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Continuity of the mean values of BMO functions and Calderón-Zygmund properties of certain singular integrals

Dedicated to Professor Shigeki YANO on his sixtieth birthday

Kôzô Yabuta*

In this note we shall show that certain bilinear singular integrals $T(a,f)$ with symmetric property in some sense are bounded bilinear operators from $\text{BMO} \times L^p$ into $L^p$, more precisely, for any BMO function $a$ the operator $T(a, \cdot)$ is a Calderón-Zygmund singular integral operator (Theorems 1 and 2). These results are, in a sense, extensions of the relating results in Baishansky and Coifman [1]. To prove the above, the Hölder continuity of the mean values of BMO functions plays the essential role. We also use “bilinear pseudo-differential operator” introduced by Coifman and Meyer [3], in order to prove $L^2$ boundedness of singular integrals. As an application we extend a recent result of J. Cohen on a bilinear singular integral [2], in the one-dimensional case.

Section 1 is for the preliminaries. In Section 2 we investigate the continuity of the mean values of BMO functions. We treat in Section 3 singular integrals such as $\text{p.v.} \int_{-\infty}^{\infty} [A(2x-y) + A(y) - 2A(x)](x-y)^{-1}f(y)dy$ for $A'(x) \in \text{BMO}$. Modifying the proof in Section 3 somewhat, we discuss, in Section 4, singular integrals such as $\text{p.v.} \int_{-\infty}^{\infty} [A(x) - A(y) - 2^{-1}(A'(x) + A'(y))(x-y)](x-y)^{-3}f(y)dy$ for $A''(x) \in \text{BMO}$. In Section 5 we extend a result of J. Cohen.

$C_0^\infty = C_0^\infty(R)$ will denote the set of all infinitely differentiable functions with compact support and $L^p = L^p(R)$ ($1 \leq p \leq \infty$) will denote the usual $L^p$ space with norm $\| \cdot \|_p$. $H^1(R)$ is the Hardy space $H^1$ in the sense of Stein-Weiss. Weak $L^p$ space is the set of all locally integrable functions $f$ satisfying $\sup_{t>0} \frac{1}{t} \int_{|f(x)|>t} |f(x)|dx < \infty$. Here $|E|$ denotes the Lebesgue measure of the set $E$. Finally we note that the letter $C$ will denote a constant which may vary from line to line.

1. Preliminaries

In the sequel, we treat only the one-dimensional case. For any interval $(a, b)$ we denote by $f_{(a,b)}$ the mean value of a function $f$ on $(a, b)$, i.e., $(b-a)^{-1} \int_a^b f(x)dx.
A locally integrable function $f$ on the real line $\mathbb{R}$ is said to be a function of bounded mean oscillation (BMO) if

$$\|f\|_{\text{BMO}} = \sup_{(a,b)} (b-a)^{-1} \int_{a}^{b} |f(x) - f_{(a,b)}| \, dx < +\infty.$$ 

Next we recall the definition of a kernel of Calderón-Zygmund in the sense of Coifman and Meyer [3, p. 79 and p. 94].

**Definition 1.** A function on $\mathbb{R} \times \mathbb{R} \setminus \{(x, x); x \in \mathbb{R}\}$ is said to be a kernel of Calderón-Zygmund, if it satisfies the following conditions.

1. For any $f \in C_0^\infty(\mathbb{R})$,

$$T(f)(x) = \lim_{\varepsilon \to 0} \int_{|y-x| > \varepsilon} K(x, y) f(y) \, dy = \text{p.v.} \int K(x, y) f(y) \, dy$$

exists for almost all $x \in \mathbb{R}$.

2. There exists $C_1 > 0$ such that

$$\|T(f)\|_2 \leq C_1 \|f\|_2, \quad f \in C_0^\infty(\mathbb{R}).$$

3. There exists $C_2 > 0$ such that

$$|K(x, y)| \leq C_2 |x-y|^{-1}.$$

4. There exist $C_3 > 0$ and $0 < \delta \leq 1$ such that for $0 < 2|y-z| \leq |x-z|$

4a. $|K(x, y) - K(x, z)| \leq C_3 |y-z|^{\delta} / |x-z|^{1+\delta}$,

4b. $|K(y, x) - K(z, x)| \leq C_3 |y-z|^{\delta} / |x-z|^{1+\delta}$.

