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Continuity of the mean values of BMO functions and Calderón-Zygmund properties of certain singular integrals

Dedicated to Professor Shigeki YANO on his sixtieth birthday

Kôzô Yabuta*

In this note we shall show that certain bilinear singular integrals $T(a,f)$ with symmetric property in some sense are bounded bilinear operators from $\text{BMO} \times L^p$ into $L^p$, more precisely, for any BMO function $a$ the operator $T(a, \cdot)$ is a Calderón-Zygmund singular integral operator (Theorems 1 and 2). These results are, in a sense, extensions of the relating results in Baishansky and Coifman [1]. To prove the above, the H"older continuity of the mean values of BMO functions plays the essential role. We also use "bilinear pseudo-differential operator" introduced by Coifman and Meyer [3], in order to prove $L^2$ boundedness of singular integrals. As an application we extend a recent result of J. Cohen on a bilinear singular integral [2], in the one-dimensional case.

Section 1 is for the preliminaries. In Section 2 we investigate the continuity of the mean values of BMO functions. We treat in Section 3 singular integrals such as $\text{p.v.} \int_{-\infty}^{\infty} \left[A(2x-y) + A(y) - 2A(x)\right](x-y)^{-2}f(y)dy$ for $A'(x) \in \text{BMO}$. Modifying the proof in Section 3 somewhat, we discuss, in Section 4, singular integrals such as $\text{p.v.} \int_{-\infty}^{\infty} \left[A(x) - A(y) - 2^{-1}(A'(x) + A'(y))(x-y)\right](x-y)^{-3}f(y)dy$ for $A''(x) \in \text{BMO}$. In Section 5 we extend a result of J. Cohen.

$C_0^\infty = C_0^\infty(\mathbb{R})$ will denote the set of all infinitely differentiable functions with compact support and $L^p = L^p(\mathbb{R})$ ($1 \leq p \leq \infty$) will denote the usual $L^p$ space with norm $\| \cdot \|_p$. $H^1(\mathbb{R})$ is the Hardy space $H^1$ in the sense of Stein-Weiss. Weak $L^p$ space is the set of all locally integrable functions $f$ satisfying $\sup_{t>0} \left(\int_{\{x \in \mathbb{R}; \ |f(x)| > t\}} |f(x)| \right) < + \infty$. Here $|E|$ denotes the Lebesgue measure of the set $E$. Finally we note that the letter $C$ will denote a constant which may vary from line to line.

1. Preliminaries

In the sequel, we treat only the one-dimensional case. For any interval $(a, b)$ we denote by $f_{(a,b)}$ the mean value of a function $f$ on $(a, b)$, i.e., $(b-a)^{-1} \int_a^b f(x)dx$. 

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A locally integrable function \( f \) on the real line \( \mathbb{R} \) is said to be a function of bounded mean oscillation (BMO) if \( \|f\|_{BMO} = \sup_{(a,b)} (b-a)^{-1} \int_a^b |f(x) - f_{(a,b)}| \, dx < +\infty \). Next we recall the definition of a kernel of Calderón-Zygmund in the sense of Coifman and Meyer [3, p. 79 and p. 94].

**Definition 1.** A function on \( \mathbb{R} \times \mathbb{R} \backslash \{(x, x); x \in \mathbb{R}\} \) is said to be a kernel of Calderón-Zygmund, if it satisfies the following conditions.

1. For any \( f \in C_0^\infty(\mathbb{R}) \),
   \[
   T(f)(x) = \lim_{\varepsilon \to 0} \int_{|y-x| > \varepsilon} K(x, y) f(y) \, dy = p.v. \int K(x, y) f(y) \, dy
   \]
   exists for almost all \( x \in \mathbb{R} \).

