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**n-Generator Property of a Polynomial Ring**

Ryûki MATSUDA*

Let $A$ be a commutative ring with identity, and let $n$ be a natural number. If each ideal of $A$ is generated by $n$ elements, $A$ is said to have rank $n$. If each finitely generated ideal of $A$ is generated by $n$ elements, $A$ is said to have $n$-generator property. Let $X$ be an indeterminate. A condition for a polynomial ring $A[X]$ to have rank $n$ was determined in [4, §5]. Let $S$ be a torsionfree cancellative commutative monoid. More generally, a condition for a semigroup ring $A[X; S]$ to have rank $n$ was determined in [4, §5]. As to $n$-generator property, a condition for $A[X; S]$ to have a 1-generator property was determined in [2]. Let $N$ be the nilradical of $A$. A condition for $A[X; S]$ to have $n$-generator property was determined in [5, §3], when $N=(0)$.

In this paper we concern conditions for a polynomial ring $A[X]$ to have $n$-generator property, when $N\neq(0)$. We give conditions for $A[X]$ to have $n$-generator property, when $A$ is a semilocal ring (i.e. one with only a finite number of maximal ideals), or $A$ has few zerodivisors, or $N$ is finitely generated, or $n=2$.

§ 1.

**Lemma 1 ([6, (11.15)])**. *Let $a$ be an element of $A$ such that $a^2=a(N)$. Then there exists an idempotent $e$ of $A$ such that $a=e(N)$.*

**Lemma 2.** Assume $A$ has dimension 0 (‘dimension’ means the Krull dimension), and let $a \in A$. Then $A$ has a direct sum decomposition $A=A_1 \oplus A_2$ such that $ae_i$ is either a unit or a nilpotent of $A_i$ for each $i$, where $e_i$ is the identity of $A_i$.

**Proof.** We set $A/N=\bar{A}$ and set $a+N=\bar{a}$. $\bar{A}$ is a regular (i.e. von Neumann regular) ring. Therefore we have $\bar{a}\bar{A}=\bar{e}\bar{A}$ for an idempotent $e$ of $\bar{A}$. By Lemma 1, we have $E=e\bar{A}$ for an idempotent $e$ of $A$. We set $A_1=eA$ and $A_2=(1-e)A$.

**Lemma 3.** Assume dim $A=0$, and let $a_1,\ldots, a_i \in A$. Then $A$ has a direct sum decomposition $A=A_1 \oplus \cdots \oplus A_m$ such that $a_ie_i$ is either a unit or a nilpotent of $A_j$ for each $i$ and $j$.

**Proof.** By Lemma 2, we have $A=A_1 \oplus A_2$ such that $a_ie_i$ is either a unit or a

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nilpotent of $A_i$ for each $i$. We apply Lemma 2 for $A_i$ and $a_2 e_i$ for each $i$. The rest is similar.

**Lemma 4 ([3, §4, Proposition 17])**. If $A[X]$ has $n$-generator property, then $\dim A = 0$.

**Lemma 5 ([1])**. If each prime ideal of $A$ is finitely generated, then $A$ is a Noetherian ring.

**Lemma 6**. Let $A$ be a local ring (i.e. one with only one maximal ideal $M$). If $A[X]$ has $n$-generator property, then $M$ is finitely generated.

**Proof**. Suppose the contrary. We take elements $p_i \in A$ such that $0 \subseteq (p_1) \subseteq (p_1, p_2) \subseteq \cdots \subseteq (p_1, \ldots, p_n) \subseteq M$. We set $a = (p_1 X, \ldots, p_n X^n, X^{n+1}) A[X]$. We have $a = (f_1, \ldots, f_n) A[X]$ for elements $f_i \in A[X]$. We have $f_i = a_i X + a_i^1 X^2 + \cdots$, where $a_1 \in (p_1)$, $a_i \in (p_1, p_2)$, $a_i \in (p_1, \ldots, p_n)$ for $1 \leq i \leq n$. Since $p_1 X \in (f_1, \ldots, f_n) A[X]$, we may take as follows:

$$
\begin{align*}
  f_1 &= p_1 X + a_1^1 X^2 + a_1^2 X^3 + \cdots \\
  f_2 &= a_2^1 X + a_2^2 X^3 + \cdots \\
  \vdots \\
  f_n &= a_n^1 X + a_n^2 X^3 + \cdots,
\end{align*}
$$