This Calderón-Zygmund singular integral operator possesses the same properties as the classical one [3, Ch. IV.], i.e., it is a bounded operator on $L^p$ ($1 < p < \infty$), from $L^\infty$ to BMO, from $H^1$ to $L^1$ and from $L^1$ to weak-$L^1$. It has $L^p$-boundedness of truncated maximal operator, too, i.e., putting

$$T_\varepsilon(f)(x) = \int_{|y-x| > \varepsilon} K(x, y) f(y) \, dy$$

and $T_\ast(f)(x) = \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)|$, we have for $1 < p < \infty$

$$\|T_\ast(f)\|_p \leq C_p \|f\|_p, \quad f \in L^p.$$

2. Continuity of the mean values of a BMO function

From the John-Nirenberg inequality for BMO functions it follows the following two facts, which are useful in the later sections.

**Lemma 1.** There exists $C > 0$ such that

$$\text{E} \{ |f(x) - f_{(x,x)}| \} < C \|f\|_{\text{BMO}}, \quad f \in L^p.$$
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for any \( a \leq c < d \leq b \) and any \( f \in \text{BMO} \).

**Lemma 2.** For any \( d > 1 \) there exists \( C > 0 \) such that

\[
|f(a, b) - f(a, c)| \leq C\|f\|_\infty \left| \frac{b-a}{c-a} \right| \log \left| \frac{b-a}{c-b} \right|
\]

for any \( f \in \text{BMO} \) and \( d|b-c| \leq |b-a| \).

The second lemma shows Hölder continuity of the mean values of a BMO function. For the sake of completeness we shall give a proof of Lemma 2, in which the proof of Lemma 1 is simultaneously contained.

**Proof of Lemma 2.** Let \( f \in \text{BMO} \). As is easily seen, \( |f(a, b) - f(a, c)| \leq (c-a) \)

\[(b-a)^{-1} \int_c^d |f(t) - f(a, b)| dt \leq C\|f\|_\infty \frac{d-c}{b-a} \left( \log \frac{b-a}{d-c} + 1 \right)
\]

for any \( a < c < d < b \) and any \( f \in \text{BMO} \).

**Remark 1.** This proof is applicable, of course, in higher dimension and due to A. Uchiyama and A. Miyachi. Our original proof was much longer.

**Remark 2.** As is easily seen, one has

\[
|f(a, b) - f(a, c)| \leq 2\|f\|_\infty \left| \frac{c-b}{c-a} \right|
\]

However, for \( 1 < p < \infty \), one has only
where 1/p + 1/q = 1, and there is no continuity when |b - c|/|b - a| tends to zero. In fact, let 0 < c < a < 1 and set f(x) = |x|^{1-a} (|x| < 1), = 0 (|x| ≥ 1). Then, f ∈ L^p(−∞, ∞) (0 < p < 1/a) and for 0 < b < 1

\[ |f_{(0,b)} - f_{(0,b^1+b^2})| = \frac{a}{1-a} b^{e-a} + O(b^{2e-a}), \]

which tends to infinity as b tends to zero. But, b^{1+e}/b tends to zero. Thus, the continuity of mean values in the sense of Lemma 2 is characteristic for bounded functions and BMO functions in the framework of L^p spaces. This reflects the dilation-invariance of the L^∞ and BMO norms.

3. Singular integrals I

We consider here the following singular integrals. For a real number s and a BMO function a we define

\[ S_0^s(f)(x) = \text{p. v. } \int_{-\infty}^{\infty} K_s(x, y)f(y)dy, \]

\[ T_0^s(f)(x) = \text{p. v. } \int_{-\infty}^{\infty} K_s(x, y) \text{sgn}(x - y)f(y)dy, \]

where \( K_s(x, y) = [A(x + s(x - y)) + A(x - s(x - y)) - 2A(x)]/(x - y)^2 \) and \( A'(x) = a(x) \). The case \( s = 1 \) and \( a \in L^\infty \) is treated in Coifman and Meyer [3, pp. 160–163].