2. There exists \( C_1 > 0 \) such that
   \[
   \|T(f)\|_2 \leq C_1 \|f\|_2, \quad f \in C_0^\infty(\mathbb{R}).
   \]

3. There exists \( C_2 > 0 \) such that
   \[
   |K(x, y)| \leq C_2 |x-y|^{-1}.
   \]

4. There exist \( C_3 > 0 \) and \( 0 < \delta \leq 1 \) such that for \( 0 < 2|y-z| \leq |x-z| \)
   \[
   \begin{align*}
   (1.4a) \quad |K(x, y) - K(x, z)| &\leq C_3 |y-z|^{\delta}/|x-z|^{1+\delta}, \\
   (1.4b) \quad |K(y, x) - K(z, x)| &\leq C_3 |y-z|^{\delta}/|x-z|^{1+\delta}.
   \end{align*}
   
   This Calderón-Zygmund singular integral operator possesses the same properties as the classical one [3, Ch. IV.], i.e., it is a bounded operator on \( L^p \) \((1 < p < \infty)\), from \( L^\infty \) to BMO, from \( H^1 \) to \( L^1 \) and from \( L^1 \) to weak-\( L^1 \). It has \( L^p \)-boundedness of truncated maximal operator, too, i.e., putting
   \[
   T_\varepsilon(f)(x) = \int_{|x-y| > \varepsilon} K(x, y) f(y) \, dy
   \]
   and \( T_\ast(f)(x) = \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)| \), we have for \( 1 < p < \infty \)
   \[
   \|T_\ast(f)\|_p \leq C_p \|f\|_p, \quad f \in L^p.
   \]

2. Continuity of the mean values of a BMO function

   From the John-Nirenberg inequality for BMO functions it follows the following two facts, which are useful in the later sections.

   **Lemma 1.** There exists \( C > 0 \) such that
for any \( a \leq c < d \leq b \) and any \( f \in \text{BMO} \).

**Lemma 2.** For any \( d > 1 \) there exists \( C > 0 \) such that

\[
|f_{(a,b)} - f_{(a,c)}| \leq C \|f\|_\text{BMO} \left| \frac{c-b}{b-a} \right| \left( \log \frac{b-a}{d-c} + 1 \right)
\]

for any \( f \in \text{BMO} \) and \( d|b-c| \leq |b-a| \).

The second lemma shows Hölder continuity of the mean values of a BMO function. For the sake of completeness we shall give a proof of Lemma 2, in which the proof of Lemma 1 is simultaneously contained.

**Proof of Lemma 2.** Let \( f \in \text{BMO} \). As is easily seen, \( |f_{(a,b)} - f_{(a,c)}| \leq (c-a) (b-a)^{-1} \|f\|_\text{BMO} \) \((a < b < c)\). So we may assume \( 1 < d < 2 \). We may also assume \( \|f\|_\text{BMO} = 1 \), \( a = 0 \), \( b = 1 \), \( 1 < c < 2 \) and \( f_{(0,1)} = 0 \). Then, since \( |f_{(0,2)}| = |f_{(0,1)} - f_{(0,2)}| \leq 2 \), by John-Nirenberg's theorem

\[
(2.1) \quad |\{x \in (0,2); |f(x)| > t\}| \leq \left| \{x \in (0,2); |f(x) - f_{(0,2)}| > t - |f_{(0,2)}|\} \right| \leq C_1 e^{-C_2 t}.
\]

Therefore, for \( 1 < p < \infty \) and \( 0 < s < 1 \) we have

\[
\left( \int_1^{1+s} |f(x)|^p \, dx \right)^{1/p} \leq C(p\Gamma(p))^{1/p} \leq C' p.
\]

Thus, taking \( p = -\log s \) and \( 1/q = 1 - 1/p \), and using Hölder's inequality we get

\[
\int_1^{1+s} f(x) \, dx \leq \frac{s^{1/q}}{\Gamma(1 + s)} \left( \int_1^{1+s} |f(x)|^p \, dx \right)^{1/p} \leq C_3 p^{1/q} = C_3 s \log s^{-1}.
\]

(This inequality can also be obtained by using (2.1) and the rearrangement of \( f \).) Therefore, since \( \int_0^1 f(x) \, dx = 0 \), we get

\[
\int_0^{1+s} f(x) \, dx - \int_0^1 f(x) \, dx \leq (1+s)^{-1} \int_1^{1+s} f(x) \, dx \leq C_3 s \log s^{-1},
\]

which completes the proof of Lemma 2.