where $a_1^i \in (p_2)$, $a_i^2 \in (p_2, p_3)$, $a_i^i \in (p_2, \ldots, p_n)$ for $2 \leq i \leq n$. Since $p_2 X^2 \in (f_1, \ldots, f_n) A[X]$, we may take as follows:

$$
\begin{align*}
  f_1 &= p_1 X + a_1^1 X^2 + a_1^2 X^3 + \cdots \\
  f_2 &= p_2 X^2 + a_2^2 X^3 + \cdots \\
  \vdots \\
  f_n &= a_n^2 X^3 + \cdots,
\end{align*}
$$

where $a_1^2 \in (p_3), \ldots, a_i^i \in (p_3, \ldots, p_n)$ for $3 \leq i \leq n$. Finally we may take as follows:

$$
\begin{align*}
  f_1 &= p_1 X + a_1^2 X^2 + \cdots + a_1^n X^n + \cdots \\
  f_2 &= p_2 X^2 + \cdots + a_2^n X^n + \cdots \\
  \vdots \\
  f_n &= p_n X^n + \cdots.
\end{align*}
$$

Then $X^{n+1} \in (f_1, \ldots, f_n) A[X]$ derives the contradiction of $1 \in M$.

**Theorem 7**. Let $A$ be a semilocal ring. Then $A[X]$ has $n$-generator property, if and only if $A[X]$ has rank $n$.

**Proof**. The necessity. By Lemma 4, we have $\dim A = 0$. By Lemma 3, we have $A = A_1 \oplus \cdots \oplus A_m$, where $A_i$ is a local ring with the maximal ideal $M_i$ for each
i. $A_i[X]$ has $n$-generator property. By Lemma 6, $M_i$ is finitely generated. By Lemma 5, $A_i$ is a Noetherian ring for each $i$. Hence $A_i[X]$ is a Noetherian ring. It follows that $A[X]$ is a Noetherian ring, and that $A[X]$ has rank $n$.

If the set of zerodivisors of $A$ is a union of a finite number of prime ideals of $A$, $A$ is said to have few zerodivisors.

**Theorem 8.** Assume that $A$ has few zerodivisors. Then $A[X]$ has $n$-generator property if and only if $A[X]$ has rank $n$.

**Proof.** The necessity. We have $\dim A = 0$. Therefore $A$ is its own total quotient ring. It follows that $A$ is a semilocal ring. By Theorem 7, $A[X]$ has rank $n$.

§ 2.

Throughout this section, we assume that $N$ is finitely generated. Let $k$ be a natural number such that $N^{k+1} = (0)$. We set as follows:

$$\begin{align*}
N^k &= (p_1, \ldots, p_{n(1)}), \\
N^{k-1}/N^k &= (p_{n(1)+1} + N^k, \ldots, p_{n(2)} + N^k), \\
N/N^2 &= (p_{n(k-1)+1} + N^2, \ldots, p_{n(k)} + N^2).
\end{align*}$$

We call $\{p_1, p_2, \ldots, p_{n(k)}\}$ a set of $*$-generators for $N$. Let $k(A)$ be the least natural number $k$ with $N^{k+1} = (0)$. If we choose above $n(1), n(2), \ldots, n(k(A))$ least, we call the set $\{p_1, p_2, \ldots, p_{n(k(A))}\}$ of $*$-generators for $N$ a minimal set of $*$-generators for $N$. And we set $n(k(A)) = n_0(k(A))$. If, for each decomposition $A = A_1 + \cdots + A_m$, we have $n_0(k(A_i)) \geq n$ for some $i$, we call $A$ bad.

**Lemma 9.** Let $\{p_1, p_2, \ldots, p_{n(k)}\}$ be a set of $*$-generators for $N$. Assume that $A$ has a decomposition $A = A_1 \oplus A_2$, and let $e_1$ be the identity of $A_1$.

1. $N^l = N^l e_1$ for each $l$, where $N_l$ denotes the nilradical of $A_l$.
2. $N^l/N^{l+1} = (p_{n(k-l)+1} e_1 + N^{l+1}), \ldots, p_{n(k-l)+1} e_1 + N^{l+1})$.
3. $k(A_1) \leq k(A)$.
4. $n_0(k(A_1)) \leq n_0(k(A))$.
5. $n_0(k(A_1)) = n_0(k(A))$ implies $k(A_1) = k(A)$.

