to estimate the integral, we divide the interval \([-1, \rho/L-1]\) into 4-parts, that is to say;
\[
\int_{-1}^{\rho/L-1} f dt = \int_{-1}^{-1/2} + \int_{-1/2}^{0} + \int_{0}^{1/2} + \int_{1/2}^{\rho/L-1} = I_1 + I_2 + I_3 + I_4.
\]
(Here, we may assume \(L < \rho/3\).)

First we estimate \(I_1\): Since \(-1 \leq t \leq -1/2\), we immediately have
\[
(3.18) \quad f(t, x_n, y_n) \leq C(\rho) y_n^{\gamma'-1} L^{(1 + t)^{\beta'-1}} g(t, x_n, y_n)
\]
\[
\times \begin{cases}
(L + M)^{\beta + \beta' - 1 - n}, & \text{Case 1.} \\
(L + M)^{\beta - n} L^{\beta - 1} |t + \tau|^{\beta - 1}, & \text{Case 2.} \\
\text{Max}[(L + M)^{\beta - n} L^{\beta - 1} |t + \tau|^{\beta - 1}, & \text{Case 3.} \\
(L + M)^{\beta - n} L^{\beta - 1} |t - \tau|^{\beta - 1}].
\end{cases}
\]
Therefore we can show, \(|\tau|\) being smaller than 1,
\[
(3.19) \quad |I_1| \leq C(\rho) y_n^{\gamma'-1} (L + M)^{\beta + \beta' - 1 - n}.
\]

Secondly we estimate \(I_2\): Since \(-1/2 \leq t \leq 0\), we can neglect the factor \((t + 1)^{\beta'-1}\), then carrying out a change of variables defined by
\[
(3.20) \quad t = -u M/L,
\]
\[
(3.21) \quad I_2 = \int_{0}^{1/2M} F(u, x_n, y_n) du.
\]
After long computations, it follows from Lemma 3.3 that the right-hand side of (3.21) is majorated by
\[
(3.22) \quad C(\rho) y_n^{\gamma'-1} L^{\beta - 1} M^{\beta + \beta' - n}
\]
\[
\left[ G(L/(2M), 1, \beta + \beta' - n - 1, 1, \gamma + \gamma' + 1) \right.
\]
\[
\quad + G(L/(2M), 1, \beta + \beta' - n - 1, 1, 0) L(x_n, y_n)^{\gamma + \gamma' + 1}, & \text{Case 1.}
\]
\[
G(L/(2M), 1, \beta - n, \beta', \gamma + \gamma' + 1)
\]
\[
\left. + G(L/(2M), 1, \beta - n, \beta', 1, 0) L(x_n, y_n)^{\gamma + \gamma' + 1}, & \text{Case 2.}
\right]
Case 3.

Similarly we have

\[ I_3 \leq C(\rho) y_n^{\alpha-1} M^{\beta+\gamma+\alpha-1} L(x_n, y_n)^{\gamma+\alpha+1}. \]

Lastly we estimate \( I_1 \). (We assume \( L < \rho / 3 \).) Since \( t \geq 2 \), we substitute \((t+1)^{\alpha-1}\) for \( t^{\alpha-1} \), and carrying out a change of variables defined by

\[ t = uM/L, \]

we have

\[ I_1 = \int_t^{\rho/L-1} f(t, x_n, y_n) dt = \text{Const.} \int_{2t/M}^{\rho/\alpha-1} F(u, x_n, y_n) du. \]

\[ \leq \text{Const.} \int_0^{\rho/\alpha} F(u, x_n, y_n) du, \]

where \( F(u, x_n, y_n) \) satisfies the following:

\[ F(u, x_n, y_n) = y_n^{\alpha-1} M^{\beta+\gamma+\alpha-1} g(uL/M, x_n, y_n) \]

\[ \times \begin{cases} u^{\alpha-1}(1+u)^{\alpha-\gamma-1}, & \text{Case 1.} \\ u^{\alpha-1}(1+u)^{\alpha-1} |u+\tilde{\tau}|^{\beta-1}, & \text{Case 2.} \\ \text{Max}[u^{\alpha-1}(1+u)^{\alpha-1} |u+\tilde{\tau}|^{\beta-1}, u^{\alpha-1}(1+u)^{\alpha-1} |u-\tilde{\tau}|^{\beta-1}], & \text{Case 3.} \end{cases} \]

And

\[ g(uL/M, x_n, y_n) \leq C(\rho) L(x_n, y_n) + \log |u+\tilde{\tau}| + \log |u-\tilde{\tau}| |^{\gamma+\alpha+1}, \]

where \( \tilde{\tau} = N/M \).