**Remark 1.** This proof is applicable, of course, in higher dimension and due to A. Uchiyama and A. Miyachi. Our original proof was much longer.

**Remark 2.** As is easily seen, one has

\[
|f_{(a,b)} - f_{(a,c)}| \leq 2 \|f\|_\text{BMO} \left| \frac{c-b}{c-a} \right|.
\]

However, for \( 1 < p < \infty \), one has only
\[ |f_{(a,b)} - f_{(a,c)}| \leq C \| f \|_p \frac{|c-b|^{1/q}}{|c-a|} \quad (d|c-b| \leq |b-a|), \]

where \( 1/p + 1/q = 1 \), and there is no continuity when \( |b-c|/|b-a| \) tends to zero. In fact, let \( 0 < c < a < 1 \) and set \( f(x) = |x|^{-a} (|x| < 1), = 0 (|x| \geq 1) \). Then, \( f \in L^p(-\infty, \infty) \) (\( 0 < p < 1/a \)) and for \( 0 < b < 1 \)

\[ |f_{(0,b)} - f_{(0,b+c/b)}| = \frac{a}{1-a} b^{b-a} + O(b^{2b-a}), \]

which tends to infinity as \( b \) tends to zero. But, \( b^{b+c/b} \) tends to zero. Thus, the continuity of mean values in the sense of Lemma 2 is characteristic for bounded functions and BMO functions in the framework of \( L^p \) spaces. This reflects the dilation-invariance of the \( L^\infty \) and BMO norms.

3. Singular integrals I

We consider here the following singular integrals. For a real number \( s \) and a BMO function \( a \) we define

\[ S_0^s(f)(x) = \text{p. v.} \int_{-\infty}^{\infty} K_s(x, y) f(y) dy, \]

and

\[ T_0^s(f)(x) = \text{p. v.} \int_{-\infty}^{\infty} K_s(x, y) \text{sgn}(x-y) f(y) dy, \]

where \( K_s(x, y) = [A(x+s(x-y)) + A(x-s(x-y)) - 2A(x)]/(x-y)^2 \) and \( A'(x) = a(x) \). The case \( s = 1 \) and \( a \in L^\infty \) is treated in Coifman and Meyer [3, pp. 160-163].

Now, one easily sees that for every \( f \in C_0^\infty(\mathbb{R}) \) the principal values of the above singular integrals exist almost everywhere. By modification of the arguments in Coifman and Meyer [3, pp. 160–163] or Corollary 4.2 in Yabuta [5], it follows that there exists \( C > 0 \) such that

\[ \| S_0^s(f) \|_2 \leq C \| a \|_s \| f \|_2, \quad f \in C_0^\infty(\mathbb{R}). \]

This also holds for \( T_0^s \). Next, the kernel \( K(x, y) = K_s(x, y) \) satisfies the following inequalities, which we shall show soon later.

(3.1) \( |K(x, y)| \leq 4 \| a \|_s |x-y|^{-1}, \)

(3.2) \( |K(x, y) - K(x, z)| \leq C \| a \|_s \frac{|y-z|}{|x-z|^2} \log \left| \frac{x-z}{y-z} \right|, \quad (2|y-z| < |x-z|), \)

(3.3) \( |K(y, x) - K(z, x)| \leq C \| a \|_s \frac{|y-z|}{|x-z|^2} \log \left| \frac{x-z}{y-z} \right|, \quad (2|y-z| < |x-z|). \)

The kernel \( K_s(x, y) \text{sgn}(x-y) \) also satisfies the same inequalities as above. (3.1) implies (3.3) and (3.2) and (3.3) imply (1.4). So, we have
THEOREM 1. For any $a \in \text{BMO}$ and $s \in \mathbb{R}$, the operators $S_0^s$ and $T_0^s$ are Calderón-Zygmund singular integral operators.