**Lemma 10.** If $A[X]$ has $n$-generator property, then we have a decomposition $A = A_1 \oplus \cdots \oplus A_m$ such that $n_0(k(A_i)) < n$ for each $i$.

**Proof.** Suppose the contrary. We use Lemma 9. For each decomposition $A = A_1 \oplus \cdots \oplus A_m$, there exists a bad factor $A_i$. We choose a bad factor $A_i$, $n_0(k(A_i))$ of which is least of all decompositions $A = A_1 \oplus \cdots \oplus A_m$ and of its all bad factors. We rewrite $A_i$ to be $A$. We set $k(A) = k$. We have $n_0(k) \geq n$. Let $\{p_1, p_2, \ldots, p_{n(k)}\}$ be a set of $*$-generators for $N$. Assume that $A$ has a decomposition $A = A_1 \oplus A_2$, and let $e_1$ be the identity of $A_1$.

1. $N^l = N^l e_1$ for each $l$, where $N_l$ denotes the nilradical of $A_l$.
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4. $n_0(k(A_1)) \leq n_0(k(A))$.
5. $n_0(k(A_1)) = n_0(k(A))$ implies $k(A_1) = k(A)$.
\(p_{n_0(k)}\) be a set of \(*\)-generators of \(N\), and set \(a = (p_1, X, \ldots, p_n, X^{n+1})A[X]\). We have \(a = (f_1, \ldots, f_n)A[X]\) for \(f_i \in A[X]\). We have \(f_i = a_1^i X + a_2^i X^2 + \cdots\), where \(a_1^i \in \langle p_1 \rangle, \ldots, a_n^i \in \langle p_1, \ldots, p_n \rangle\) for \(1 \leq i \leq n\). We set \(a_1^i = p_1 b_1^i, \ldots, a_n^i = p_1 b_n^i\) for \(b_i \in A\). We have \(p_1 X = \Sigma f_i g_i\), where \(g_i = c_0^i + c_1^i X + \cdots \in A[X]\). Applying Lemma 3 for \(\{b_1^i, \ldots, b_n^i, c_0^i, \ldots, c_n^i\}\), we have \(A = A_1 \oplus \cdots \oplus A_m\). Choose a bad factor \(A_i\). It follows that \(n_0(k(A_i)) = n_0(k)\) and that \(\{p_1 e_i, p_2 e_i, \ldots, p_{n_0(k)} e_i\}\) is a minimal set of \(*\)-generators for \(N_i\). We rewrite \(A_i\) to be \(A\). By \(p_1 X = \Sigma f_i g_i\), we may take as follows:

\[
\begin{align*}
f_1 &= p_1 X + a_1^1 X^2 + \cdots \\
f_2 &= a_1^2 X^2 + \cdots \\
f_n &= a_1^n X^2 + \cdots,
\end{align*}
\]

where \(a_1^i \in \langle p_2 \rangle, \ldots, a_1^i \in \langle p_2, \ldots, p_n \rangle\) for \(2 \leq i \leq n\). We set \(a_2^i = p_2 b_2^i, \ldots, a_n^i = p_2 b_n^i\). We have \(p_2 X^2 = \Sigma f_i g_i\), where \(g_i = c_0^i + c_1^i X + \cdots \in A[X]\). Applying Lemma 3 for \(\{b_2^i, \ldots, b_n^i, c_0^i, \ldots, c_n^i\}\), we have \(A = A_1 \oplus \cdots \oplus A_m\). Choose a bad factor \(A_i\). It follows that \(n_0(k(A_i)) = n_0(k)\) and that \(\{p_1 e_i, p_2 e_i, \ldots, p_{n_0(k)} e_i\}\) is a minimal set of \(*\)-generators for \(N_i\). We rewrite \(A_i\) to be \(A\). By \(p_2 X^2 = \Sigma f_i g_i\), we may take as follows:

\[
\begin{align*}
f_1 &= p_1 X + a_1^2 X^2 + a_1^1 X^3 + \cdots \\
f_2 &= p_2 X^2 + a_1^2 X^3 + \cdots \\
f_3 &= a_1^3 X^3 + \cdots \\
f_n &= a_1^n X^3 + \cdots,
\end{align*}
\]

