<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>Bounded Approximate Identities and Quasicentrality of Banach Modules</td>
</tr>
<tr>
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</tr>
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</tr>
</tbody>
</table>

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Bounded Approximate Identities and Quasicentrality of Banach Modules

Sin-ei Takahasi*

1. Introduction.

In [2] we gave a new interpretation of the theorem of Doran and Wichmann which asserts that if \( A \) is a Banach algebra with bounded approximate identity and if \( A^* = A^* A + AA^* \), then \( A \) has a bounded quasicentral approximate identity which is contained in the convex hull of the original approximate identity (See [1, Theorem (29.2)]). In this note we introduce a notation of quasicentrality for two-sided Banach modules and present a similar result as in [2].

2. Results.

Throughout this note let \( A \) be a Banach algebra with bounded approximate identity \( \{ e_\lambda \} \) and \( X \) a two-sided Banach \( A \)-module. A net \( \{ u_\lambda \} \) in \( A \) is said to be quasicentral for \( X \) if \( \lim_{\lambda} \| u_\lambda x - xu_\lambda \| = 0 \) for all \( x \in X \). We now consider a necessary and sufficient condition for \( A \) to have a bounded approximate identity which is quasicentral for \( X \). To do this we are going to introduce two canonical module multiplications which make \( X^{**} \) into a two-sided Banach \( A^{**} \)-module, where \( X^{**} \) (resp. \( A^{**} \)) denotes the second dual of \( X \) (resp. \( A \)). These multiplications are defined as follows: Given \( a \in A, \phi \in A^{**}, x \in X, f \in X^* \) (the dual space of \( X \)) and \( F \in X^{**} \), define

\[
\langle x, f \star a \rangle = \langle ax, f \rangle : X^* \star A \subset X^*,
\]

\[
\langle a, F \star f \rangle = \langle f \star a, F \rangle : X^{**} \star X^* \subset A^*,
\]

\[
\langle f, \phi \star F \rangle = \langle F \star f, \phi \rangle : A^{**} \star X^{**} \subset X^{**},
\]

\[
\langle a, f \star x \rangle = \langle xa, f \rangle : X^* \star X \subset A^*,
\]

\[
\langle x, \phi \star f \rangle = \langle f \star x, \phi \rangle : A^{**} \star X^* \subset X^*,
\]

\[
\langle f, F \star \phi \rangle = \langle \phi \star f, F \rangle : X^{**} \star A^{**} \subset X^{**},
\]

\[
\langle a, x \star f \rangle = \langle ax, f \rangle : X \star X^* \subset A^*.
\]

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We note that if $A=X$ then the module multiplication $\ast$ is equal to the first Arens product in $A^{**}$ and $\ast$ is also equal to the second Arens product in $A^{**}$. Recall that $\text{supp} A$, the support of $A$ is the weak* -limit point of $|e_a|$ in $A^{**}$ (cf. [2]). Further given a Banach space $Y$, let us denote by $\pi_Y$ the canonical embedding of $Y$ into $Y^{**}$. In this setting we have the following

**Theorem.** (I) The following conditions are equivalent:

(i) $\text{supp} A$ commutes with each element of $\pi_X(X)$ under the module multiplication $\ast$.

(ii) $\text{supp} A$ commutes with each element of $\pi_X(X)$ under the module multiplication $\ast$.

(iii) $A$ has a bounded approximate identity which is quasicentral for $X$ and is containd in the convex hull of $|e_a|$.

(II) If $XA=AX$ and if $Y^*=(Y^* \ast A)+(A \ast Y^*)$, then (i) holds, where $AX=\{ax : a \in A, x \in X\}$, $XA=\{xa : a \in A, x \in X\}$ and $Y=AX$.

Note that by the Cohen-Hewitt's factorization theorem, $Y$ is norm-closed and hence $Y$ becomes a two-sided Banach $A$-module. The proof of the above result is essentially obtained in a way similar to that in [1], but for completeness we sketch it in the next section.

3. Proof of Results.

(I) It is straightforward that (i) and (ii) are equivalent. We now set $\Phi = \text{supp} A$ and so (i) is equivalent to the following condition:

(iv) $f \ast \Phi = \Phi \ast f$ for all $f \in X^*$.

In fact, let $x \in X$ and $f \in X^*$. Then

\[ \langle f, \pi_x(x) \ast \Phi \rangle = \langle \Phi \ast f, \pi_x(x) \rangle = \langle x, \Phi \ast f \rangle. \]

Note also that $\pi_x(x) \ast f = x \ast f$, so that

\[ \langle f, \Phi \ast \pi_x(x) \rangle = \langle \pi_x(x) \ast f, \Phi \rangle = \langle x \ast f, \Phi \rangle = \langle x, f \ast \Phi \rangle. \]
These equations show that (i) and (iv) are equivalent.