In order to prove the above theorem we have to show the inequalities (3.1), (3.2) and (3.4). (3.1) is easy. As for (3.2)

$$K(x, y) - K(x, z)$$

$$= \left[ A(x+s(x-z)) + A(x-s(x-z)) - 2A(x) \right] \left[ (x-y)^{-1} - (x-z)^{-1} \right] (x-z)^{-1}$$

$$+ \left[ \frac{A(x+s(x-y)) + A(x-s(x-y)) - 2A(x)}{x-y} \right] \times (x-y)^{-1} = I_1 + I_2.$$ 

Then, we have

$$|I_1| \leq 4\|a\|_{\text{BMO}} \frac{|y-z|}{|x-y||x-z|} \quad \text{(by (3.1))}$$

$$\leq C \frac{|y-z|}{|x-z|^2} \quad (2|y-z| < |x-z|).$$

And

$$|x-y||I_2| \leq \left| \frac{1}{x-y} \int_{x}^{x+s(x-y)} a(t) dt - \frac{1}{x-z} \int_{x}^{x+s(x-z)} a(t) dt \right|$$

$$+ \left| \frac{1}{x-y} \int_{x}^{x-s(x-y)} a(t) dt - \frac{1}{x-z} \int_{x}^{x-s(x-z)} a(t) dt \right|.$$ 

Hence by Lemma 2

$$|I_2| \leq C \frac{|y-z|}{|x-z|^2} \log \left| \frac{x-z}{y-z} \right|, \quad (2|y-z| < |x-z|).$$

This proves (3.2). Similar calculations give (3.3). This completes the proof of Theorem 1.

4. Singular integrals II

For a function $A$ on the real line let $P_k(A; x, y) = A(x) - A(y) - \cdots - \frac{A^{(k-1)}(y)}{(k-1)!}$ be the $k$-th Taylor series remainder of $A$ at $x$ expanded about $y$. Let us consider the following singular integrals,

$$T^k(a, f)(x) = 4\pi i \text{ p.v.} \int_{-\infty}^{\infty} \frac{P_k(A; x, y)}{(x-y)^{k+1}} f(y) dy,$$

$$S^k(a, f)(x) = 4\pi i \text{ p.v.} \int_{-\infty}^{\infty} \frac{P_k(A; x, y) - A^{(k-1)}(x) - A^{(k-1)}(y))(x-y)^{k-1}/k!}{(x-y)^{k+1}} f(y) dy,$$
where \(a(x)\) is the \(k\)-th derivative of \(A\). Let \(\chi_1(\alpha, \xi)\) be the characteristic function of the sector \(\{(r, \theta); r > 0, -\pi/4 < \theta < \pi/2\}\), \(\chi_2\) for \(\{(r, \theta); r > 0, \pi/2 < \theta < 3\pi/4\}\), \(\chi_3\) for \(\{(r, \theta); r > 0, 3\pi/4 < \theta < 3\pi/2\}\), and \(\chi_4\) for \(\{(r, \theta); r > 0, 3\pi/2 < \theta < 7\pi/4\}\). Then for \(a, f \in C_0^\infty\) we have

\[
T^k(a, f)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x+y)} \sigma(T^k)(x, \xi) \hat{a}(\xi) \hat{f}(\xi) d\xi d\xi,
\]