Again by Lemma 3.3, we have

\[ I_1 \leq C(\rho) y_n^{\alpha-1} M^{\beta+\gamma+\alpha-1} \]

\[ \times \begin{cases} G(\rho/M, \alpha, \beta+\gamma+\alpha-1, 1, \gamma+\alpha+1) \\ + G(\rho/M, \alpha, \beta+\gamma+\alpha-1, 1, 0) L(x_n, y_n)^{\gamma+\alpha+1}, & \text{Case 1.} \end{cases} \]
Case 2.

$$\text{Case 3.}$$

After all, adding (3.19), (3.22), (3.23) and (3.26), we obtain the desired estimate.

Q.E.D.

§4. **Proof of Proposition 2**

We begin with the following elementary lemma, where the proof is omitted.

**Lemma 4.1.** We have, for $$\rho > 0$$ and $$r > 0$$,

$$\int_{|x' - x| < r} |x - z|^\beta \, dz' \leq C(\beta) \left\{ \begin{array}{ll}
r^{\beta - 1}, & \beta > 1, \\
\log (1 + r/|x_n - z_n|), & \beta = 1, \\
|x_n - z_n|^{\beta - 1}, & 0 < \beta < 1. \end{array} \right.$$

Here $$(x', x_n), z=(z', z_n)$$ and $$C(\beta)$$ is a positive constant independent of $$r$$.

Let us set

$$B(x, r)=|(z', z_n)| \cdot |z' - x'| < r, |z_n - x_n| < r \text{ and } z_n \geq 0|$$

We note that we may suppose $$x \neq y$$ in estimating $$h \# h_{\#}(x, y)$$ with $$\beta + \beta' \geq n + 1$$, if not, we may replace $$B(\rho)$$ by $$B(x, 2\rho)$$ and use the same argument as below.

We set

$$|x - y| = 2r > 0,$$

and divide $$B(\rho)$$ into 3-regions as follows:

$$B(\rho)=B_1 \cup B_2 \cup B_3,$$

where
\[ B_i = \{(z', z_n) | |z - x| < r \cap B(\rho) \}, \]
\[ B_i = \{(z', z_n) | |z - y| < r \cap B(\rho) \}, \]
and
\[ B_i = B(\rho) - B_1 \cup B_2. \]

In \( B_i \), we have

\[
| h \# h'_n(x, y) | \leq C \text{Min} \{ r^{\sigma - n}, (2\rho)^{\sigma - n} | y_n^{\sigma - 1} \}
\times \int_{\mathcal{M}} z_n^{\sigma - 1} |x - z|^{\sigma - n} \mathcal{L}(x_n, z_n)^\gamma \mathcal{L}(x_n, y_n)^\gamma dz
\]
\[
\leq C \text{Min} \{ r^{\sigma - n}, (2\rho)^{\sigma - n} | y_n^{\sigma - 1} \}
\times \int_{z_n - x_n < r} z_n^{\sigma - 1} |\mathcal{L}(x_n, z_n)^\gamma + \mathcal{L}(z_n, y_n)^\gamma| dz_n
\]
\[
\times \int_{|z_n - x_n| < r} |x - z|^{\sigma - n} dz', \quad \text{(lemma 4.1)}
\]
\[
\leq C \text{Min} \{ r^{\sigma - n}, (2\rho)^{\sigma - n} | y_n^{\sigma - n} \}
\times \int_{z_n - x_n < r} z_n^{\sigma - 1} |\mathcal{L}(x_n, z_n)^\gamma + \mathcal{L}(z_n, y_n)^\gamma| dz_n
\]
\[
\times \left[ \begin{array}{c}
(1 + r/|x_n - z_n|) \\
\log (1 + r/|x_n - z_n|)
\end{array} \right] dz_n.
\]

Here we note that

\[
\log (1 + r/|x_n - z_n|) \leq C(\rho)(1 + |\log |x_n - z_n||) \quad \text{and}
\]
\[
\int_{z_n - x_n < r} z_n^{\sigma - 1} |x_n - z_n|^{\sigma - 1} |\log |y_n - z_n|| \gamma dz_n
\]
\[
\leq G(2\rho, \alpha, \beta, 0, \gamma) \quad \text{(by lemma 3.3)}
\]
\[
\leq \text{Const. } \rho^{\alpha + \beta - 1}(\log \rho)^{\gamma + 1}.
\]

(Here we may assume \( \rho < 1/2 \) without loss of generality.)