where \(a_1^i \in \langle p_3 \rangle, \ldots, a_1^i \in \langle p_3, \ldots, p_n \rangle\) for \(3 \leq i \leq n\). We set \(a_3^i = p_3 b_3^i, \ldots, a_n^i = p_3 b_n^i\). We have \(p_3 X^3 = \Sigma f_i g_i\), where \(g_i = c_0^i + c_1^i X + \cdots \in A[X]\). Applying Lemma 3 for \(\{b_3^i, \ldots, b_n^i, c_0^i, \ldots, c_n^i\}\), we have \(A = A_1 \oplus \cdots \oplus A_m\). Finally we may take as follows:

\[
\begin{align*}
f_1 &= p_1 X + a_1^2 X^2 + a_1^1 X^3 + \cdots + a_2^n X^n + \cdots \\
f_2 &= p_2 X^2 + a_2^1 X^3 + \cdots + a_2^n X^n + \cdots \\
f_n &= p_3 X^n + \cdots,
\end{align*}
\]

We have \(X^{n+1} = \Sigma f_i g_i\), where \(g_i = c_0^i + c_1^i X + \cdots \in A[X]\). Applying Lemma 3 for \(\{c_0^i, \ldots, c_n^i\}\), we have \(A = A_1 \oplus \cdots \oplus A_m\). Choose a bad factor \(A_i\). It follows that \(n_0(k(A_i)) = n_0(k)\) and that \(\{p_1 e_i, p_2 e_i, \ldots, p_{n_0(k)} e_i\}\) is a minimal set of \(*\)-generators for \(N_i\). We rewrite \(A_i\) to be \(A\). By \(X^{n+1} = \Sigma f_i g_i\), we have \(1 \in N\); a contradiction.

**Lemma 11.** Assume that \(\dim A = 0\), and let \(\{p_1, p_2, \ldots, p_{n_0(k)}\}\) be a set of \(*\)-generators for \(N\). Let \(a \in A\). Then we have a decomposition \(A = A_1 \oplus \cdots \oplus A_m\) such that \(ae_i\) is either a unit of \(A_i\) or of the form \((p_1 e_{i_1} + p_2 e_{i_2} + \cdots + p_{n_0(k)} e_{i_{n_0(k)}}) e_i\).
where $\mathcal{E}_{ij}$ is either zero or a unit of $A$ for each $i$ and $j$.

**Proof.** We may suppose that $a$ is neither zero nor a unit of $A$. By Lemma 2, we have $A = A_1 \oplus A_2$, where $ae_1$ is a unit of $A_1$ and $ae_2$ is a nilpotent of $A_2$. Hence we may suppose that $a$ is a nilpotent of $A$. We have $a = p_1 a_1 + p_2 a_2 + \ldots + p_n(a_n)$ for $a_i \in A$. We use Lemma 9. Applying Lemma 2 for $a_{n(k)}$, we have $A = A_1 \oplus A_2$. Since $\{p_1 e_i, p_2 e_i, \ldots, p_{n(k)} e_i\}$ is a set of $*$-generators for $N$, and since $p_{n(k)} N \subset (p_1, p_2, \ldots, p_{n(k)} - 1)$, we have $ae_i = (p_1 b_1 + p_2 b_2 + \ldots + p_{n(k)} b_{n(k)}) e_i$, where $b_{n(k)}$ is either zero or a unit of $A$. We rewrite $A_1$ to $A$. Then $a_{n(k)}$ is either zero or a unit of $A$. Applying Lemma 2 for $a_{n(k)} - 1$, we have $A = A_1 \oplus A_2$. Since $\{p_1 e_i, p_2 e_i, \ldots, p_{n(k)} e_i\}$ a set of $*$-generators for $N$, and since $p_{n(k) - 1} N \subset (p_1, p_2, \ldots, p_{n(k) - 2})$, we have $ae_i = (p_1 b_1 + \ldots + p_{n(k) - 1} b_{n(k) - 1} + p_{n(k)} b_{n(k)}) e_i$, where $b_{n(k) - 1}$ and $b_{n(k)}$ are either zero or a unit of $A$. The rest is similar.

