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On Semigroups of the Poisson Transforms and
Groups of Boundary Values of the Poisson
Transforms on Weighted \( L^p \) Spaces

Katsuo Takano*

In this note by applying the weighted norm inequalities for maximal functions
and singular integrals obtained by B. Muckenhoupt [7] and R. Hunt, B.
Muckenhoupt and R. Wheeden [6] we obtain the general extension of the semi-
groups of the Poisson transforms on \( L^p(\mathbb{R}) \) and the group of boundary values of
the Poisson transforms on \( L^2(\mathbb{R}) \) obtained by E. Hille [3] and [4 Example 5 (50)].

Throughout in this note let \( p > 1 \). Let us introduce weight functions
\[ \mu(x) = w(x) \text{ or } (1 + x^2)^{-p/2} w(x) \geq 0 \]
and weighted \( L^p_w \) spaces
\[ L^p_w(\mathbb{R}) = \{ f : f(x) \text{ is Lebesgue measurable and} \}
\[ \int_{-\infty}^{\infty} |f(x)|^p \mu(x) dx < \infty \}
with the norm
\[ \| f \|_{p,w} = \left[ \int_{-\infty}^{\infty} |f(x)|^p \mu(x) dx \right]^{1/p}. \]

Suppose that the weight \( w(x) \) is Lebesgue measurable, almost everywhere positive
and \( w(x), w(x)^{-1/(p-1)} \) are locally integrable and Muckenhoupt's condition holds,
that is,
\[ \sup_I \left( \frac{1}{|I|} \int_I w(x) dx \right) \left( \frac{1}{|I|} \int_I [w(x)]^{-1/(p-1)} dx \right)^{p-1} < \infty, \]
where \( I \) denotes a bounded interval and \( |I| \) is the length of the interval \( I \). It is
known in [7] that \( w(x) = |x|^a \) for \( -1/p < a < 1 - 1/p \) satisfies Muckenhoupt's condition.
Note that the space \( L^p_w(\mathbb{R}) \) is the case \( \mu(x) = 1 \) and that \( \mu(x) = (1 + x^2)^{-1} \)
does not satisfy Muckenhoupt's condition. The space \( L^p_w(\mathbb{R}) \) is a Banach space.
We use \( p' \) to denote the index conjugate to \( p \); \( 1/p + 1/p' = 1 \). It is known in [6],
[8] that

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* Department of Mathematics, College of General Education, Ibaraki University, Mito,
Ibaraki 310, Japan.
It is easily seen that the dual space of $L_p^p(\mathbb{R})$ is $L_{p-1/(p-1)}^p(\mathbb{R})$ and by (1), (2) that Schwartz's testing function space $S(\mathbb{R})$ is dense in $L_p^p(\mathbb{R})$ and in $L_{p-1/(p-1)}^p(\mathbb{R})$.

Since the system of functions
\[ A_0 + \sum_{j=1}^{\infty} \frac{A_j}{x - \xi_j - i\eta_j} \]
is dense in $C_\infty$ provided that $\xi_j + i\eta_j \neq \xi_k + i\eta_k$ ($j \neq k$), $\eta_j \neq 0$ ($j = 1, 2, \ldots$) and
\[ \sum_{\eta_j > 0} \frac{\eta_j}{1 + \xi_j^2 + \eta_j^2} \leq \sum_{\eta_j < 0} \frac{-\eta_j}{1 + \xi_j^2 + \eta_j^2} = \infty, \]
the space $L_p^p(\mathbb{R})$ is separable (cf. [1]). Let us define a maximal function $f^*$ of $f$ in $L_{p}^p(\mathbb{R})$ as the definition in [2]. Then we will make use of the following results (cf. [2], [6], [7], [9]).

1. The Hilbert transform of $f$ in $L_{p}^p(\mathbb{R})$ can be defined by
\[ \lim_{\eta \to +0} \int_{|y| \geq \eta} \frac{f(x-y)}{\pi y} dy = (Hf)(x), \]  
(3) where the convergence is taken in the $L_{p}^p$ norm.