Now, one easily sees that for every \( f \in C_0^\infty(\mathbb{R}) \) the principal values of the above singular integrals exist almost everywhere. By modification of the arguments in Coifman and Meyer [3, pp. 160–163] or Corollary 4.2 in Yabuta [5], it follows that there exists \( C > 0 \) such that

\[ \|S_0^s(f)(x)\|_2 \leq C\|a\|_\infty\|f\|_2, \quad f \in C_0^\infty(\mathbb{R}). \]

This also holds for \( T_0^s \). Next, the kernel \( K(x, y) = K_s(x, y) \) satisfies the following inequalities, which we shall show soon later.

\[ |K(x, y)| \leq 4\|a\|_\infty|x - y|^{-1}, \]

\[ |K(x, y) - K(x, z)| \leq C\|a\|_\infty \frac{|y - z|}{|x - z|^3} \log \left| \frac{x - z}{y - z} \right|, \quad (2|y - z| < |x - z|), \]

\[ |K(y, x) - K(z, x)| \leq C\|a\|_\infty \frac{|y - z|}{|x - z|^3} \log \left| \frac{x - z}{y - z} \right|, \quad (2|y - z| < |x - z|). \]

The kernel \( K_s(x, y) \, \text{sgn}(x - y) \) also satisfies the same inequalities as above. (3.1) implies (1.3) and (3.2) and (3.3) imply (1.4). So, we have
THEOREM 1. For any \( a \in \text{BMO} \) and \( s \in \mathbb{R} \), the operators \( S_0^s \) and \( T_0^s \) are Calderón-Zygmund singular integral operators.

In order to prove the above theorem we have to show the inequalities (3.1), (3.2) and (3.4). (3.1) is easy. As for (3.2)

\[
K(x, y) - K(x, z) = \left[ A(x + s(x - z)) + A(x - s(x - z)) - 2A(x) \right] [(x - y)^{-1} - (x - z)^{-1}] (x - z)^{-1} \\
+ \left[ \frac{A(x + s(x - y)) + A(x - s(x - y)) - 2A(x)}{x - y} \right] (x - y)^{-1} = I_1 + I_2.
\]

Then, we have

\[
|I_1| \leq 4 \|a\| \frac{|y - z|}{|x - y||x - z|} \quad \text{(by (3.1))}
\]

\[
\leq C \frac{|y - z|}{|x - z|^2} \quad \text{(2|y - z| < |x - z|)}.
\]

And

\[
|x - y||I_2| \leq \left| \frac{1}{x - y} \int_x^{x+s(x-y)} a(t) dt - \frac{1}{x - z} \int_x^{x+s(x-z)} a(t) dt \right| + \frac{1}{x - y} \int_x^{x-s(x-y)} a(t) dt - \frac{1}{x - z} \int_x^{x-s(x-z)} a(t) dt \right|.
\]

Hence by Lemma 2

\[
|I_2| \leq C \frac{|y - z|}{|x - z|^2} \log \frac{|x - z|}{|y - z|}, \quad \text{(2|y - z| < |x - z|)}.
\]

This proves (3.2). Similar calculations give (3.3). This completes the proof of Theorem 1.

4. Singular integrals II

For a function \( A \) on the real line let \( P_k(A; x, y) = A(x) - A(y) - \cdots - \frac{A^{(k-1)}(y)}{(k-1)!} \)

\( (x - y)^{k-1} \) be the \( k \)-th Taylor series remainder of \( A \) at \( x \) expanded about \( y \). Let us consider the following singular integrals,

\[
T^k(a, f)(x) = 4\pi i \text{ p.v.} \int_{-\infty}^{\infty} \frac{P_k(A; x, y)}{(x - y)^{k+1}} f(y) dy,
\]

\[
S^k(a, f)(x) = 4\pi i \text{ p.v.} \int_{-\infty}^{\infty} \frac{P_k(A; x, y) - \frac{(A^{(k-1)}(x) - A^{(k-1)}(y))(x - y)^{k-1}/k!}{(x - y)^{k+1}}}{(x - y)^{k+1}} f(y) dy,
\]
where $a(x)$ is the $k$-th derivative of $A$. Let $\chi_1(\alpha, \xi)$ be the characteristic function of the sector $\{(r, \theta); r > 0, -\pi/4 < \theta < \pi/2\}$, $\chi_2$ for $\{(r, \theta); r > 0, \pi/2 < \theta < 3\pi/4\}$, $\chi_3$ for $\{(r, \theta); r > 0, 3\pi/4 < \theta < 3\pi/2\}$, and $\chi_4$ for $\{(r, \theta); r > 0, 3\pi/2 < \theta < 7\pi/4\}$.