**Lemma 12.** Assume that $\dim A = 0$, and let $\{p_1, p_2, \ldots, p_{n(k)}\}$ be a set of $*$-generators for $N$. Let $a_1, \ldots, a_l \in A$. Then we have a decomposition $A = A_1 \oplus \cdots \oplus A_m$ such that $a_{ij}$ is either a unit of $A_j$ or of the form $(p_1 \mathcal{E}_{i1} + p_2 \mathcal{E}_{i2} + \cdots + p_{n(k)} \mathcal{E}_{in(k)}) e_j$, where $\mathcal{E}_{ij}$ is either zero or a unit of $A$ for each $i, j$ and $h$.

**Proof.** Applying Lemma 11 for $a_1$, we have a decomposition $A = A_1 \oplus \cdots \oplus A_m$ such that $a_{1j}$ is either a unit of $A_j$ or of the form $(p_1 \mathcal{E}_{1j1} + p_2 \mathcal{E}_{1j2} + \cdots + p_{n(k)} \mathcal{E}_{1jm(k)}) e_j$, where $\mathcal{E}_{ij}$ is either zero or a unit of $A$ for each $i$ and $j$. We apply Lemma 11 for $A_i$ and $a_{2j} e_i (1 \leq i \leq m)$. We have a new decomposition $A = A_1 \oplus \cdots \oplus A_m$ such that $a_{ij}$ is either a unit of $A_j$ or of the form $(p_1 \mathcal{E}_{i1j1} + p_2 \mathcal{E}_{i1j2} + \cdots + p_{n(k)} \mathcal{E}_{i1jm(k)}) e_j$, where $\mathcal{E}_{ijh}$ is either zero or a unit for each $j$ and $h$ ($1 \leq i \leq 2$). The rest is similar.

**Lemma 13.** Let $f_i = a_0^i + a_1^i X + \cdots + a_{d_i}^i X^{d_i} \in A[X]$ for $1 \leq i \leq n + 1$. Let $l$ be a natural number less than $n$. Assume that $a_{d_i}^i$ is either a unit or of the form $p_1 \mathcal{E}_{i1} + \cdots + p_{l} \mathcal{E}_{il}$ for each $i$. Then we have $(f_1, \ldots, f_{n+1}) A[X] = (g_1, \ldots, g_{n+1}) A[X]$ for some $g_i \in A[X]$ such that $\Sigma \deg (g_i) < \Sigma d_i$.

**Proof.** Suppose that $d_1 \leq d_2$ and that $\mathcal{E}_{i1}$ is a unit of $A$ for instance. Then we replace $f_2$ by $f_2 - \mathcal{E}_{11}^{-1} \mathcal{E}_{21} X^{d_2 - d_1} f_1$. The rest is similar.

**Lemma 14.** If $\dim A = 0$ and if $n_0(k(A)) < n$, then $A[X]$ has $n$-generator property.

**Proof.** We show that an ideal $a$ of $A[X]$ generated by $n + 1$ elements $f_1, \ldots, f_{n+1}$ is generated by $n$ elements by the induction on $\Sigma \deg (f_i)$. We set $f_i = a_{0i}^i + a_{1i}^i X + \cdots + a_{d_i}^i X^{d_i}$ for $1 \leq i \leq n + 1$. Set $k(A) = k$, and let $\{p_1, p_2, \ldots, p_{n(k)}\}$ be a minimal set of $*$-generators for $N$. Applying Lemma 12 for $\{a_{d_i}^1 + \cdots + a_{d_i}^n\}$, we have $A = A_1 \oplus \cdots \oplus A_m$. We rewrite $A_i$ by $A(1 \leq i \leq n)$. Then $a_{d_i}^i$ is either a unit of $A$ or of the form $p_1 \mathcal{E}_{i11} + p_2 \mathcal{E}_{i12} + \cdots + p_{n(k)} \mathcal{E}_{i1n(k)}$, where $\mathcal{E}_{ij}$ is either zero or a unit.
of $A$. By Lemma 13, we may choose $g_i \in A[X]$ such that $(f_1, \ldots, f_{n+1})A[X] = (g_1, \ldots, g_{n+1})A[X]$ and that $\sum \deg(g_i) < \sum d_i$. Hence $a$ is generated by $n$ elements.