2. \[ \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \frac{x-y}{(x-y)^2 + r^2} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} (Hf)(y) \frac{r}{(x-y)^2 + r^2} dy, \]  
(4)

3. \[ \left| \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \frac{r}{(x-y)^2 + r^2} dy \right| \leq Cf^*(x), \]  
(5)
for $r > 0$ and $f$ in $L_{p}^p(\mathbb{R})$.

4. $\|f^*\|_{p,w} \leq C \|f\|_{p,w}$,  
(6)

5. $\|Hf\|_{p,w} \leq C \|f\|_{p,w}$  
(7)
for $f$ in $L_{p}^p(\mathbb{R})$. Here $C$ denotes the different constant numbers in (5), (6) and (7).

Let us denote the complex numbers with positive real part by $\mathbb{C}^+$. Let us consider the following function with parameter $t$ in $\mathbb{C}^+$,

\[ p(t, x) = \frac{t}{\pi(x^2 + t^2)}. \]

**Lemma 1** For any fixed $t$ in $\mathbb{C}^+$, $p(t, x)$ belongs to $L_p^p(\mathbb{R})$ and $L_{p-1/(p-1)}^p(\mathbb{R})$.

**Proof.** Let $t = r - iq$, $r > 0$. Then
\[ |p(r - iq, u)| = \frac{(r^2 + q^2)^{1/2}}{\pi[(r^2 - q^2) + u^2]^{1/2} + 4r^2q^2} \]
(8)
holds and for all $|u|$ sufficiently large we obtain
\[ |p(r-iq, u)| \leq (r^2 + q^2)^{1/2} [\pi(u^2 + (r^2 - q^2))]^{-1}. \] (7)

Hence by (1) we see that
\[
\int_{|u| \geq k} |p(r-iq, u)|^p w(u) du \leq \sup_{u \in \mathbb{R}} \left[ |p(r-iq, u)| (1 + |u|) \right]^p \cdot \int_{|u| \geq k} \frac{w(u)}{(1 + |u|)^p} du < \infty.
\]
for sufficiently large $k$. This shows $p(r-iq, u) \in L^p_w(\mathbb{R})$. Also by (2) we see that
\[
\int_{|u| \geq k} |p(r-iq, u)|^{p'} w^{-1/(p-1)}(u) du \leq \sup_{u \in \mathbb{R}} \left[ |p(r-iq, u)| (1 + |u|) \right]^{p'} \int_{|u| \geq k} \frac{w(u)^{1/(p-1)}}{(1 + |u|)^{p'}} du < \infty
\]
for sufficiently large $k$. This shows $p(r-iq, u) \in L^{p'}_{w^{-1/(p-1)}}(\mathbb{R})$. For the case $\mu(x) = (1+x^2)^{-p/2}w(x)$ we also obtain
\[
\int_{|u| \geq k} |p(r-iq, u)|^{p'} \left[ (1 + u^2)^{-p/2} w(u) \right]^{-1/(p-1)} du \leq \sup_{u \in \mathbb{R}} |p(r-iq, u)(1 + u^2)|^{p'} \int_{|u| \geq k} \frac{w(u)^{-1/(p-1)}}{(1 + u^2)^{p'/2}} du < \infty
\]
for sufficiently large $k$. Consequently $p(r-iq, u) \in L^{p}_{\mu^{-1/(p-1)}}(\mathbb{R})$. Q.E.D.

For $t$ in $\mathbb{C}^+$ and $f$ in $L^p_{\mu}(\mathbb{R})$ let
\[
(T(t)f)(x) = \int_{-\infty}^{\infty} p(t, x-u)f(u)du,
\]
\[
(T(0)f)(x) = f(x).
\]

By Lemma 1 and the H"{o}lder inequality the above integral exists for all $x$.