Then for $a, f \in C_0^\infty$ we have

$$T^k(a, f)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x+t)} \sigma(T^k)(\alpha, \xi) \hat{a}((\xi))(\xi) d\alpha d\xi,$$

where $\hat{a}(\alpha) = \int_{-\infty}^{\infty} e^{ix\alpha} a(x) dx$ and $\sigma(T^k)(\alpha, \xi) = [\chi_1 - \chi_2 + \left(1 - 2\left(\frac{-\xi}{\alpha}\right)^k\right)\chi_2 - \chi_4]/k!$. In case $k = 1$ this formula is given in [5, Section 2], and calculated in the same way as in [3, p. 162]. General case can be obtained inductively, noticing $\sigma(T^k(-\alpha, -\xi) = \sigma(T^k)(\alpha, \xi)$. We leave the detailed proof to the reader. Hence we get

$$S^k(a, f)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x+t)} \sigma(S^k)(\alpha, \xi) \hat{a}((\xi))(\xi) d\alpha d\xi,$$

in which $\sigma(S^k)(\alpha, \xi) = 2\left(\frac{\xi}{\alpha} - \left(\frac{-\xi}{\alpha}\right)^k\right)(\chi_2 - \chi_4)/k!$. Since $\sigma(S^k)$ is homogeneous of degree zero and $\sigma(S^k)(\alpha, \xi) = \sigma(S^k)(r, \theta)$ is continuous and piecewise $C^\infty$ with respect to $\theta$ $(r, \theta)$ is the polar coordinates) and $\sigma(S^k)(0, \xi) = 0$, we have by Theorem 3.1 in Yabuta [5]

\begin{equation}
\|S^k(a, f)\|_2 \leq C\|a\|_\infty \|f\|_2, \quad a \in \text{BMO and } f \in L^2.
\end{equation}

This implies (1.2). (1.1) is easy. Next let $K(x, y)$ be the integral kernel for $S^k$. Then

$$K(x, y) = (k!)^{-1} \left[ \int_y^x (k(x-t)^{k-1} - (x-y)^{k-1})a(t) dt \right]/(x-y)^{k+1}.$$

Put $F(x, y, t) = (k(x-t)^{k-1} - (x-y)^{k-1})/k!$. Then

\begin{equation}
\int_y^x F(x, y, t) dt = 0,
\end{equation}

(4.3)\hspace{1cm}|F(x, y, t)| \leq C|x-y|^{k-1}, \text{ and}

(4.4)\hspace{1cm}|F(x, y, t) - F(x, z, t)|, |F(y, x, t) - F(z, x, t)| \leq C|y-z||x-y|^{k-2}.

To show (1.3) and (1.4) in Definition 1 we may assume $y < z < x$. Then we get by (4.2)

$$K(x, y) = \int_y^x F(x, y, t)(a(t) - a(y, x)) dt/(x-y)^{k+1}.$$

Hence by (4.3)

$$|K(x, y)| \leq C|x-y|^{-2} \int_y^x |a(t) - a(y, x)| dt \leq C \|a\|_\infty |x-y|^{-1}.$$

This means (1.3). Next,
Calderón-Zygmund properties

\[ K(x, y) - K(x, z) = (x - y)^{-k} \left[ \frac{1}{x - y} \int_y^x F(x, y, t) a(t) dt - \frac{1}{x - z} \int_z^x F(x, z, t) a(t) dt \right] \]

\[ + \left( (x - y)^{-k} - (x - z)^{-k} \right) \frac{1}{x - z} \int_z^x F(x, z, t) a(t) dt \]

\[ = I_1 + I_2. \]

For \( I_2 \) as in the above
\[ |I_2| \leq C \|a\|_{\text{BMO}} |y - z|/|x - z|^2, \quad (2|y - z| < |x - z|). \]

