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引用文献
Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 15: 25-28

発行年
1983

URL
http://hdl.handle.net/10109/2960

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On the Probability Density Functions and the Oscillations

Hiroshi Onose*

1. Introduction

Consider the first order delay differential equation

\[ y'(t) + a(t)y(t) + p(t)y(\tau(t)) = 0 \quad \text{for} \quad t \in [T, \infty), \]

where \( p(t) > 0 \), \( a(t) \) and \( \tau(t) \) are continuous functions, \( T \in \mathbb{R}_+ \) and \( \lim_{t \to \infty} \tau(t) = +\infty \).

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros.

Very recently, Ladas and Stavroulakis [4] and Koplatadze and Chanturija [2] proved some interesting results concerning the oscillations which were caused by the retarded argument and which did not appear in the corresponding ordinary differential equation. But they did not mention about real examples in their papers.

In the present paper, we treat with the statistical examples. In section 2, we mention to the statistical examples. In section 3, we mention to the applications of section 2. In section 4, we propose two oscillation theorems.

2. The statistical examples

Karl Pearson ([7], pp. 49–50) considered the following differential equation

\[ \frac{dy}{dt} = \frac{(t+a)y}{b+ct+gt^2}, \quad \text{where} \quad a, b, c \text{ and } g \text{ are constants.} \]

We call the set of distribution which has \( y(t) \) being a solution of (2) as a probability density function, for Pearson distribution system. Pearson classified the distributions for some types according as the shapes of (2). For example, when \( c = g = 0 \) and \( b < 0 \) in (2), we have the Normal distribution. He used this set as the \( \chi^2 \) test of goodness of fit. In the followings, we consider the above facts on the functional differential equations.

**PROPERTY 1.** Consider the functional differential equation with deviating argument,

\[ \frac{dy(t)}{dt} + \frac{(t-\mu)}{\sigma^2} \exp\left(\frac{-2\mu t + 3\mu^2}{2\sigma^2}\right)y(t-\mu) = 0 \quad \text{for} \quad -\infty < t < +\infty, \]

Received March 11, 1983.

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where $\sigma > 0$ and $\mu$ are constants.

Equation (3) has a solution $y(t) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)$, which is the Gauss distribution (the Normal distribution).

**PROPERTY 2.** Consider the functional differential equation with deviating argument

$$
\frac{dy(t)}{dt} + \lambda \exp(-\lambda t)y(t-\mu) = 0 \quad \text{for} \quad t \geq 0,
$$

where $\lambda$ and $\mu$ are constants.

Equation (4) has a solution $y(t) = \lambda \exp(-\lambda t)$, which is the Laplace distribution (the Exponential distribution).

The verifications of Properties 1 and 2 are easy, so that we omit them here. These Properties 1 and 2 work as useful examples to illustrate the nonoscillation theorem in the next section.

### 3. Applications of section 2

Consider the following first order functional differential equation

$$
y'(t) + \sum_{i=1}^{N} q_i(t)f_i(y(\tau_i(t))) = 0,
$$

where the following conditions are assumed to hold:

(a) $q_i, \tau_i \in C([T, \infty), \mathbb{R})$, $q_i(t) \geq 0$, and $\lim_{t \to \infty} \tau_i(t) = \infty$, $1 \leq i \leq N$;

(b) $f_i \in C(\mathbb{R}, \mathbb{R})$, $f_i$ is nondecreasing and $uf_i(u) > 0$ for $u \neq 0$, $1 \leq i \leq N$.

**THEOREM A[1].** If

$$
\sum_{i=1}^{N} \int_{t_0}^{\infty} q_i(t)dt < \infty,
$$

then equation (5) have nonoscillatory solutions.

Consider the delay differential equation

$$
y'(t) + py(t-\tau) = 0,
$$

where $p$ and $\tau$ are positive constants.

**THEOREM B([3], [5]).** The condition

$$
p\tau > e^{-1}
$$

is a necessary and sufficient one for all proper solutions of (7) to be oscillatory.

**EXAMPLE 1.** Consider the equation (3) for $t \in [T, \infty)$. We can see that
where $B = \mu^2/\sigma^2$, $C = \exp\left(3\mu^2/(2\sigma^2)\right)$ and $k \in \left[T, \infty\right)$ is a constant. Hence, by Theorem A, equation (3) has in real a nonoscillatory solution $y(t)$. In fact, the solution $y(t)$ exists as the Normal distribution.

**Example 2.** Consider the equation (4) for $t \in \left[T, \infty\right)$. We see that

$$\lambda \mu \exp\left(-\lambda \mu\right) \leq e^{-1}. \tag{8}$$

If we set $f(k) = ke^{-k} - e^{-1}$ where $k = \lambda \mu$, then we have $f'(k) = e^{-k}(1 - k)$. From this, we obtain $f(k) \leq f(1) = 0$. This proves (8). Hence, by Theorem B, equation (4) has in real a nonoscillatory solution $y(t)$. In fact, the solution $y(t)$ exists as the Exponential distribution.

If we consider equation (7), then

$$\int_{x(t)}^{\infty} p(t) dt = \infty$$

holds. This shows that Theorem A can not be applied to equation (7) and equation (4). This means that Theorem A may be sharpened when equation (5) is the linear case. This was done firstly by Ladas [3] for equation (7).

4. The oscillations

**Theorem 1.** Consider the delay differential equation

$$y'(t) + a(t)y(t) + p(t)y(\tau(t)) = 0 \text{ for } t \in \left[T, \infty\right). \tag{1}$$

Suppose that

$$\tau(t) < t \text{ for } t \in \left[T, \infty\right), \tag{9}$$

and $k$ is a constant and that

$$\int_{\tau(t)}^{t} a(s) ds > k > -\infty \text{ for all } t \in \left[T, \infty\right) \text{ and that} \tag{10}$$

$$\int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{s} a(u) du\right) ds \leq e^{-1} \text{ for } t \geq t_0 > T.$$

Then equation (1) has a nonoscillatory solution.

This is obtained easily by using Koplatadze and Chanturiya’s [2] method and Tychonoff-Schauder’s fixed point theorem. (cf. [2], Theorem 3)

**Theorem 2.** Consider the retarded differential equation

$$y'(t) + a(t)y(t) + p(t)y(t-\tau) = 0 \text{ for } t \in \left[T, \infty\right). \tag{11}$$

Assume that $p(t) > 0$ (at least) on a sequence of disjoint intervals $\{(\xi_n, t_n)\}_{n=1}^{\infty}$ with $t_n - \xi_n = 2\tau$. If (9) and
(12) \[ \liminf_{t \to \infty} \int_{t_{n-1}}^{t_n} p(s) \exp \left( \int_{s-\tau}^{s} a(u) \, du \right) \, ds > 1 \]

hold, then equation (11) is oscillatory.

PROOF. From equation (11) we obtain

(13) \[ \left( y(t) \exp \left( \int_{c}^{t} a(s) \, ds \right) \right)' + p(t) \left( \exp \left( \int_{c}^{t} a(s) \, ds \right) \right) y(\tau(t)) = 0 \quad \text{for} \quad t \geq c > T. \]

If we set \( Y(t) = y(t) \left( \int_{c}^{t} a(s) \, ds \right) \) and \( P(t) = p(t) \exp \left( \int_{c}^{t} a(s) \, ds \right) \), then, from (13), we have

(14) \[ Y'(t) + Q(t) Y(t-\tau) = 0 \quad \text{for} \quad t \geq c. \]

The rest of the proof proceeds as same as Ladas, Sficas and Stavroulakis ([6], Theorem 1).

Q. E. D.

References