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A Note on Homomorphisms of C*-algebras

Sin-ei Takahasi*

J. Anderson [2] investigated the extension questions for arbitrary C*-algebras. Moreover H. Kim and Y. Kim [3] generalized some results in [2]. One of their results is the following: If \( f \) is a pure state on a unital C*-algebra \( A \) and \( G^+(f) \) commutes with every element of \( A \), then \( f \) is a homomorphism. Here \( G^+(f) \) denotes the set of all positive elements \( t \) of \( A \) such that \( f(t) = \|t\| = 1 \).

The purpose of this note is to generalize the above result. Our method is quite different from that given in [3]. In fact they have proved this by using the map: \( x \mapsto \alpha_x(f) = \inf \{ \|t x t^*\| : t \in G(f) \} \) which is considered in [2]. However we shall prove this result by considering the supports of states on C*-algebras.

Throughout the remainder of the note let \( A \) be a C*-algebra with an identity \( e \), \( B \) be a C*-subalgebra of \( A \) containing \( e \) and \( f \) be a state on \( A \). Also let \( L(f) \) be the left ideal associated with \( f \) and \( L^*(f) = \{ x \in A : x^* \in L(f) \} \). Furthermore let \( E(f) \) be the support of \( f \) in the second dual \( A^{**} \) of \( A \) and \( \overline{E(f)} \) be the closure of \( E(f) \). Actually \( \overline{E(f)} \) is the smallest closed projection majorizing \( E(f) \) (c.f. [1, Definition II. 11]). For arbitrary subsets \( X \) and \( Y \) of \( A \), let \( X \bowtie Y \) mean that \( xy = yx \) for all \( x \in X \) and \( y \in Y \).

Under the above notations we have the following statements.

(i) \( G^+(f) \bowtie B \) if and only if \( L(f) \bowtie L^*(f) \bowtie B \).

(ii) If \( L(f) \bowtie L^*(f) \bowtie B \), then \( \overline{E(f)} \bowtie B \).

(iii) If \( E(f) \bowtie B \) and \( f \) is a pure state, then \( f|B \) is a homomorphism.

In fact observe that \( \{ x \in A : e - x \in G^+(f) \} = \{ x \in L(f) \cap L^*(f) : 0 \leq x \leq e \} \).

Since \( e \in B \), (i) follows from the above equality. Now since \( \overline{E(f)} \) is closed, there exists a positive net \( \{ e_\lambda \} \subset A \) such that \( e_\lambda \leq L(f) \) in the weak*-topology of \( A^{**} \).

Note that \( (e - e_\lambda) \overline{E(f)} = 0 \) and hence \( e - e_\lambda \in L(f) \cap L^*(f) \) for each \( \lambda \). Therefore if \( L(f) \bowtie L^*(f) \bowtie B \), then \( e_\lambda \bowtie B \) for each \( \lambda \) and so \( \overline{E(f)} \bowtie B \). Thus (ii) is proved. If \( f \) is a pure state, then \( E(f) \) is minimal and so \( E(f) x E(f) = f(x) E(f) \) for all \( x \in A^{**} \). Therefore we have that \( b^* b E(f) = b^* E(f) b E(f) = f(b^* b) E(f) \) and so \( f(b^* b) = f(b^*) f(b) \) whenever \( E(f) \bowtie B \) and \( b \in B \). This implies easily (iii).

The following result is our promised generalization of the result of H. Kim and Y. Kim.

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Proposition. If $G^+(f) \supset B$ and if $f$ is a pure state, then $f\{B$ is a homomorphism.

Proof. If $G^+(f) \supset B$, then $E(f) \supset B$ from (i) and (ii). Furthermore if $f$ is a pure state, then $E(f)$ is a closed projection and hence $E(f) \supset B$ from the above argument. Therefore our result follows from the statement (iii).

References