For \( I_1 \) we get
\[
(x - y)^k I_1 = \frac{1}{x - y} \int_y^x F(x, y, t) (a(t) - a_{(y, z)}) dt \\
+ \frac{1}{x - y} \int_z^x F(x, y, t) (a(t) - a_{(y, z)}) dt \\
+ \left( \frac{1}{x - y} - \frac{1}{x - z} \right) \int_z^x F(x, y, t) (a(t) - a_{(z, x)}) dt \\
+ \frac{1}{x - z} \int_z^x (F(x, y, t) - F(x, z, t)) (a(t) - a_{(z, x)}) dt \\
= I_3 + I_4 + I_5 + I_6.
\]

Then we have for \( 2|y - z| < |x - z| \)
\[ |I_3| \leq C \|a\|_{\text{BMO}} |x - y|^{k-2} |y - z| \log \left| \frac{x - y}{y - z} \right| \quad \text{(by Lemma 1 and (4.3))}, \]
\[ |I_4| \leq C \|a\|_{\text{BMO}} |x - y|^{k-2} |y - z| \log \left| \frac{x - y}{y - z} \right| \quad \text{(by Lemma 2 and (4.3))}, \]
\[ |I_5| \leq C \|a\|_{\text{BMO}} |x - y|^{k-2} |y - z| \quad \text{(by (4.3))}, \]
\[ |I_6| \leq C \|a\|_{\text{BMO}} |x - y|^{k-2} |y - z| \quad \text{(by (4.4))}. \]

These show (1.4a). Similarly (1.4b) holds. Therefore we have proved the following

**Theorem 2.** For any \( a \in \text{BMO}, S^k(a, \cdot) \) is a Calderón-Zygmund singular integral operator.

**Remark 1.** Naturally one can consider other combinations of Taylor series remainder terms.

**Remark 2.** For bounded \( a \), the corresponding results are known [1].
5. **Singular integrals III**

In this section we apply Theorem 2 to an extension of a recent result of J. Cohen on a singular integral [2]. The operators considered here are

\[ K^k(a, f)(x) = 4\pi i \text{ p.v.} \int_{-\infty}^{\infty} \frac{P_k(A; x, y)}{(x-y)^k} f(y) \, dy \]

and their truncated maximal operators \( K_k^*(a, f) \), where \( a \) is the \((k-1)\)-th derivative of \( A \). Then we have

**Theorem 3.** If \( a \in \text{BMO} \), \( 1 < p < \infty \), and \( k=2, 3, \ldots \), then for any \( f \in L^p \) \( K^k(a, f)(x) \) exists for almost every \( x \in \mathbb{R} \) and there exists \( C > 0 \) such that

\[ \|K^k(a, f)\|_p \leq C \|a\|_\infty \|f\|_p \quad \text{and} \]

\[ \|K_k^*(a, f)\|_p \leq C \|a\|_\infty \|f\|_p, \quad f \in L^p. \]

**Proof.** The case \( k=2 \) is contained in Corollary (1.2) in [2, p. 694]. So, let \( k=3, 4, \ldots \). Then, we have by easy calculation

\[ (5.1) \quad K^k(a, f) = S^{k-1}(a, f) + \frac{1}{(k-1)!} K^2(a, f). \]

Therefore, by the case \( k=2 \) and Theorem 2 we obtain the desired inequalities.

**Remark 1.** \( K^k(a, \cdot) \) is, in general, neither a bounded operator from \( H^1 \) into \( L^1 \) nor from \( L^1 \) into weak-\( L^1 \). To show them, it is enough to treat the case \( k=2 \), by Theorem 2 and (5.1). An example for the first case is given by \( a(x)=x|0, \infty)-x(-\infty,0) \) and \( f(x)=a(x)x(-1,1). \) For the second case \( a(x)=\log |x|, f(x)=x^{1<|x|<2}. \) We leave to the reader to check the above.

**Remark 2.** We note here that in Theorem (1.1) and Corollary (1.2) in [2, p. 694] \( C(\mathcal{F} A, \cdot) \) is not a bounded operator from \( L^1 \) into weak-\( L^1 \), if \( \mathcal{F} A \in \text{BMO}(\mathbb{R}^n) \) \((n \geq 2)\). An example is given by \( A(x)=x_1 \log |x|^2 \) and \( f(x)=x_{1<|x|<2}. \)

**References**