**Lemma 2**

1) $T(t)$ is a linear bounded operator on $L^p_{\mu}(\mathbb{R})$ to itself, and
2) $\|T(t)\| \leq B$ holds for $t = r-iq$, $0 < r \leq 1$, $|q| \leq 1$, where $B$ is a constant number.

**Proof.** 1) Let us prove the case $\mu(x) = (1+x^2)^{-p/2}w(x)$. We use the following decomposition of the kernel $p(t, x-u)$,
\[
p(t, x-u) = \frac{1}{2\pi i} \left[ -\frac{x-ia}{(u-ia+i\theta)(u-x+i\theta)} + \frac{x-ia}{(u-ia-i\theta)(u-x-i\theta)} + \frac{2it}{(u-ia-i\theta)(u-ia+i\theta)} \right]
\]
for a positive number $a$ and $\text{Re } t = a$. Let $t = r-iq$, $r > 0$ and...
\[ f(u; a, r, q) = \frac{f(u)}{u - i(a + r)} = \frac{f(u)}{u + q - i(a - r)}. \]

Then \( f(u; a, r, q) \) belongs to \( L^p_\omega(\mathbb{R}) \). Using (10) we obtain

\[
(T(t)f)(x) = -\frac{x}{2\pi} \int_{-\infty}^{\infty} \frac{f(u; a, r, q)}{u - x + q + 1r} du + x \int_{-\infty}^{\infty} \frac{f(u; a, r, -q)}{u - x - q - 1r} du + \frac{r - iq}{\pi} \int_{-\infty}^{\infty} \frac{f(u; a, r, q)}{u - q - i(a + r)} du. \tag{11}
\]

Let us show that

\[
\| T(t) \| \leq C \left[ \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u + q - i(a - r)} \right| + \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - q - i(a + r)} \right| \right] + \frac{1}{\pi} \left[ (r^2 + q^2)^{1/2} \sup_{w \in \mathbb{R}} \left| \frac{u - i}{u + q - i(a - r)} \right| \left[ \int_{-\infty}^{\infty} \left[ \frac{w(u)}{(u - q)^2 + (r + a)^2} \right]^{1/2} du \right]^{1/p'} \cdot \left[ \int_{-\infty}^{\infty} w(u) \left( 1 + u^2 \right)^{-p/2} du \right]^{1/p}, \tag{12}\right.
\]

where \( C \) is a constant number. With the first term of (11) we see by (4)–(7) that

\[
\left[ \int_{-\infty}^{\infty} \frac{1}{\pi} \left| \int_{-\infty}^{\infty} \frac{f(u; a, r, q)}{u - x + q + 1r} du \right|^p w(x) dx \right]^{1/p} \leq \left[ \int_{-\infty}^{\infty} \frac{1}{\pi} \left| \int_{-\infty}^{\infty} (Hf(\cdot; a, r, q))(u) \left( \frac{r}{u - x + q + 1r} \right) du \right|^p w(x) dx \right]^{1/p} + \left[ \int_{-\infty}^{\infty} \frac{1}{\pi} \left| \int_{-\infty}^{\infty} f(u; a, r, q) \left( \frac{r}{u - x + q + 1r} \right) du \right|^p w(x) dx \right]^{1/p} \leq C \left[ \| Hf(\cdot; a, r, q) \|_{p, w} + \| f(\cdot; a, r, q) \|_{p, w} \right] \leq C \left[ \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u + q - i(a - r)} \right| \right] \| f \|_{p, u}. \tag{13}\]

With the second term of (11) we also obtain

\[
\left[ \int_{-\infty}^{\infty} \frac{1}{\pi} \left| \int_{-\infty}^{\infty} \frac{f(u; a, r, q)}{u - x - q - 1r} du \right|^p w(x) dx \right]^{1/p} \leq C \left[ \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - q - i(a + r)} \right| \right] \| f \|_{p, u}. \tag{14}\]

