タイトル
A Remark on the Littlewood Conjecture

著者
YABUTA, Kozo

引文
Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 14: 19-21

発行日
1982

URL
http://hdl.handle.net/10109/2952

権利
このリポジトリに収録されているコンテンツの著作権は、それぞれの著作権者に帰属します。引用、転載、複製等される場合は、著作権法を遵守してください。
A Remark on the Littlewood Conjecture

Kôzô Yabuta*

Recently McGehee, Pigno and Smith [1, 2] have solved affirmatively the Littlewood conjecture on the $L^1$ norm of exponential sums. In this short note we improve their method somewhat and obtain a somewhat better constant arising in the inequality in the conjecture. That is, we will show the following

**Theorem 1.** Let $r \in \{2, 3, \ldots\}$ and $b > 0$ such that $B(r, b) < 1$. Then it holds

\[(1) \quad (2\pi)^{-1} \int_0^{2\pi} |\sum_{j=1}^n c_j e^{i n_j x}| dx \geq A(r, b) \log (1 + (r-1)N)\]

for any integers $n_1 < n_2 < \cdots < n_N$ and complex numbers $c_j$ with $|c_j| \geq 1 (j = 1, 2, \ldots, N)$. Here $B(r, b) = \sqrt{2br(\sqrt{r} + 1)}/(r-1)^2$ and $A(r, b) = (1 - B(r, b))(1 - e^{-b})/\log r$.

If we take $r = 90$ and $b = 1.7$, then $A(r, b) = 0.129590\cdots$. This number is almost optimal with respect to $r$ and $b$. What is the best estimate in the inequality (1)? $L^1$ norm of the Dirichlet kernel is asymptotically equal to $\sqrt{2}/4 = 0.4049\cdots$. Is $\sqrt{2}/4 \log N$ the best one?

1. To prove Theorem 1 we follow McGehee, Pigno and Smith [1]. $Z$ will denote the set of all integers and $T$ the circle group, identified with $[0, 2\pi)$ with the normalized Haar measure $(2\pi)^{-1} dx$. Now let $r \in \{2, 3, \ldots\}$. For any $S = \{n_1 < n_2 < \cdots < n_N\} \subseteq Z$ we divide $S$ as follows. $S_0 = \{n_1\}$, $S_1 = \{n_2 < \cdots < n_{r+1}\}$, $S_M = \{n_L < \cdots < n_N\}$ so that $S = \bigcup_{j=0}^M S_j$, $\# S_j = r^j$ ($j = 0, 1, \ldots, M - 1$) and $\# S_M \leq r^M$, where $M$ is the integer satisfying $M < \log(1 + (r-1)N)/\log r \leq M + 1$, $L = (r^M - 1)/(r-1)$ and $\# S_j$ denotes the cardinal number of $S_j$. Then we have the following lemma.

**Lemma 1.** Let $r \in \{2, 3, \ldots\}$, $b > 0$ and $c = 1 - e^{-b}$. Then for any $S = \{n_1, n_2, \ldots, n_N\} \subseteq Z$ and $a_n \in \mathbb{C}$ with $|a_n| = 1$ ($n \in S$), there exists an $F \in L^\infty(T)$ with $|F| \leq 1$ such that

\[(2) \quad |ca_n - r^j \hat{F}(n)| \leq cB(r, b), \quad n \in S_j (j = 0, 1, \ldots, M - 1),\]

\[(3) \quad ca_n = \# S_M \hat{F}(n), \quad n \in S_M.\]
where \( \tilde{F}(n) \) is the \( n \)-th Fourier coefficient of \( F \).

PROOF. We introduce the following as in [1]. For \( f \in L^2(T) \) we set \( P_n(f) = \sum_{k \leq n} f(k)e^{ikx} \) and \( Q_n(f) = \sum_{k > n} f(k)e^{ikx} \). Now put \( f_j(x) = \sum_{k \in S_j} a_k(n(S_j))^{-1}e^{inx} \), \( F_0 = cf_0 \), and \( F_k = F_{k-1} \times \exp(-bh_k) + cf_k \) \((k=1, \ldots, M)\). Finally put \( F = F_M \). Then, arguing as in [1, p. 615] we get \( |F| \leq 1 \). We use here, however, the inequality \( \exp(-bx) + cx \leq 1 \) \((0 \leq x \leq 1)\) in place of the inequality \( \exp(-x/4) + x/5 \leq 1 \) \((0 \leq x \leq 1)\). Clearly \( \tilde{F}(n) = c\tilde{f}_M(n) = c(n(S_M))^{-1}a_n \) \((n \in S_M)\). For \( n \in S_j \) \((j=0, 1, \ldots, M-1)\), we get

\[
|c^{r-1}a_n - \tilde{F}(n)| = \sum_{k=j}^{M-1} \left| (1 - \exp(-bh_{k+1})) F_k \right| (n).
\]

If we put \( m_j = \min S_j \), we can replace \( F_k \) by \( P_{m_j}(F_k) \) and arguing as in [1, p. 615] we get

\[
D_{j,k}(n) = \left| \left(1 - \exp(-bh_{k+1}) \right) F_k \right| (n) |
\leq \sqrt{2}bh_{k+1} \times c \sum_{q=j}^{k} \| P_{m_j}(f_q) \|_2
= \sqrt{2}bc r^{(k+1)/2} \sum_{q=j}^{k} r^{-q/2}.
\]

Hence

\[
|c^{r-1}a_n - \tilde{F}(n)| \leq \sum_{k=j}^{M-1} D_{j,k}(n) \leq \sqrt{2}bc r^{1-(r-1)/(r+1)}(r-1)^2,
\]

which completes the proof of the lemma.

PROOF OF THE theorem. Let \( f(x) = \sum_{j=1}^{N} c_j e^{inx} \). Taking \( a_n = c_j |c_j| \) and applying Lemma 1, we can choose an \( F \in L^\infty \) with \( \| F \|_\infty \leq 1 \) such that

\[
|\tilde{F}(n)| - c(n(S_j))^{-1} |\tilde{f}(n)| \leq cB(r, b)(n(S_j))^{-1} |\tilde{f}(n)| \quad (n \in S_j),
\]

\( j = 0, 1, \ldots, M \). Hence

\[
\sum_{j=1}^{\infty} \text{Re} \tilde{F}(n) \tilde{f}(n) \geq c(1 - B(r, b))(M + 1).
\]

Therefore, if \( B(r, b) < 1 \), we get

\[
\| f \|_1 \geq \left| \sum_{j=1}^{\infty} \text{Re} \tilde{F}(n) \tilde{f}(n) \right| \geq c(1 - B(r, b)) \log (1 + (r-1)N) / \log r.
\]

2. If we apply Lemma 1 to the generalization of Hardy's inequality in [1, Theorem 2], we can take \( C = c(1 - B(r, b))(r-1) \). That is, for any \( S = \{ n_1 < n_2 < \cdots \} \subset \mathbb{Z} \) and any Borel measure \( \mu \) on \( T \) with support \( \mu \subset S \), we have

\[
\sum_{j=1}^{\infty} |\mu(n_j)|/j \leq c(1 - B(r, b))(r-1)^{-1} \| \mu \|.
\]

If \( r = 3 \) and \( b = 0.16 \), the constant in the above inequality is equal to 25.219..., and this is almost optimal with respect to the choice of \( r \) and \( b \). What is the best possible constant in the generalized Hardy inequality? Is it \( \pi \)?
References