Thus we have

\[
| h \# h'_n(x, y) | \leq \text{Const. } y_n^{\sigma - 1} \rho^{\alpha + \beta - n - 1}(\log \rho)^{\gamma + \gamma + 1}.
\]

We also have

\[
| h \# h'_n(x, y) | \leq \text{Const. } y_n^{\sigma - 1} \rho^{\alpha + \beta - n - 1}(\log \rho)^{\gamma + \gamma + 1}.
\]

As for \( B_3 \), we note
A Certain Integral Equation

\[ r \leq |z - y| \leq \text{Min}(2\rho, 3|x - z|). \]

Hence we have

\[ |h \# h_n(x, y)| \leq \text{Const. } y_n^{\sigma - 1} \int_{|z_n| < \rho} \int_{|y_n - z_n| < \delta} |z - y|^\sigma |z_n - y_n|^{\sigma - n} |\log (1 + \rho/|y_n - z_n|)|, \]

using (4.9) and an easy variant of Lemma 4.1,

\[ \leq \text{Const. } y_n^{\sigma - 1} \int_0^\rho \int_{|z_n| < \rho} \int_{|y_n - z_n| < \delta} |\log (1 + \rho/|y_n - z_n|)|, \]

again by Lemma 3.3,

\[ \leq \text{Const. } y_n^{\sigma - 1} \rho^{\sigma - \delta - n - 1}(\log \rho)^{\sigma + \delta + n + 1}. \]

Finally from (4.7), (4.8) and (4.12), the desired estimate follows. Q.E.D.

§ 5. Application

In this section, we shall improve the theorem 3 in [3] as an application of §1 of this paper.

In [3], we have constructed a parametrix \( E_\alpha(x, y) \) for the degenerated elliptic operator \( A \) defined on a domain \( \Omega \) in \( \mathbb{R}^n \). In the case \( \text{Re } \alpha(x) \leq -1 \), we have derived from \( E_\alpha(x, y) \) a fundamental solution \( E_\alpha(x, y) \) for \( A \) by the method of E.E. Levi. Now we can treat the unsolved part of the problem, i.e. the case \( \text{Re } \alpha(x) < 0 \). For the sake of simplicity, we shall discuss in the half space \( \mathbb{R}^n_+ \) in place of \( \Omega \), and we may assume without loss of generality that there is a positive constant \( \delta < 1 \) such that

\[ -1 \leq \sup_{x \in Q} \text{Re } \alpha(x) \leq -\delta < 0 \]

for a sufficiently small \( Q \). Consequently, the problem is equivalent to solve the following integral equation for a sufficiently small region \( Q \) in \( \mathbb{R}^n_+ \):

\[ S(x, y) + h(x, y) + \int Q h(x, z)S(x, y)dV(z) = 0, \]

where \( dV \) is a bounded positive measure, and \( h(x, y) \in C^\infty (Q \times Q \text{-diagonal}) \) satisfying

\[ |h(x, y)| \leq \text{Const. } y_n^{\sigma - 1} |x - y|^{\sigma - n}(1 + |\log |x - y||), \]

for \( (x, y) \in Q \times Q \) and \( 0 < \delta \leq 1 \).

Since \( h(x, y) \) can be regarded as an element of \( K^\sigma_1(Q \times Q) \), it follows from
Theorem 1 that (5.1) has a solution $S(x, y)$ satisfying, for $(x, y) \in Q \times Q$,

$$|S(x, y)| \leq \text{Const. } y_n^{\sigma-1} |x-y|^{1-n} L(x_n, y_n)^k,$$

for some non-negative number $k$.

After all, we can improve the theorem 3 in [3] as follows.

**Theorem 3’.** We assume that there is a positive constant $\delta < 1$ satisfying $-1 \leq \sup_{x \in Q} \text{Re} \alpha(x) \leq -\delta < 0$ for a sufficiently small $Q$. Then the operator $A$ has a fundamental solution $E_a(x, y)$ in $Q \times Q$ satisfying the following property.

$$E_a(x, y) = \tilde{E}_a(x, y) + R(x, y),$$

where $R(x, y)$ is defined by

$$R(x, y) = \int_{Q} \tilde{E}_a(x, z) S(z, y) dV(z),$$

using the solution kernel $S(x, y)$ of (5.1) with $h(x, y)$ being

$$A \tilde{E}_a(x, y) \in K_{\delta, 1}^\flat,$$

and $R(x, y)$ satisfies, for $(x, y) \in Q \times Q$,

$$|R(x, y)| \leq \text{Const. } y_n^{\sigma-1} |x-y|^{1+\sigma-n} L(x_n, y_n)^k,$$

where $k$ is a non-negative number independent of $(x, y) \in Q \times Q$.

REMARK. $\tilde{E}_a(x, y)$ itself is a kernel in $K_{\delta, 1}^\flat$, so the last inequality (5.6) is obvious from Proposition 1.

**References**