where \(\hat{a}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} a(x) dx\) and \(\sigma(T^k)(x, \xi) = [\chi_1 - \chi_2 + \left(1 - 2 \left(\frac{-\xi}{\alpha}\right)^k\right)(\chi_2 - \chi_4)]/k!\). In case \(k=1\) this formula is given in [5, Section 2], and calculated in the same way as in [3, p. 162]. General case can be obtained inductively, noticing \(\sigma(T^k)(-\alpha, -\xi) = \sigma(T^k)(\alpha, \xi)\). We leave the detailed proof to the reader. Hence we get

\[
S^k(a, f)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x+y)} \sigma(S^k)(x, \xi) \hat{a}(\xi) \hat{f}(\xi) d\xi d\xi,
\]

in which \(\sigma(S^k)(\alpha, \xi) = 2 + \left(\frac{\xi}{\alpha} - \left(\frac{-\xi}{\alpha}\right)^k\right)(\chi_2 - \chi_4)/k!\). Since \(\sigma(S^k)\) is homogeneous of degree zero and \(\sigma(S^k)(\alpha, \xi) = \sigma(S^k)(r, \theta)\) is continuous and piecewise \(C^\infty\) with respect to \(\theta\) \((r, \theta)\) is the polar coordinates) and \(\sigma(S^k)(0, \xi) = 0\), we have by Theorem 3.1 in Yabuta [5]

\[
(4.1) \quad \|S^k(a, f)\|_2 \leq C\|a\|_\infty \|f\|_2, \quad a \in \text{BMO} \quad \text{and} \quad f \in L^2.
\]

This implies (1.2). (1.1) is easy. Next let \(K(x, y)\) be the integral kernel for \(S^k\). Then

\[
K(x, y) = (k!)^{-1} \left[ \int_y^x (k(x-t)^{k-1} - (x-y)^{k-1}) a(t) dt \right]/(x-y)^{k+1}.
\]

Put \(F(x, y, t) = (k(x-t)^{k-1} - (x-y)^{k-1})/k!\). Then

\[
(4.2) \quad \int_y^x F(x, y, t) dt = 0,
\]

\[
(4.3) \quad |F(x, y, t)| \leq C|x-y|^{k-1}, \quad \text{and}
\]

\[
(4.4) \quad |F(x, y, t) - F(x, z, t)|, \quad |F(y, x, t) - F(z, x, t)| \leq C|y-z| |x-y|^{k-2}.
\]

To show (1.3) and (1.4) in Definition 1 we may assume \(y < z < x\). Then we get by (4.2)

\[
K(x, y) = \int_y^x F(x, y, t)(\alpha(t) - \alpha(y, x)) dt/(x-y)^{k+1}.
\]

Hence by (4.3)

\[
|K(x, y)| \leq C|x-y|^{-2} \int_y^x |\alpha(t) - \alpha(y, x)| dt \leq C\|a\|_\infty |x-y|^{-1}.
\]

This means (1.3). Next,
Calderón-Zygmund properties

\[ K(x, y) - K(x, z) = (x - y)^{-k} \mathcal{L} \left[ \frac{1}{x - y} \int_y^z F(x, y, t) a(t) \, dt - \frac{1}{x - z} \int_z^x F(x, z, t) a(t) \, dt \right] \]

\[ + ((x - y)^{-k} - (x - z)^{-k}) \frac{1}{x - z} \int_z^x F(x, z, t) a(t) \, dt \]

\[ = I_1 + I_2. \]

For \( I_2 \) as in the above

\[ |I_2| \leq C \|a\|_\infty |y - z|/|x - z|^2, \quad (2|y - z| < |x - z|). \]

For \( I_1 \) we get

\[ (x - y)^k I_1 = \frac{1}{x - y} \int_y^x F(x, y, t)(a(t) - a(y, x)) \, dt \]

\[ + \frac{1}{x - y} \int_x^z F(x, y, t)(a(t) - a(y, x)) \, dt \]

\[ + \left( \frac{1}{x - y} - \frac{1}{x - z} \right) \int_z^x F(x, y, t)(a(t) - a(z, x)) \, dt \]

\[ + \frac{1}{x - z} \int_z^x (F(x, y, t) - F(x, z, t))(a(t) - a(z, x)) \, dt \]

\[ = I_3 + I_4 + I_5 + I_6. \]