With the third term of (11) we see by the Hölder inequality that

\[
\left| \int_{-\infty}^{\infty} \frac{f(u; a, r, q)}{u - q - i(a + r)} du \right| \leq \| f(\cdot; a, r, q) \|_{p, w} \cdot \left[ \int_{-\infty}^{\infty} \left[ w(u) \right]^{1/(p - 1)} \left[ (u - q)^2 + (r + a)^2 \right]^{-p/2} du \right]^{1/p'},
\]

and consequently we obtain
Summarizing (13), (14) and (15) we obtain (12). In the case \( \mu(x) = w(x) \) by using the decomposition of \( p(t, x-u) \), and by the same arguments as the above case \( \mu(x) = (1+x^2)^{-p/2}w(x) \) we can show that \( \|\mathcal{T}(t)\| \leq B \) holds for all \( t \) in \( C^+ \), where \( B \) is a constant number.

2) If we take \( a=2 \) in (12) we see that there exists a constant number \( B \) such that \( \|\mathcal{T}(t)\| \leq B \) holds for \( t=r-iq, \, 0<r\leq 1, \, |q|\leq 1 \). Q.E.D.

**THEOREM 1** The family \( \{\mathcal{T}(t): t \in C^+\} \) is a holomorphic semigroup of linear operators of class \( (C_0) \) on \( (0, \infty) \).

2) The infinitesimal generator \( A \) and its domain \( D(A) \) of the semigroup of \( \{\mathcal{T}(t): 0\leq t<\infty\} \) are given as follows.

i) If \( \mu(x) = w(x) \),

\[
(\mathcal{A}f)(x) = -\frac{d}{dx} (\mathcal{H}f)(x) = -\left( \mathcal{H} \frac{d}{dx} f \right)(x)
\]

for \( f \) in \( D(A) = \{ f \in L^p_c(\mathbb{R}) : (\mathcal{H}f)(x) \text{ is absolutely continuous and } \frac{d}{dx} (\mathcal{H}f)(x) \in L^p_c(\mathbb{R}) \} \).

ii) If \( \mu(x) = (1+x^2)^{-p/2}w(x) \),

\[
(\mathcal{A}f)(x) = -(x-i) \frac{d}{dx} \left( \mathcal{H} \left( \frac{f}{-i} \right) \right)(x) - \left( \mathcal{H} \left( \frac{f}{-i} \right) \right)(x)
\]

for \( f \) in \( D(A) = \{ f \in L^p_c(\mathbb{R}) : \mathcal{H} \left( \frac{f}{-i} \right)(x) \text{ is absolutely continuous and } \frac{d}{dx} \left( \mathcal{H} \left( \frac{f}{-i} \right) \right)(x) \in L^p_c(\mathbb{R}) \} \).

3) Let \( -\infty < q < \infty \).

i) If \( \mu(x) = w(x) \) let

\[
(\mathcal{T}(-iq)f)(x) = \frac{1}{2} \left[ f(x-q) - i(\mathcal{H}f)(x-q) \right] + \frac{1}{2} \left[ f(x+q) + i(\mathcal{H}f)(x+q) \right]
\]

for \( f \) in \( L^p_c(\mathbb{R}) \).

ii) If \( \mu(x) = (1+x^2)^{-p/2}w(x) \) let
\[(T(-iq)f)(x) = \frac{1}{2} \left[ f(x-q) - i(x-i) \left( H \left( \frac{f}{i+q} \right) \right)(x-q) \right] + \frac{1}{2} \left[ f(x+q) + i(x-i) \left( H \left( \frac{f}{i-q} \right) \right)(x+q) \right] - \frac{iq}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(u-i+q)(u-i-q)} \, du \quad (17)\]

for \(f\) in \(L^p_{\mu}(R)\).

Then \(T(r-iq)f\) converges to \(T(-iq)f\) as \(r\to+0\) in the \(L^p_{\mu}\) norm.