We next assume that (i) (and hence (iv)) holds. Let \( \epsilon > 0 \), \( a_1, \ldots, a_n \in A \) and \( x_1, \ldots, x_n \in X \). Choose an element \( \lambda_0 \) such that \( \|e_{\lambda}a_j - a_j\| < \epsilon \) and \( \|a_{\lambda}e_{\lambda} - a_{\lambda}\| < \epsilon \) for \( \lambda \geq \lambda_0 \) and \( j = 1, \ldots, n \). Let \( W \) be the norm-closure of the set

\[
\bigoplus_{j=1}^{n} \mathbb{C} x_j - x_j e : e \in E(\lambda_0)
\]

in the Banach space direct sum of \( n \)-copies of \( X \) where \( E(\lambda_0) \) is the convex hull of \( \{ e_{\lambda} : \lambda \geq \lambda_0 \} \). Then \( W \) is a closed convex set. If \( 0 \notin W \), then there are \( f_1, \ldots, f_n \in X^* \) such that

\[
\sum_{j=1}^{n} \langle e_{\lambda}x_j - x_j e, f_j \rangle < 1
\]

for all \( e \in E(\lambda_0) \) from the Hahn-Banach separation theorem. However

\[
\lim_{\lambda} \langle e_{\lambda}x_j - x_j e, f_j \rangle = \lim_{\lambda} \langle e_{\lambda}, x_j * f_j - f_j * x_j \rangle = \lim_{\lambda} \langle x_j * f_j - f_j * x_j, \pi_{\lambda}(e_{\lambda}) \rangle = \langle x_j * f_j - f_j * x_j, \Phi \rangle = \langle x_j, f_j * \Phi - \Phi * f_j \rangle = 0,
\]

because \( f_j * \Phi = \Phi * f_j \) by (iv). This contradiction shows (iii) holds.

Conversely assume (iii) holds. Then \( A \) has a bounded approximate identity \( \{ u_{\alpha} \} \) which is quasicentral for \( X \). So we assume, without loss of generality, that \( \{ \pi_{\lambda}(u_{\alpha}) \} \) converges to \( \Phi = \text{supp } A \) for the weak*-topology. Then

\[
\langle x, f * \Phi \rangle = \lim_{\alpha} \langle x * f, \pi_{\lambda}(u_{\alpha}) \rangle = \lim_{\alpha} \langle u_{\alpha} x, f \rangle = \lim_{\alpha} \langle u_{\alpha} x, f \rangle = \lim_{\alpha} \langle f * x, \pi_{\lambda}(u_{\alpha}) \rangle = \langle x, \Phi * f \rangle
\]

for all \( x \in X \) and \( f \in X^* \). Thus (iv) and so (i) holds.

(II) Suppose that \( AX = XA = Y \) and \( Y^* = (Y^* * A) + (A * Y^*) \). Let \( x \in X \) and \( f \in X^* \). Then there are \( \varphi, \psi \in Y^* \) and \( a, b \in A \) such that \( f | Y = \varphi * a + b * \psi \). Therefore

\[
\langle x, f * \Phi \rangle = \lim_{\lambda} \langle e_{\lambda} x, f \rangle = \lim_{\lambda} \langle e_{\lambda} x, \varphi * a + b * \psi \rangle = \lim_{\lambda} \langle a e_{\lambda} x, \varphi \rangle + \lim_{\lambda} \langle e_{\lambda} x b, \psi \rangle = \langle a x, \varphi \rangle + \langle x b, \psi \rangle.
\]
Similarly, $\langle x, \Phi \cdot f \rangle = \langle ax, \phi \rangle + \langle xb, \phi \rangle$. Then $\langle x, f \cdot \Phi \rangle = \langle x, \Phi \cdot f \rangle$ for all $x \in X$ and $f \in X^*$. Thus (iv) and so (i) holds.

4. Remarks.

If $X$ is a Banach algebra containing $A$ as a subalgebra, then $X$ becomes naturally a two-sided Banach $A$-module. In that case, $A^{**}$ can be regarded as a subalgebra of $X^{**}$ the Banach algebra with the first Arens product and the following equations hold:

$$F\phi = F \cdot \phi \quad \text{and} \quad \phi F = \phi \cdot F \quad (\phi \in A^{**}, \, F \in X^{**}),$$

where $F\phi$ and $\phi F$ mean the products of $F$ and $\phi$ under the first Arens product in $X^{**}$. Therefore we see easily that the theorem is a generalization of one given in [2].

References