Then we have for \( 2|y - z| < |x - z| \)

\[ |I_3| \leq C \|a\|_\infty |x - y|^{k-2}|y - z| \log \left| \frac{x - y}{y - z} \right| \quad (\text{by Lemma 1 and (4.3)}), \]

\[ |I_4| \leq C \|a\|_\infty |x - y|^{k-2}|y - z| \log \left| \frac{x - y}{y - z} \right| \quad (\text{by Lemma 2 and (4.3)}), \]

\[ |I_5| \leq C \|a\|_\infty |x - y|^{k-2}|y - z| \quad (\text{by (4.3)}), \]

\[ |I_6| \leq C \|a\|_\infty |x - y|^{k-2}|y - z| \quad (\text{by (4.4)}). \]

These show (1.4a). Similarly (1.4b) holds. Therefore we have proved the following

**Theorem 2.** For any \( a \in BMO \), \( S^k(a, \cdot) \) is a Calderón-Zygmund singular integral operator.

**Remark 1.** Naturally one can consider other combinations of Taylor series remainder terms.

**Remark 2.** For bounded \( a \), the corresponding results are known [1].
5. Singular integrals III

In this section we apply Theorem 2 to an extension of a recent result of J. Cohen on a singular integral [2]. The operators considered here are

\[ K^k(a, f)(x) = 4\pi i \text{ p.v.} \int_{-\infty}^{\infty} \frac{P_k(A; x, y)}{(x-y)^k} f(y) \, dy \]

and their truncated maximal operators \( K^*_k(a, f) \), where \( a \) is the \((k-1)\)-th derivative of \( A \). Then we have

**Theorem 3.** If \( a \in BMO \), \( 1 < p < \infty \), and \( k=2, 3, \ldots \), then for any \( f \in L^p \) \( K^k(a, f)(x) \) exists for almost every \( x \in \mathbb{R} \) and there exists \( C > 0 \) such that

\[
\| K^k(a, f) \|_p \leq C \| a \|_\infty \| f \|_p \quad \text{and} \\
\| K^*_k(a, f) \|_p \leq C \| a \|_\infty \| f \|_p, \quad f \in L^p.
\]

**Proof.** The case \( k=2 \) is contained in Corollary (1.2) in [2, p. 694]. So, let \( k=3, 4, \ldots \). Then, we have by easy calculation

\[ (5.1) \quad K^k(a, f) = S^{k-1}(a, f) + \frac{1}{(k-1)!} K^2(a, f). \]

Therefore, by the case \( k=2 \) and Theorem 2 we obtain the desired inequalities.

**Remark 1.** \( K^k(a, \cdot) \) is, in general, neither a bounded operator from \( H^1 \) into \( L^1 \) nor from \( L^1 \) into weak-\( L^1 \). To show them, it is enough to treat the case \( k=2 \), by Theorem 2 and (5.1). An example for the first case is given by \( a(x) = \chi_{(0,\infty)} - \chi_{(-\infty,0)} \), and \( f(x) = a(x) \chi_{(-1,1)} \). For the second case \( a(x) = \log |x| \), \( f(x) = \chi_{(1,2)} \). We leave to the reader to check the above.

**Remark 2.** We note here that in Theorem (1.1) and Corollary (1.2) in [2, p. 694] \( C(FA, \cdot) \) is not a bounded operator from \( L^1 \) into weak-\( L^1 \), if \( FA \in BMO(\mathbb{R}^n) \) \((n \geq 2)\). An example is given by \( A(x) = x_1 \log |x|^2 \) and \( f(x) = \chi_{\{1 < |x| < 2\}} \).

**References**