4) The family \(\{T(iq): -\infty < q < \infty\}\) is a strongly continuous group of linear bounded operators on \(L^p_{\mu}(R)\) with the infinitesimal generator \(iA\) and the domain \(D(A)\) in the above 2).

5) If \(t=r-iq, r>0, -\infty < q < \infty\), then

\[\|T(t)\| \leq K_p e^{\alpha |q|}, \quad (18)\]

where \(K_p, \alpha, \beta\) are some constant numbers.

**Proof.**

1) For \(t_1, t_2 \in C^+\) let us show

\[T(t_1)T(t_2)f = T(t_1+t_2)f \quad (19)\]

for \(f\) in \(L^p_{\mu}(R)\). If \(f\) belongs to \(S(R)\), (19) holds. Since \(S(R)\) is dense in \(L^p_{\mu}(R)\) and \(T(t_1), T(t_2)\) and \(T(t_1+t_2)\) are bounded operators (19) holds for all \(f\) in \(L^p_{\mu}(R)\). Let us show that

\[\lim_{t \in C^+, t \to t_0} T(t)f = T(t_0)f \quad (20)\]

for \(t_0 \in C^+\) in the \(L^p_{\mu}\) norm. If \(f\) belongs to \(S(R)\) we obtain (20). Since \(S(R)\) is dense in \(L^p_{\mu}(R)\), by (12) we can show that (20) also holds for all \(f\) in \(L^p_{\mu}(R)\). Next let us show that \(T(t)\) is holomorphic in \(C^+\). By Morera's theorem it suffices to show that when \(\Gamma\) is any triangular path in \(C^+\)

\[\int_{\Gamma} \left\{ \int_{-\infty}^{\infty} (T(t)f)(x)g(x) \, dx \right\} \, dt = 0 \quad (21)\]

holds for any \(g\) in \(L^{p-1/(p-1)}_{\mu}(R)\). Since \(p(t,u)\) is holomorphic in \(C^+\) and bounded over \(\Gamma \times R\), by the Fubini theorem we see that (21) holds for \(f\) and \(g\) in \(S(R)\). Hence by taking two sequences of functions \(\{f_n\}, \{g_n\}\) in \(S(R)\) which converge to \(f\) and \(g\) in the \(L^p_{\mu}\) norm and in the \(L^{p-1/(p-1)}_{\mu}\) norm, respectively and by using the dominated convergence theorem we obtain (21) for \(f\) in \(L^p_{\mu}(R)\) and \(g\) in \(L^{p-1/(p-1)}_{\mu}(R)\).

2) i) Set \(A = -\frac{d}{dx} H\) and \(D(A) = \{f \in L^p_{\mu}(R): (Hf)(x)\) is absolutely continuous and \(\frac{d}{dx} (Hf)(x) \in L^p_{\mu}(R)\}\). Let us denote the infinitesimal generator of the semigroup of \(\{T(t): t \geq 0\}\) by \(C\) and its domain \(D(C)\). By (5) and (6) it is seen that
It is known in [5] that the resolvent $R(\lambda, C)$ of the generator $C$ is given by

$$R(\lambda, C) = \int_0^\infty e^{-\lambda t} T(t) dt$$

for $\lambda > \omega_0$. Consequently

$$D(C) = R(1, C)[L^p_w(R)]$$

holds. We shall prove that $D(A) = D(C)$ and $Af = Cf$ for $f$ in $D(A) = D(C)$. Let $f$ be in $S(R)$. We see that

$$\text{cf. [3. (42)], [10].}$$

Hence we have

$$(\lambda - A)R(\lambda, C)f = f$$

for $f$ in $S(R)$. Since $S(R)$ is dense in $L^p_w(R)$ and $A$ is the closed operator the equality (24) holds for all $f$ in $L^p_w(R)$. That $A$ is closed follows from that if $(f, \varphi') = (g, \varphi)$ holds for $f, g$ in $L^p_w(R)$ and for all $\varphi$ in $S(R)$, $f'(x) = g(x)$ holds for almost all $x$. By (22) we obtain that