**Theorem 15.** Assume that $N$ is finitely generated. Then $A[X]$ has $n$-generator property if and only if $\dim A = 0$ and $A$ has a decomposition $A_1 \oplus \cdots \oplus A_m$ such that $n_0(k(A_i)) < n$ for each $i$.

**Proof.** The necessity. By Lemma 4, we have $\dim A = 0$. By Lemma 10, $A$ has a decomposition $A_1 \oplus \cdots \oplus A_m$ such that $n_0(k(A_i)) < n$ for each $i$. The sufficiency. By Lemma 14, $A_i[X]$ has $n$-generator property. It follows that $A[X]$ has $n$-generator property.

§3.

**Proposition 16.** If $A[X]$ has $n$-generator property, then $N^n = (0)$.

**Proof.** Suppose the contrary. We have $p_1 \cdots p_n \neq 0$ for some $p_i \in N$. There exists a prime ideal $P$ of $A$ containing $(0 : p_1 \cdots p_n)$. $A_p[X]$ has $n$-generator property. By Theorem 7, $A_p[X]$ has rank $n$. By [4, §5], we have $P^n A_p = (0)$. Hence $p_1 \cdots p_n A_p = (0)$, which is a contradiction.

**Lemma 17.** Assume that $A[X]$ has $n$-generator property. If $a$ is a finitely generated ideal of $A$ contained in $N$, then $a$ is generated by $n - 1$ elements.

**Proof.** We work for $a$ what we worked for $N$ in §2.

We have another proof of Proposition 16 from the above proof of Lemma 17.

**Proposition 18.** If $A[X]$ has $n$-generator property, then $A$ has $(n - 1)$-generator property.

**Proof.** Let $a = (a_1, \ldots, a_l)$ be a finitely generated ideal of $A$. Applying Lemma 3 for $\{a_1, \ldots, a_l\}$, we have $A = A_1 \oplus \cdots \oplus A_m$. If $a_j e_i$ is a unit of $A_i$ for some $j$, then $a A_i$ is generated by 1 element of $A_i$. If $\{a_{i_1} e_{j_1}, \ldots, a_{l_1} e_{j_l}\} \subset N_0$, then $a A_i$ is generated by $n - 1$ elements by Lemma 17. Hence $a$ is generated by $n - 1$ elements.

**Theorem 19.** $A[X]$ has 2-generator property if and only if $A$ has 1-generator property, of dimension 0 and $N^2 = (0)$.

**Proof.** The necessity. By Lemma 4, we have $\dim A = 0$. By Proposition 16, we have $N^2 = (0)$. By Proposition 18, $A$ has 1-generator property. The sufficiency. We show that an ideal $a = (f_1, f_2, f_3)A[X]$ of $A[X]$ generated by 3 elements is generated by 2 elements. We set $f_i = a_{i_1} + a_{i_2} X + \cdots + a_{i_3} X^{d_i}$. We rely on the induction on $d_1 + d_2 + d_3$. We may assume that $d_1 \leq d_2 \leq d_3$. Applying Lemma 3 for $\{a_{i_1}, a_{i_2}, a_{i_3}\}$, we have $A = A_1 \oplus \cdots \oplus A_m$. Hence we may assume
that \( a^{d_i} \) is either a unit or a nilpotent of \( A \) (\( 1 \leq i \leq 3 \)). If either \( a^{d_1} \) or \( a^{d_2} \) is a unit, we may find \( g_i \) such that \( a=(g_1, g_2, g_3)A[X] \) and \( \sum \deg (g_i)<d_1+d_2+d_3 \). Then \( a \) is generated by 2 elements. Suppose \( \{a^{d_1}, a^{d_2}\} \subset N \). We have \( (a^{d_1}, a^{d_2})=(p) \) for some \( p \in N \). We set \( a^{d_1}=pb_1 \) and \( a^{d_2}=pb_2 \). By Lemma 3, we may assume that \( b_i \) is either a unit or a nilpotent of \( A \) (\( 1 \leq i \leq 2 \)). If \( b_1 \) is a unit, we may find \( g_i \in A[X] \) such that \( a=(g_1, g_2, g_3)A[X] \) and \( \sum \deg (g_i)<d_1+d_2+d_3 \). Hence \( a \) is generated by 2 elements. If \( b_1 \in N \), we have \( a^{d_1}=0 \).

References