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<td>引用</td>
<td>Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 15: 17-18</td>
</tr>
<tr>
<td>発行年</td>
<td>1983</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10109/2958">http://hdl.handle.net/10109/2958</a></td>
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Quasi-Centrality of Banach Algebras and Approximate Identity

Sin-ei Takahasi*

In this note, we give a different proof of Archbold's result characterizing quasi-central C*-algebras [1]. However we also see that our method is useful for considering more general Banach algebras. A Banach algebra is said to be quasi-central if no primitive ideal contains its center (cf. [1, Definition 1] or [5, Definition 3.5]). R. J. Archbold actually proved that in order for a C*-algebra A to be quasi-central it is necessary and sufficient that A has an approximate identity each element of which belongs to the center Z(A) of A.

The sufficiency of the condition is immediate as observed in this proof. In fact, if \{z_\lambda\} is an approximate identity of A each element of which belongs to Z(A) and if P is a primitive ideal of A which contains Z(A), then for any x \in A, we have that z_\lambda x \in P and lim_\lambda \|z_\lambda x - x\| = 0, and hence P = A, a contradiction. He also proved the necessity making use of the classical Dini Theorem. However we will here give the following different proof for the necessity: Choose a bounded approximate identity \{z_\lambda\} of Z(A) and set

$$I = \{x \in A: \lim_\lambda \|z_\lambda x - x\| = 0\}.$$  

It is easy to see that I is a two-sided ideal of A which contains Z(A). Also since \{z_\lambda\} is bounded, I is a closed set. Now if \{z_\lambda\} is not an approximate identity of A, then I \neq A and so there exists a primitive ideal P of A which contains I and hence Z(A). Consequently, A is not quasi-central.

Notice that we have, from the above proof, the following

**Proposition.** Let A be a Banach algebra such that every proper closed two-sided ideal in A is contained in some primitive ideal. If A is quasi-central, then any bounded approximate identity of Z(A) is already an approximate identity of A. Conversely, if A has an approximate identity each element of which belongs to Z(A), then A is quasi-central.

**Remark.** If A is a C*-algebra, then A has obviously the property that every proper closed two-sided ideal in A is contained in some primitive ideal. Any strongly semi-simple Tauberian Banach algebra has such a property, too (cf. [2, Theorem 2.7.25]). Also Archbold's result was extended by the author [3, 4] to

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Received February 28, 1983. Research partially supported by the Grant-in-Aid for Scientific Research C-57540005 (1982) from the Ministry of Education.

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another more general context.

Finally we will consider the quasi-centrality of a general algebra. The quasi-centrality can be defined as well as previous definition (cf. [6, p. 70]).

Let \( A \) be a semi-simple algebra and suppose that the center \( Z(A) \) has an identity \( e \). If \( A \) is quasi-central, then \( e \) is already an identity of \( A \). In fact, for any primitive ideal \( P \) of \( A \), \( AeA(1-e) = \{0\} \subset P \) and hence either \( Ae \subset P \) or \( A(1-e) \subset P \) from [2, Theorem 2.2.9 (iv)]. If \( A \) is quasi-central, then the first inclusion is impossible and so it follows that \( A(1-e) \) is contained in the radical of \( A \). Therefore \( e \) is an identity of \( A \) because \( A \) is semi-simple.

We thank Professor K. Yabuta for helpful discussions.

References