$$D(A) \subseteq D(C) \text{ and } Af = Cf \text{ for } f \text{ in } D(C).$$

Let us show $D(A) \subseteq D(C)$. It is known in [5] that

$$\lambda R(\lambda, C)f \longrightarrow f \text{ as } \lambda \longrightarrow \infty$$

in the $L^p_w$ norm for all $f$ in $L^p_w(R)$. Let $f$ be in $D(A)$ and let us show

$$C\lambda R(\lambda, C)f = A\lambda R(\lambda, C)f = \lambda R(\lambda, C)Af.$$ (27)

Then by (26) $Cf = Af$ holds for $f$ in $D(A)$. Thus $D(A) \subseteq D(C)$ holds and so we obtain that $D(A) = D(C)$ and $Af = Cf$ for $f$ in $D(A) = D(C)$. At first let $f$ be in $S(R)$. By (23), $H[\lambda R(\lambda, C)f] = \lambda R(\lambda, C)[Hf]$ holds. Since the operators $H$ and $R(\lambda, C)$ are continuous this equality holds for all $f$ in $L^p_w(R)$. By (25) we see that

$$C\lambda R(\lambda, C)f, \varphi = (A\lambda R(\lambda, C)f, \varphi) = (\frac{d}{dx} [\lambda R(\lambda, C)Hf], \varphi) = (\lambda R(\lambda, C)Af, \varphi)$$

for $\varphi$ in $S(R)$ and $f$ in $D(A)$. Since $S(R)$ is dense in $L^p_{w-1/(p-1)}(R)$ we obtain (27).

Set $A_1 = -\frac{d}{dx}$ and $D(A_1) = \{f \in L^p_w(R) : f(x) \text{ is absolutely continuous and } f' \in L^p_w(R)\}$. We can also show that the operator $A_1$ is closed, and that $D(A_1) \supset D(C)$ and $A_1f = Cf$ for $f$ in $D(C)$. Hence we obtain the assertion i).

ii) Set $a = 1$ and $q = 0$ in (11). By (4)–(7) and by i) and the fact that
\[ H\left( \frac{f}{-1} \right)(x) = (x - i)\left( H\left( \frac{f}{(x - i)^2} \right) \right)(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(x)(x - i)^{-2} dx \]

We can obtain the assertion ii).

3) ii) Set \( a = 1 \) in (11). By the Minkowski inequality we see that

\[
\| T(r - \iota q) f - T(-\iota q) f \|_{p, \mu} \\
\leq \left[ \int_{-\infty}^{\infty} \left| \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u; 1, -r, q)}{u - x - i(r - \iota q)} \, du \right|^p w(x) \, dx \right]^{1/p} \\
+ \left[ \int_{-\infty}^{\infty} \left| -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u; 1, r, q)}{u - x + i(r - \iota q)} \, du \right|^p w(x) \, dx \right]^{1/p} \\
+ \left[ \int_{-\infty}^{\infty} \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(u; 1, r, q)}{u - i - i(r - \iota q)} \, du \right|^p \mu(x) \, dx \right]^{1/p}.
\]

(28)

By (4) and by the strong continuity (20) the first and the second terms of (28) converge to 0 as \( r \to +0 \). By the dominated convergence theorem the third term of (22) also converges to 0 as \( r \to +0 \).

i) This follows from the same arguments as the above ii).

4) From the semigroup property and the strong continuity (20) the group property of the \( \{ T(iq): -\infty < q < \infty \} \) follows. By Lemma 2 and 1), 2) and by [5. Theorem 17.9.2] we see that the infinitesimal generator of the group of the family \( \{ T(iq): -\infty < q < \infty \} \) is the operator \( iA \) with the domain \( D(A) \).

5) By Lemma 2 and 1), 3), 4) and [5. Theorem 17.9.1–17.9.2] we obtain (18).

Q.E.D.

References


